Problem 1 (Netherlands). Find all triples (a, b, c) of real numbers such that $ab + bc + ca = 1$ and

$$a^2b + c = b^2c + a = c^2a + b.$$

Solution 1. First suppose that $a = 0$. Then we have $bc = 1$ and $c = b^2c = b$. So $b = c$, which implies $b^2 = 1$ and hence $b = \pm 1$. This leads to the solutions $(a, b, c) = (0, 1, 1)$ and $(a, b, c) = (0, -1, -1)$. Similarly, $b = 0$ gives the solutions $(a, b, c) = (1, 0, 1)$ and $(a, b, c) = (-1, 0, -1)$, while $c = 0$ gives $(a, b, c) = (1, 1, 0)$ and $(a, b, c) = (-1, -1, 0)$.

Now we may assume that $a, b, c \neq= 0$. We multiply $ab + bc + ca = 1$ by a to find $a^2b + abc + ca^2 = a$, hence $a^2b = a - abc - a^2c$. Substituting this in $a^2b + c = b^2c + a$ yields $a - abc - a^2c + c = b^2c + a$, so $b^2c + abc + a^2c = c$. As $c \neq= 0$, we find $b^2 + ab + a^2 = 1$.

Analogously we have $b^2 + bc + c^2 = 1$ and $a^2 + ac + c^2 = 1$. Adding these three equations yields $2\left(a^2 + b^2 + c^2\right) + ab + bc + ca = 3$, which implies $a^2 + b^2 + c^2 = 1$. Combining this result with $b^2 + ab + a^2 = 1$, we get $1 - ab = 1 - c^2$, so $c^2 = ab$.

Analogously we also have $b^2 = ac$ and $a^2 = bc$. In particular we now have that ab, bc and ca are all positive. This means that a, b and c must all be positive or all be negative. Now assume that $|c|$ is the largest among $|a|$, $|b|$ and $|c|$, then $c^2 \geq |ab| = ab = c^2$, so we must have equality. This means that $|c| = |a|$ and $|c| = |b|$. Since (a, b, c) must all have the same sign, we find $a = b = c$. Now we have $3a^2 = 1$, hence $a = \pm\frac{1}{3}\sqrt{3}$. We find the solutions $(a, b, c) = \left(\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}\right)$ and $(a, b, c) = \left(-\frac{1}{3}\sqrt{3}, -\frac{1}{3}\sqrt{3}, -\frac{1}{3}\sqrt{3}\right)$.

We conclude that all possible triples (a, b, c) are $(0, 1, 1)$, $(0, -1, -1)$, $(1, 0, 1)$, $(-1, 0, -1)$, $(1, 1, 0)$, $(-1, -1, 0)$, $\left(\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}\right)$ and $\left(-\frac{1}{3}\sqrt{3}, -\frac{1}{3}\sqrt{3}, -\frac{1}{3}\sqrt{3}\right)$.

Solution 2. From the problem statement $ab = 1 - bc - ca$ and thus $b^2c + a = a^2b + c = a - abc - a^2c + c$, $c\left(b^2 + a^2 + ab - 1\right) = 0$. If $c = 0$ then $ab = 1$ and $a^2b = b$, which implies $a = b = \pm 1$. Otherwise $b^2 + a^2 + ab = 1$. Cases $a = 0$ and $b = 0$ are completely analogous to $c = 0$, so we may suppose that $a, b, c \neq 0$. In this case we end up with

$$\begin{cases} a^2 + b^2 + ab = 1, \\ b^2 + c^2 + bc = 1, \\ c^2 + a^2 + ca = 1, \\ ab + bc + ca = 1. \end{cases}$$

Adding first three equations and subtracting the fourth yields $2(a^2 + b^2 + c^2) = 2 = 2(ab + bc + ca)$. Consequently, $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$. Now we can easily conclude that $a = b = c = \pm\frac{1}{\sqrt{3}}$.

Solution by Achilleas Sinefakopoulos, Greece. We have

$$c(1 - b^2) = a(1 - ab) = a(bc + ca) = c(ab + a^2),$$

and so

$$c(a^2 + ab + b^2 - 1) = 0.$$

Similarly, we have

$$b(a^2 + ac + c^2 - 1) = 0 \quad \text{and} \quad a(b^2 + bc + c^2 - 1) = 0.$$

If $c = 0$, then we get $ab = 1$ and $a^2b = a = b$, which give us $a = b = 1$, or $a = b = -1$. Similarly, if $a = 0$, then $b = c = 1$, or $b = c = -1$, while if $b = 0$, then $a = c = 1$, or $a = c = -1$.

So assume that $abc \neq 0$. Then

$$a^2 + ab + b^2 = b^2 + bc + c^2 = c^2 + ca + a^2 = 1.$$

Adding these gives us

$$2(a^2 + b^2 + c^2) + ab + bc + ca = 3,$$

and using the fact that $ab + bc + ca = 1$, we get

$$a^2 + b^2 + c^2 = 1 = ab + bc + ca.$$

Hence

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 2(a^2 + b^2 + c^2) - 2(ab + bc + ca) = 0$$

and so $a = b = c = \pm\dfrac{1}{\sqrt{3}}$.

Therefore, the solutions (a, b, c) are $(0, 1, 1)$, $(0, -1, -1)$, $(1, 0, 1)$, $(-1, 0, -1)$, $(1, 1, 0)$, $(-1, -1, 0)$, $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$

Solution by Eirini Miliori (HEL2). It is $ab + bc + ca = 1$ and

$$a^2b + c = b^2c + a = c^2a + b. \tag{1}$$

We have

$$\begin{aligned} a^2b + c = b^2c + a &\iff a^2b - a = b^2c - c \\ &\iff a(ab - 1) = c(b^2 - 1) \\ &\iff a(-bc - ac) = c(b^2 - 1) \\ &\iff -ac(a + b) = c(b^2 - 1) \end{aligned} \tag{2}$$

First, consider the case where one of a, b, c is equal to 0. Without loss of generality, assume that $a = 0$. Then $bc = 1$ and $b = c$ from (1), and so $b^2 = 1$ giving us $b = 1$ or -1. Hence $b = c = 1$ or $b = c = -1$.

Therefore, (a, b, c) equals one of the triples $(0, 1, 1)$, $(0, -1, -1)$, as well as their rearrangements $(1, 0, 1)$ and $(-1, 0, -1)$ when $b = 0$, or $(1, 1, 0)$ and $(-1, -1, 0)$ when $c = 0$.

Now consider the case where $a \neq 0$, $b \neq 0$ and $c \neq 0$. Then (2) gives us

$$-a(a + b) = b^2 - 1 \iff -a^2 - ab = b^2 - 1 \iff a^2 + ab + b^2 - 1 = 0.$$

The quadratic $P(x) = x^2 + bx + b^2 - 1$ has $x = a$ as a root. Let x_1 be its second root (which could be equal to a in the case where the discriminant is 0). From Vieta's formulas we get

$$\begin{cases} x_1 + a = -b & \iff x_1 = -b - a, \text{ and} \\ x_1 a = b^2 - 1 & \iff x_1 = \frac{b^2-1}{a}. \end{cases}$$

Using $a^2b + c = c^2a + b$ we obtain $b(a^2 - 1) = c(ac - 1)$ yielding $a^2 + ac + c^2 - 1 = 0$ in a similar way. The quadratic $Q(x) = x^2 + cx + c^2 - 1$ has $x = a$ as a root. Let x_2 be its second root (which could be equal to a in the case where the discriminant is 0). From Vieta's formulas we get

$$\begin{cases} x_2 + a = -c & \iff x_2 = -c - a, \text{ and} \\ x_2 a = c^2 - 1 & \iff x_2 = \frac{c^2-1}{a}. \end{cases}$$

Then

$$\begin{cases} x_1 + x_2 = -b - a - c - a, \text{ and} \\ x_1 + x_2 = \frac{b^2-1}{a} + \frac{c^2-1}{a}, \end{cases}$$

which give us

$$\begin{aligned} -(2a + b + c) = \frac{b^2 - 1}{a} + \frac{c^2 - 1}{a} &\iff -2a^2 - ba - ca = b^2 + c^2 - 2 \\ &\iff bc - 1 - 2a^2 = b^2 + c^2 - 2 \\ &\iff 2a^2 + b^2 + c^2 = 1 + bc. \qquad (3) \end{aligned}$$

By symmetry, we get

$$2b^2 + a^2 + c^2 = 1 + ac, \text{ and} \qquad (4)$$
$$2c^2 + a^2 + b^2 = 1 + bc \qquad (5)$$

Adding equations (3), (4), and (5), we get

$$4(a^2 + b^2 + c^2) = 3 + ab + bc + ca \iff 4(a^2 + b^2 + c^2) = 4 \iff a^2 + b^2 + c^2 = 1.$$

From this and (3), since $ab + bc + ca = 1$, we get

$$a^2 = bc = 1 - ab - ac \iff a(a + b + c) = 1.$$

Similarly, from (4) we get

$$b(a + b + c) = 1,$$

and from (4),

$$c(a + b + c) = 1.$$

Clearly, it is $a + b + c \neq 0$ (for otherwise it would be $0 = 1$, a contradiction). Therefore,

$$a = b = c = \frac{1}{a + b + c},$$

and so $3a^2 = 1$ giving us $a = b = c = \pm\frac{1}{\sqrt{3}}$.

In conclusion, the solutions (a, b, c) are $(0, 1, 1)$, $(0, -1, -1)$, $(1, 0, 1)$, $(-1, 0, -1)$, $(1, 1, 0)$, $(-1, -1, 0)$, $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

Solution by ISR5. First, homogenize the condition $a^2b + c = b^2c + a = c^2a + b$ by replacing c by $c(ab + bc + ca)$ (etc.), yielding

$$a^2b + c = a^2b + abc + bc^2 + c^2a = abc + \sum_{cyc} a^2b + (c^2b - b^2c) = abc + \sum_{cyc} a^2b + bc(c - b).$$

Thus, after substracting the cyclicly symmetric part $abc + \sum_{cyc} a^2b$ we find the condition is eqivalent to

$$D := bc(c-b) = ca(a-c) = ab(b-a).$$

Ending 1. It is easy to see that if e.g. $a=0$ then $b=c=\pm 1$, and if e.g. $a=b$ then either $a=b=c=\pm\frac{1}{\sqrt{3}}$ or $a=b=\pm 1, c=0$, and these are indeed solutions. So, to show that these are *all* solutions (up to symmetries), we may assume by contradiction that a,b,c are pairwise different and non-zero. All conditions are preserved under cyclic shifts and under simultaenously switching signs on all a,b,c, and by applying these operations as necessary we may assume $a<b<c$. It follows that $D^3 = a^2b^2c^2(c-b)(a-c)(b-a)$ must be negative (the only negative term is $a-c$, hence D is negative, i.e. $bc, ab < 0 < ac$. But this means that a,c have the same sign and b has a different one, which clearly contradicts $a<b<c$! So, such configurations are impossible.

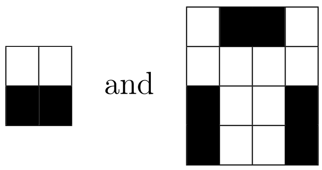
Ending 2. Note that $3D = \sum c^2b - \sum b^2c = (c-b)(c-a)(b-a)$ and $D^3 = a^2b^2c^2(c-b)(a-c)(b-a) = -3a^2b^2c^2D$. Since $3D$ and D^3 must have the same sign, and $-3a^2b^2c^2$ is non-positive, necessarily $D=0$. Thus (up to cyclic permutation) $a=b$ and from there we immediately find either $a=b=\pm 1, c=0$ or $a=b=c=\pm\frac{1}{\sqrt{3}}$.

Problem 2 (Luxembourg). Let n be a positive integer. Dominoes are placed on a $2n \times 2n$ board in such a way that every cell of the board is adjacent to exactly one cell covered by a domino. For each n, determine the largest number of dominoes that can be placed in this way.

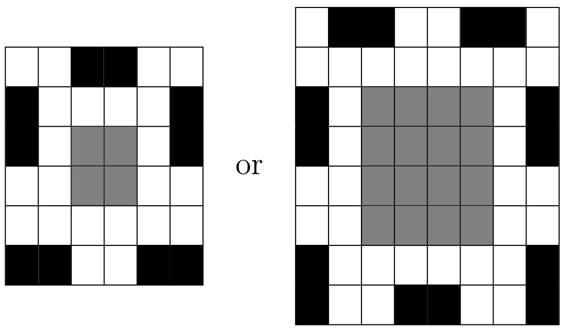
(A *domino* is a tile of size 2×1 or 1×2. Dominoes are placed on the board in such a way that each domino covers exactly two cells of the board, and dominoes do not overlap. Two cells are said to be *adjacent* if they are different and share a common side.)

Solution 1. Let M denote the maximal number of dominoes that can be placed on the chessboard. We claim that $M = n(n+1)/2$. The proof naturally splits into two parts: we first prove that $n(n+1)/2$ dominoes can be placed on the board, and then show that $M \leq n(n+1)/2$ to complete the proof.

We construct placings of the dominoes by induction. The base cases $n = 1$ and $n = 2$ correspond to the placings



Next, we add dominoes to the border of a $2n \times 2n$ chessboard to obtain a placing of dominoes for the $2(n+2) \times 2(n+2)$ board,



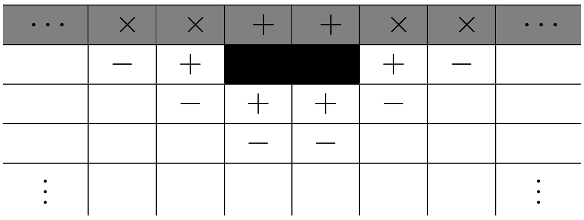
depending on whether n is odd or even. In these constructions, the interior square is filled with the placing for the $2n \times 2n$ board. This construction adds $2n + 3$ dominoes, and therefore places, in total,

$$\frac{n(n+1)}{2} + (2n+3) = \frac{(n+2)(n+3)}{2}$$

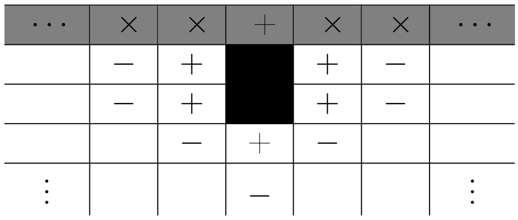
dominoes on the board. Noticing that the contour and the interior mesh together appropriately, this proves, by induction, that $n(n+1)/2$ dominoes can be placed on the $2nn$ board.

To prove that $M \leq n(n+1)/2$, we border the $2n \times 2n$ square board up to a $(2n+2) \times (2n+2)$ square board; this adds $8n + 4$ cells to the $4n^2$ cells that we have started with. Calling a cell covered if it belongs to a domino or is adjacent to a domino, each domino on the $2n \times 2n$ board is seen to cover exactly 8 cells of the $(2n+2) \times (2n+2)$ board (some of which may belong to the border). By construction, each of the $4n^2$ cells of the $2n \times 2n$ board is covered by precisely one domino.

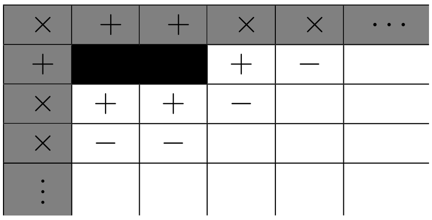
If two adjacent cells on the border, away from a corner, are covered, then there will be at least two uncovered cells on both sides of them; if one covered cell lies between uncovered cells, then again, on both sides of it there will be at least two uncovered cells; three or more adjacent cells cannot be all covered. The following diagrams, in which the borders are shaded, $\times$ marks an uncovered cell on the border, $+$ marks a covered cell not belonging to a domino, and $-$ marks a cell which cannot belong to a domino, summarize the two possible situations,



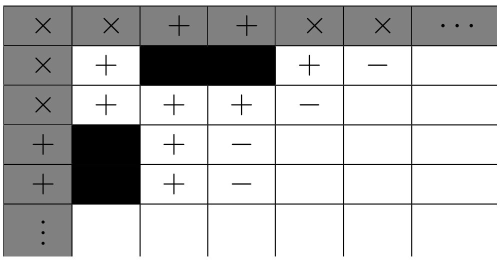
or



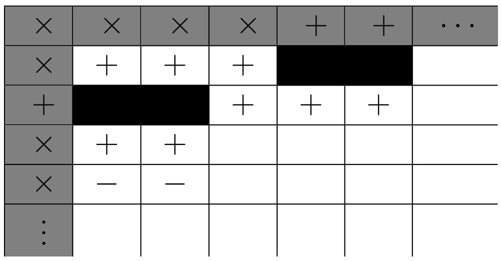
Close to a corner of the board, either the corner belongs to some domino,



or one of the following situations, in which the corner cell of the original board is not covered by a domino, may occur:



or

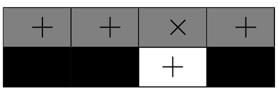


It is thus seen that at most half of the cells on the border, i.e. $4n+2$ cells, may be covered, and hence

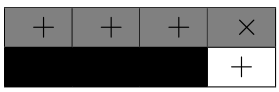
$$M \leq \left[\frac{4n^2+(4n+2)}{8}\right] = \left[\frac{n(n+1)}{2}+\frac{1}{2}\right] = \frac{n(n+1)}{2},$$

which completes the proof of our claim.

Solution 2. We use the same example as in Solution 1. Let M denote the maximum number of dominoes which satisfy the condition of the problem. To prove that $M \leq n(n+1)/2$, we again border the $2n \times 2n$ square board up to a $(2n+2)\times(2n+2)$ square board. In fact, we shall ignore the corner border cells as they cannot be covered anyway and consider only the $2n$ border cells along each side. We prove that out of each four border cells next to each other at most two can be covered. Suppose three out of four cells A, B, C, D are covered. Then there are two possibilities below:



or



The first option is that A, B and D are covered (marked with $+$ in top row). Then the cells inside the starting square next to A, B and D are covered by the dominoes, but the cell in between them has now two adjacent cells with dominoes, contradiction. The second option is that A, B and C are covered. Then the cells inside the given square next to A, B and C are covered by the dominoes. But then the cell next to B has two adjacent cells with dominoes, contradiction.

Now we can split the border cells along one side in groups of 4 (leaving one group of 2 if n is odd). So when n is even, at most n of the $2n$ border cells along one side can be covered, and when n is odd, at most $n+1$ out of the $2n$ border cells can be covered. For all four borders together, this gives a contribution of $4n$ when n is even and $4n+4$ when n is odd. Adding $4n^2$ and dividing by 8 we get the desired result.

Solution (upper bound) by ISR5. Consider the number of pairs of adjacent cells, such that one of them is covered by a domino. Since each cell is adjacent to one covered cell, the number of such pairs is exactly $4n^2$. On the other hand, let n_2 be the number of covered corner cells, n_3 the number of covered edge cells (cells with 3 neighbours), and n_4 be the number of covered interior cells (cells with 4 neighbours). Thus the number of pairs is $2n_2 + 3n_3 + 4n_4 = 4n^2$, whereas the number of dominoes is $m = \frac{n_2+n_3+n_4}{2}$.

Considering only the outer frame (of corner and edge cells), observe that every covered cell dominates two others, so at most half of the cells are ccovered. The frame has a total of $4(2n-1)$ cells, i.e. $n_2+n_3 \leq 4n-2$. Additionally $n_2 \leq 4$ since there are only 4 corners, thus

$$8m = 4n_2+4n_3+4n_4 = (2n_2+3n_3+4n_4)+(n_2+n_3)+n_2 \leq 4n^2+(4n-2)+4 = 4n(n+1)+2$$

Thus $m \leq \frac{n(n+1)}{2} + \frac{1}{4}$, so in fact $m \leq \frac{n(n+1)}{2}$.

Solution (upper and lower bound) by ISR5. We prove that this is the upper bound (and also the lower bound!) by proving that any two configurations, say A and B, must contain exactly the same number of dominoes.

Colour the board in a black and white checkboard colouring. Let W be the set of white cells covered by dominoes of tiling A. For each cell $w \in W$ let N_w be the set of its adjacent (necessarily black) cells. Since each black cell has exactly one neighbour (necessarily white) covered by a domino of tiling A, it follows that each black cell is contained in exactly one N_w, i.e. the N_w form a partition of the black cells. Since each white cell has exactly one (necessarily black) neighbour covered by a tile of B, each B_w contains exactly one black tile covered by a domino of B. But, since each domino covers exactly one white and one black cell, we have

$$|A| = |W| = |\{N_w : w \in W\}| = |B|$$

as claimed.

Problem 3 (Poland). Let ABC be a triangle such that $\angle CAB > \angle ABC$, and let I be its incentre. Let D be the point on segment BC such that $\angle CAD = \angle ABC$. Let ω be the circle tangent to AC at A and passing through I. Let X be the second point of intersection of ω and the circumcircle of ABC. Prove that the angle bisectors of $\angle DAB$ and $\angle CXB$ intersect at a point on line BC.

Solution 1. Let S be the intersection point of BC and the angle bisector of $\angle BAD$, and let T be the intersection point of BC and the angle bisector of $\angle BXC$. We will prove that both quadruples A, I, B, S and A, I, B, T are concyclic, which yields $S = T$.

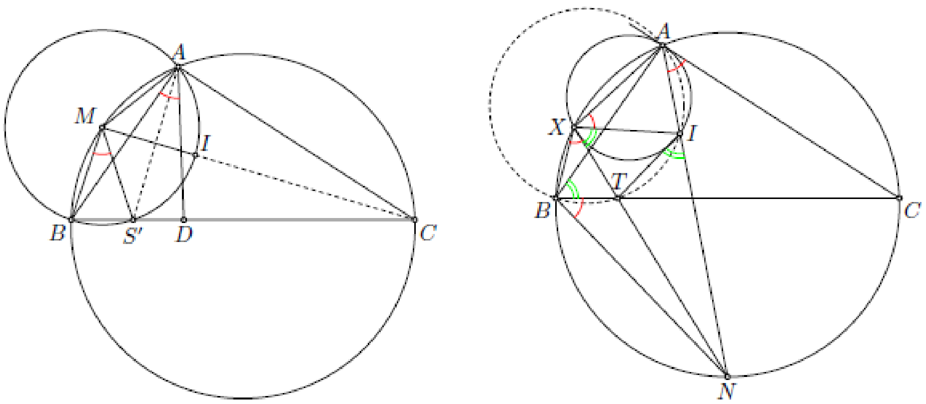
Firstly denote by M the middle of arc AB of the circumcenter of ABC which does not contain C. Consider the circle centered at M passing through A, I and B (it is well-known that $MA = MI = MB$); let it intersect BC at B and S'. Since $\angle BAC > \angle CBA$ it is easy to check that S' lies on side BC. Denoting the angles in ABC by α, β, γ we get

$$\angle BAD = \angle BAC - \angle DAC = \alpha - \beta.$$

Moreover since $\angle MBC = \angle MBA + \angle ABC = \frac{\gamma}{2} + \beta$, then

$$\angle BMS' = 180^\circ - 2\angle MBC = 180^\circ - \gamma - 2\beta = \alpha - \beta.$$

It follows that $\angle BAS' = 2\angle BMS' = 2\angle BAD$ which gives us $S = S'$.



Secondly let N be the middle of arc BC of the circumcenter of ABC which does not contain A. From $\angle BAC > \angle CBA$ we conclude that X lies on the arc AB of circumcircle of ABC not containing C. Obviously both AI and XT are passing through N. Since $\angle NBT = \frac{\alpha}{2} = \angle BXN$ we obtain $\triangle NBT \sim \triangle NXB$, therefore

$$NT \cdot NX = NB^2 = NI^2.$$

It follows that $\triangle NTI \sim \triangle NIX$. Keeping in mind that $\angle NBC = \angle NAC = \angle IXA$ we get

$$\angle TIN = \angle IXN = \angle NXA - \angle IXA = \angle NBA - \angle NBC = \angle TBA.$$

It means that A, I, B, T are concyclic which ends the proof.

Solution 2. Let $\angle BAC = \alpha$, $\angle ABC = \beta$, $\angle BCA = \gamma$ $\angle ACX = \phi$. Denote by W_1 and W_2 the intersections of segment BC with the angle bisectors of $\angle BXC$ and $\angle BAD$ respectively. Then $BW_1/W_1C = BX/XC$ and $BW_2/W_2D = BA/AD$. We shall show that $BW_1 = BW_2$.

Since $\angle DAC = \angle CBA$, triangles ADC and BAC are similar and therefore

$$\frac{DC}{AC} = \frac{AC}{BC}.$$

By the Law of sines

$$\frac{BW_2}{W_2D} = \frac{BA}{AD} = \frac{BC}{AC} = \frac{\sin\alpha}{\sin\beta}.$$

Consequently

$$\frac{BD}{BW_2} = \frac{W_2D}{BW_2} + 1 = \frac{\sin\beta}{\sin\alpha} + 1,$$

$$\frac{BC}{BW_2} = \frac{BC}{BD} \cdot \frac{BD}{BW_2} = \frac{1}{1 - DC/BC} \cdot \frac{BD}{BW_2} = \frac{1}{1 - AC^2/BC^2} \cdot \frac{BD}{BW_2} =$$

$$\frac{sin^2\alpha}{\sin^2\alpha - \sin^2\beta} \cdot \frac{\sin\beta + \sin\alpha}{\sin\alpha} = \frac{\sin\alpha}{\sin\alpha - \sin\beta}.$$

Note that $AXBC$ is cyclic and so $\angle BXC = \angle BAC = \alpha$. Hence, $\angle XBC = 180° - \angle BXC - \angle BCX = 180° - \alpha - \phi$. By the Law of sines for the triangle BXC, we have

$$\frac{BC}{W_1B} = \frac{W_1C}{W_1B} + 1 = \frac{CX}{BX} + 1 = \frac{\sin\angle CBX}{\sin\phi} + 1 =$$

$$\frac{\sin(\alpha+\phi)}{\sin\phi} + 1 = \sin\alpha\cot\phi + \cos\alpha + 1.$$

So, it's enough to prove that

$$\frac{\sin\alpha}{\sin\alpha - \sin\beta} = \sin\alpha\cot\phi + \cos\alpha.$$

Since AC is tangent to the circle AIX, we have $\angle AXI = \angle IAC = \alpha/2$. Moreover $\angle XAI = \angle XAB + \angle BAI = \phi + \alpha/2$ and $\angle XIA = 180° - \angle XAI - \angle AXI = 180° - \alpha - \phi$. Applying the Law of sines again XAC, XAI, IAC we obtain

$$\frac{AX}{\sin(\alpha+\phi)} = \frac{AI}{\sin\alpha/2},$$

$$\frac{AX}{\sin(\gamma-\phi)} = \frac{AC}{\sin\angle AXC} = \frac{AC}{\sin\beta},$$

$$\frac{AI}{\sin\gamma/2} = \frac{AC}{\sin(\alpha/2+\gamma/2)}.$$

Combining the last three equalities we end up with

$$\frac{\sin(\gamma-\phi)}{\sin(\alpha+\phi)} = \frac{AI}{AC} \cdot \frac{\sin\beta}{\sin\alpha/2} = \frac{\sin\beta}{\sin\alpha/2} \cdot \frac{\sin\gamma/2}{\sin(\alpha/2+\gamma/2)},$$

$$\frac{\sin(\gamma-\phi)}{\sin(\alpha+\phi)} = \frac{\sin\gamma\cot\phi - \cos\gamma}{\sin\alpha\cot\phi + \cos\alpha} = \frac{2\sin\beta/2\sin\gamma/2}{\sin\alpha/2},$$

$$\frac{\sin\alpha\sin\gamma\cot\phi-\sin\alpha\cos\gamma}{\sin\gamma\sin\alpha\cot\phi+\sin\gamma\cos\alpha}=\frac{2\sin\beta/2\cos\alpha/2}{\cos\gamma/2}$$

Subtracting 1 from both sides yields

$$\frac{-\sin\alpha\cos\gamma-\sin\gamma\cos\alpha}{\sin\gamma\sin\alpha\cot\phi+\sin\gamma\cos\alpha}=\frac{2\sin\beta/2\cos\alpha/2}{\cos\gamma/2}-1=$$

$$\frac{2\sin\beta/2\cos\alpha/2-\sin{(\alpha/2+\beta/2)}}{\cos\gamma/2}=\frac{\sin\beta/2\cos\alpha/2-\sin\alpha/2\cos\beta/2}{\cos\gamma/2},$$

$$\frac{-\sin(\alpha+\gamma)}{\sin\gamma\sin\alpha\cot\phi+\sin\gamma\cos\alpha}=\frac{\sin{(\beta/2-\alpha/2)}}{\cos\gamma/2},$$

$$\frac{-\sin\beta}{\sin\alpha\cot\phi+\cos\alpha}=2\sin\gamma/2\sin{(\beta/2-\alpha/2)}=$$

$$2\cos{(\beta/2+\alpha/2)}\sin{(\beta/2-\alpha/2)}=\sin\beta-\sin\alpha,$$

and the result follows. We are left to note that none of the denominators can vanish.

Solution by Achilleas Sinefakopoulos, Greece. We first note that

$$\angle BAD=\angle BAC-\angle DAC=\angle A-\angle B.$$

Let CX and AD meet at K. Then $\angle CXA=\angle ABC=\angle KAC$. Also, we have $\angle IXA=\angle A/2$, since ω is tangent to AC at A. Therefore,

$$\angle DAI=|\angle B-\angle A/2|=|\angle KXA-\angle IXA|=\angle KXI,$$

(the absolute value depends on whether $\angle B\geq\angle A/2$ or not) which means that $XKIA$ is cyclic, i.e. K lies also on ω.

Let IK meet BC at E. (If $\angle B=\angle A/2$, then IK degenerates to the tangent line to ω at I.) Note that $BEIA$ is cyclic, because

$$\angle EIA=180^\circ-\angle KXA=180^\circ-\angle ABE.$$

We have $\angle EKA=180^\circ-\angle AXI=180^\circ-\angle A/2$ and $\angle AEI=\angle ABI=\angle B/2$. Hence

$$\begin{aligned}\angle EAK&=180^\circ-\angle EKA-\angle AEI\\&=180^\circ-(180^\circ-\angle A/2)-\angle B/2\\&=(\angle A-\angle B)/2\\&=\angle BAD/2.\end{aligned}$$

This means that AE is the angle bisector of $\angle BAD$. Next, let M be the point of intersection of AE and BI. Then

$$\angle EMI=180^\circ-\angle B/2-\angle BAD/2=180^\circ-\angle A/2,$$

and so, its supplement is

$$\angle AMI = \angle A/2 = \angle AXI,$$

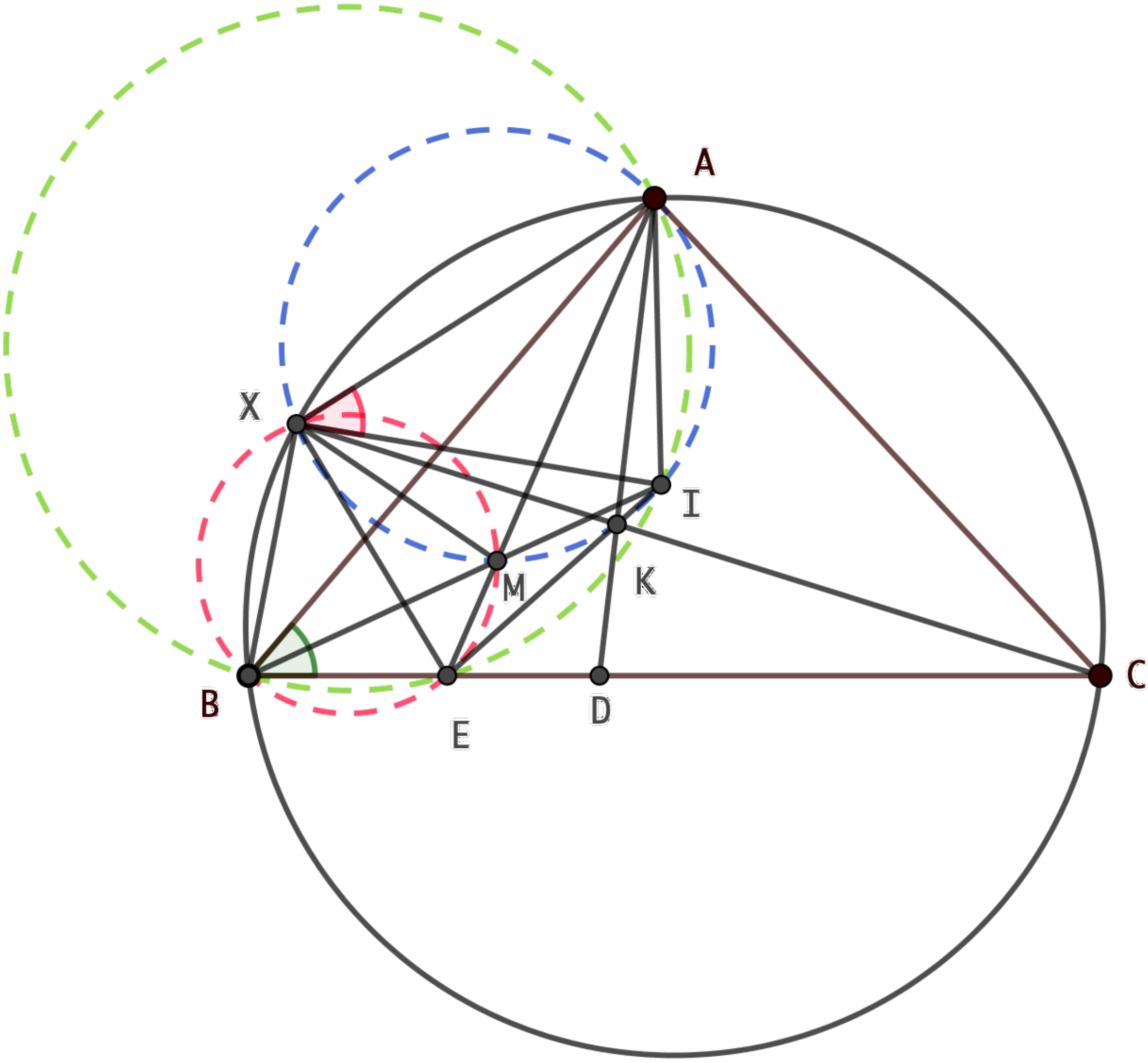
so X, M, K, I, A all lie on ω. Next, we have

$$\begin{aligned}
\angle XMA &= \angle XKA \\
&= 180^\circ - \angle ADC - \angle XCB \\
&= 180^\circ - \angle A - \angle XCB \\
&= \angle B + \angle XCA \\
&= \angle B + \angle XBA \\
&= \angle XBE,
\end{aligned}$$

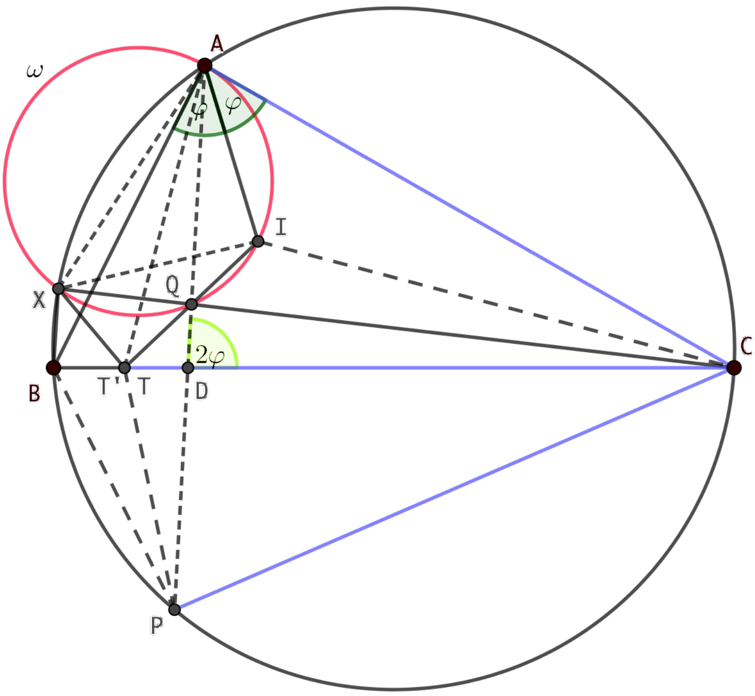
and so X, B, E, M are concyclic. Hence

$$\begin{aligned}
\angle EXC &= \angle EXM + \angle MXC \\
&= \angle MBE + \angle MAK \\
&= \angle B/2 + \angle BAD/2 \\
&= \angle A/2 \\
&= \angle BXC/2.
\end{aligned}$$

This means that XE is the angle bisector of $\angle BXC$ and so we are done!



Solution based on that by Eirini Miliori (HEL2), edited by A. Sinefakopoulos, Greece. It is $\angle ABD = \angle DAC$, and so $\overline{AC}$ is tangent to the circumcircle of $\triangle BAD$ at A. Hence $CA^2 = CD \cdot CB$.



Triangle $\triangle ABC$ is similar to triangle $\triangle CAD$, because $\angle C$ is a common angle and $\angle CAD = \angle ABC$, and so $\angle ADC = \angle BAC = 2\varphi$.

Let Q be the point of intersection of $\overline{AD}$ and $\overline{CX}$. Since $\angle BXC = \angle BAC = 2\varphi$, it follows that $BDQX$ is cyclic.Therefore, $CD \cdot CB = CQ \cdot CX = CA^2$ which implies that Q lies on ω.

Next let P be the point of intersection of $\overline{AD}$ with the circumcircle of triangle $\triangle ABC$. Then $\angle PBC = \angle PAC = \angle ABC = \angle APC$ yielding $CA = CP$. So, let T be on the side $\overline{BC}$ such that $CT = CA = CP$. Then

$$\angle TAD = \angle TAC - \angle DAC = \left(90^\circ - \frac{\angle C}{2}\right) - \angle B = \frac{\angle A - \angle B}{2} = \frac{\angle BAD}{2},$$

that is, line $\overline{AT}$ is the angle bisector of $\angle BAD$. We want to show that $\overline{XT}$ is the angle bisector of $\angle BXC$. To this end, it suffices to show that $\angle TXC = \varphi$.

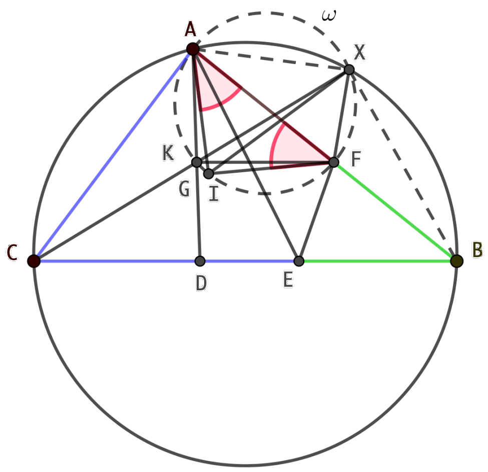
It is $CT^2 = CA^2 = CQ \cdot CX$, and so $\overline{CT}$ is tangent to the circumcircle of $\triangle XTQ$ at T. Since $\angle TXQ = \angle QTC$ and $\angle QDC = 2\varphi$, it suffices to show that $\angle TQD = \varphi$, or, in other words, that I, Q, and T are collinear.

Let T' is the point of intersection of $\overline{IQ}$ and $\overline{BC}$. Then $\triangle AIC$ is congruent to $\triangle T'IC$, since they share $\overline{CI}$ as a common side, $\angle ACI = \angle T'CI$, and

$$\angle IT'D = 2\varphi - \angle T'QD = 2\varphi - \angle IQA = 2\varphi - \angle IXA = \varphi = \angle IAC.$$

Therefore, $CT' = CA = CT$, which means that T coincides with T' and completes the proof.

Solution based on the work of Artemis-Chrysanthi Savva (HEL4), completed by A. Sinefakopoulos, Greece. Let G be the point of intersection of $\overline{AD}$ and $\overline{CX}$. Since the quadrilateral $AXBC$ is cyclic, it is $\angle AXC = \angle ABC$.



Let the line $\overline{AD}$ meet ω at K. Then it is $\angle AXK = \angle CAD = \angle ABC$, because the angle that is formed by a chord and a tangent to the circle at an endpoint of the chord equals the inscribed angle to that chord. Therefore, $\angle AXK = \angle AXC = \angle AXG$. This means that the point G coincides with the point K and so G belongs to the circle ω.

Let E be the point of intersection of the angle bisector of $\angle DAB$ with $\overline{BC}$. It suffices to show that

$$\frac{CE}{BE} = \frac{XC}{XB}.$$

Let F be the second point of intersection of ω with $\overline{AB}$. Then we have $\angle IAF = \frac{\angle CAB}{2} = \angle IXF$, where I is the incenter of $\triangle ABC$, because $\angle IAF$ and $\angle IXF$ are inscribed in the same arc of ω. Thus $\triangle AIF$ is isosceles with $AI = IF$. Since I is the incenter of $\triangle ABC$, we have $AF = 2(s-a)$, where $s = (a+b+c)/2$ is the semiperimeter of $\triangle ABC$. Also, it is $CE = AC = b$ because in triangle $\triangle ACE$, we have

$$\begin{aligned}
\angle AEC &= \angle ABC + \angle BAE \\
&= \angle ABC + \frac{\angle BAD}{2} \\
&= \angle ABC + \frac{\angle BAC - \angle ABC}{2} \\
&= 90^\circ - \frac{\angle ACE}{2},
\end{aligned}$$

and so $\angle CAE = 180^\circ - \angle AEC - \angle ACE = 90^\circ - \frac{\angle ACE}{2} = \angle AEC$. Hence

$$BF = BA - AF = c - 2(s-a) = a - b = CB - CE = BE.$$

Moreover, triangle $\triangle CAX$ is similar to triangle $\triangle BFX$, because $\angle ACX = \angle FBX$ and

$$\angle XFB = \angle XAF + \angle AXF = \angle XAF + \angle CAF = \angle CAX.$$

Therefore

$$\frac{CE}{BE} = \frac{AC}{BF} = \frac{XC}{XB},$$

as desired. The proof is complete.

Solution by IRL1 and IRL 5. Let ω denote the circle through A and I tangent to AC. Let Y be the second point of intersection of the circle ω with the line AD. Let L

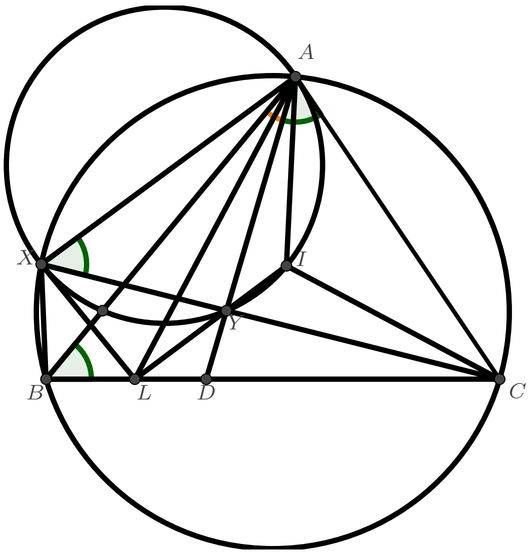
be the intersection of BC with the angle bisector of $\angle BAD$. We will prove $\angle LXC = 1/2\angle BAC = 1/2\angle BXC$.

We will refer to the angles of $\triangle ABC$ as $\angle A, \angle B, \angle C$. Thus $\angle BAD = \angle A - \angle B$.

On the circumcircle of $\triangle ABC$, we have $\angle AXC = \angle ABC = \angle CAD$, and since AC is tangent to ω, we have $\angle CAD = \angle CAY = \angle AXY$. Hence C, X, Y are collinear.

Also note that $\triangle CAL$ is isosceles with $\angle CAL = \angle CLA = \frac{1}{2}(\angle BAD) + \angle ABC = \frac{1}{2}(\angle A + \angle B)$ hence $AC = CL$. Moreover, CI is angle bisector to $\angle ACL$ so it's the symmetry axis for the triangle, hence $\angle ILC = \angle IAC = 1/2\angle A$ and $\angle ALI = \angle LIA = 1/2\angle B$. Since AC is tangent to ω, we have $\angle AYI = \angle IAC = 1/2\angle A = \angle LAY + \angle ALI$. Hence L, Y, I are collinear.

Since AC is tangent to ω, we have $\triangle CAY \sim \triangle CXA$ hence $CA^2 = CX \cdot CY$. However we proved $CA = CL$ hence $CL^2 = CX \cdot CY$. Hence $\triangle CLY \sim \triangle CXL$ and hence $\angle CXL = \angle CLY = \angle CAI = 1/2\angle A$.



Solution by IRL 5. Let M be the midpoint of the arc BC. Let ω denote the circle through A and I tangent to AC. Let N be the second point of intersection of ω with AB and L the intersection of BC with the angle bisector of $\angle BAD$. We know $\frac{DL}{LB} = \frac{AD}{AB}$ and want to prove $\frac{XB}{XC} = \frac{LB}{LC}$.

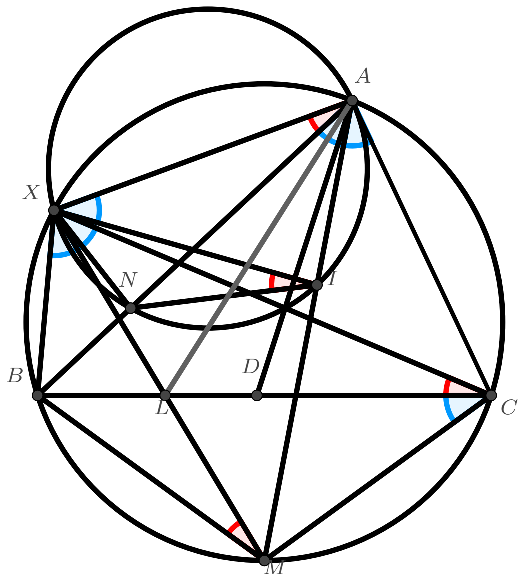
First note that $\triangle CAL$ is isosceles with $\angle CAL = \angle CLA = \frac{1}{2}(\angle BAD) + \angle ABC$ hence $AC = CL$ and $\frac{LB}{LC} = \frac{LB}{AC}$.

Now we calculate $\frac{XB}{XC}$:

Comparing angles on the circles ω and the circumcircle of $\triangle ABC$ we get $\triangle XIN \sim \triangle XMB$ and hence also $\triangle XIM \sim \triangle XNB$ (having equal angles at X and proportional adjoint sides). Hence $\frac{XB}{XM} = \frac{NB}{IM}$.

Also comparing angles on the circles ω and the circumcircle of $\triangle ABC$ and using the tangent AC we get $\triangle XAI \sim \triangle XCM$ and hence also $\triangle XAC \sim \triangle XIM$. Hence $\frac{XC}{XM} = \frac{AC}{IM}$.

Comparing the last two equations we get $\frac{XB}{XC} = \frac{NB}{AC}$. Comparing with $\frac{LB}{LC} = \frac{LB}{AC}$, it remains to prove $NB = LB$.



We prove $\triangle INB \equiv \triangle ILB$ as follows:

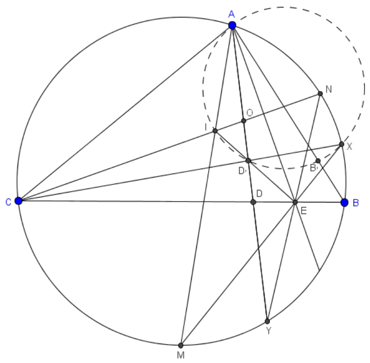
First, we note that I is the circumcentre of $\triangle ALN$. Indeed, CI is angle bisector in the isosceles triangle ACL so it's perpendicular bisector for AL. As well, $\triangle IAN$ is isosceles with $\angle INA = \angle CAI = \angle IAB$ hence I is also on the perpendicular bisector of AN.

Hence $IN = IL$ and also $\angle NIL = 2\angle NAL = \angle A - \angle B = 2\angle NIB$ (the last angle is calculated using that the exterior angle of $\triangle NIB$ is $\angle INA = \angle A/2$. Hence $\angle NIB = \angle LIB$ and $\triangle INB \equiv \triangle ILB$ by SAS.

Solution by ISR5 (with help from IRL5). Let M, N be the midpoints of arcs BC, BA of the circumcircle ABC, respectively. Let Y be the second intersection of AD and circle ABC. Let E be the incenter of triangle ABY and note that E lies on the angle bisectors of the triangle, which are the lines YN (immediate), BC (since $\angle CBY = \angle CAY = \angle CAD = \angle ABC$) and the angle bisector of $\angle DAB$; so the question reduces to showing that E is also on XM, which is the angle bisector of $\angle CXB$.

We claim that the three lines CX, ADY, IE are concurrent at a point D'. We will complete the proof using this fact, and the proof will appear at the end (and see the solution by HEL5 for an alternative proof of this fact).

To show that XEM are collinear, we construct a projective transformation which projects M to X through center E. We produce it as a composition of three other projections. Let O be the intersection of lines $AD'DY$ and CIN. Projecting the points $YNCM$ on the circle ABC through the (concyclic) point A to the line CN yields the points $ONCI$. Projecting these points through E to the line AY yields $OYDD'$ (here we use the facts that D' lies on IE and AY). Projecting these points to the circle ABC through C yields $NYBX$ (here we use the fact that D' lies on CX). Composing, we observe that we found a projection of the circle ABC to itself sending $YNCM$ to $NYBX$. Since the projection of the circle through E also sends YNC to NYB, and three points determine a projective transformation, the projection through E also sends M to X, as claimed.



Let B', D' be the intersections of AB, AD with the circle AXI, respectively. We wish to show that this D' is the concurrency point defined above, i.e. that $CD'X$ and $ID'E$ are collinear. Additionally, we will show that I is the circumcenter of $AB'E$.

Consider the inversion with center C and radius CA. The circles AXI and ABD are tangent to CA at A (the former by definition, the latter since $\angle CAD = \angle ABC$), so they are preserved under the inversion. In particular, the inversion transposes D and B and preserves A, so sends the circle CAB to the line AD. Thus X, which is the second intersection of circles ABC and AXI, is sent by the inversion to the second intersection of AD and circle AXI, which is D'. In particular $CD'X$ are collinear.

In the circle AIB', AI is the angle bisector of $B'A$ and the tangent at A, so I is the midpoint of the arc AB', and in particular $AI = IB'$. By angle chasing, we find that ACE is an isosceles triangle:

$$\angle CAE = \angle CAD + \angle DAE = \angle ABC + \angle EAB = \angle ABE + \angle EAB = \angle AEB = \angle AEC,$$

thus the angle bisector CI is the perpendicular bisector of AE and $AI = IE$. Thus I is the circumcenter of $AB'E$.

We can now show that $ID'E$ are collinear by angle chasing:

$$\angle EIB' = 2\angle EAB' = 2\angle EAB = \angle DAB = \angle D'AB' = \angle D'IB'.$$

Solution inspired by ISR2. Let W be the midpoint of arc BC, let D' be the second intersection point of AD and the circle ABC. Let P be the intersection of the angle bisector XW of $\angle CXB$ with BC; we wish to prove that AP is the angle bisector of DAB. Denote $\alpha = \frac{\angle CAB}{2}$, $\beta = \angle ABC$.

Let M be the intersection of AD and XC. Angle chasing finds:

$$\begin{aligned}\angle MXI &= \angle AXI - \angle AXM = \angle CAI - \angle AXC = \angle CAI - \angle ABC = \alpha - \beta\\ &= \angle CAI - \angle CAD = \angle DAI = \angle MAI\end{aligned}$$

And in particular M is on ω. By angle chasing we find

$$\angle XIA = \angle IXA + \angle XAI = \angle ICA + \angle XAI = \angle XAC = \angle XBC = \angle XBP$$

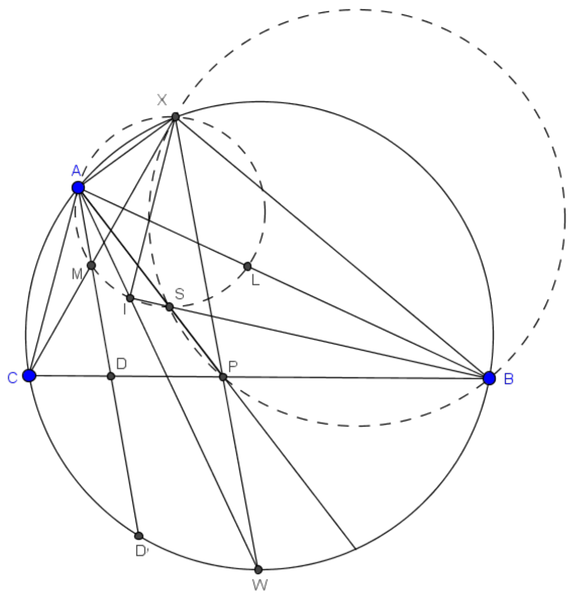
and $\angle PXB = \alpha = \angle CAI = \angle AXI$, and it follows that $\triangle XIA \sim \triangle XBP$. Let S be the second intersection point of the cirumcircles of XIA and XBP. Then by the spiral map lemma (or by the equivalent angle chasing) it follows that ISB and ASP are collinear.

Let L be the second intersection of ω and AB. We want to prove that ASP is the angle bisector of $\angle DAB = \angle MAL$, i.e. that S is the midpoint of the arc ML of ω. And this follows easily from chasing angular arc lengths in ω:

$$\begin{aligned}
\widehat{AI} &= \angle CAI = \alpha \\
\widehat{IL} &= \angle IAL = \alpha \\
\widehat{MI} &= \angle MXI = \alpha - \beta \\
\widehat{AI} - \widehat{SL} &= \angle ABI = \tfrac{\beta}{2}
\end{aligned}$$

And thus

$$\widehat{ML} = \widehat{MI} + \widehat{IL} = 2\alpha - \beta = 2(\widehat{AI} - \tfrac{\beta}{2}) = 2\widehat{SL}.$$



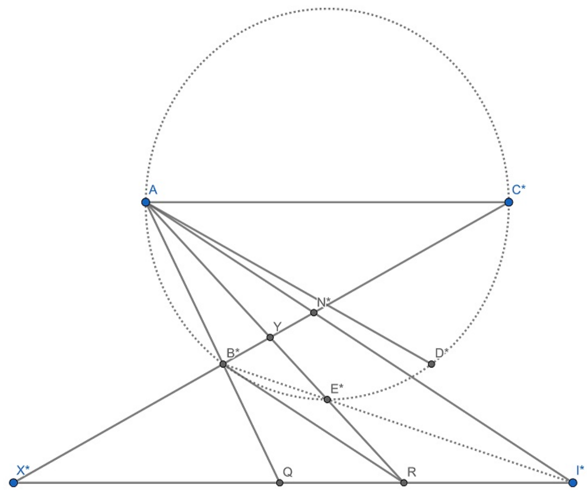
Solution by inversion, by JPN Observer A, Satoshi Hayakawa. Let E be the intersection of the bisector of $\angle BAD$ and BC, and N be the middle point of arc BC of the circumcircle of ABC. Then it suffices to show that E is on line XN.

We consider the inversion at A. Let P^* be the image of a point denoted by P. Then A, B^*, C^*, E^* are concyclic, X^*, B^*, C^* are colinear, and X^*I^* and AC^* are parallel. Now it suffices to show that A, X^*, E^*, N^* are concyclic. Let Y be the intersection of B^*C^* and AE^*. Then, by the power of a point, we get

$$\begin{aligned}
A, X^*, E^*, N^* \text{ are concyclic} &\iff YX^* \cdot YN^* = YA \cdot YE^* \\
&\iff YX^* \cdot YN^* = YB^* \cdot YC^*. \\
&\qquad (A, B^*, C^*, E^* \text{ are concyclic})
\end{aligned}$$

Here, by the property of inversion, we have

$$\angle AI^*B^* = \angle ABI = \frac{1}{2}\angle ABC = \frac{1}{2}\angle C^*AD^*.$$



Define Q, R as described in the figure, and we get by simple angle chasing

$$\angle QAI^* = \angle QI^*A, \quad \angle RAI^* = \angle B^*I^*A.$$

Especially, B^*R and AI^* are parallel, so that we have

$$\frac{YB^*}{YN^*} = \frac{YR}{YA} = \frac{YX^*}{YC^*},$$

and the proof is completed.

Problem 4 (Poland). Let ABC be a triangle with incentre I. The circle through B tangent to AI at I meets side AB again at P. The circle through C tangent to AI at I meets side AC again at Q. Prove that PQ is tangent to the incircle of ABC.

Solution 1. Let QX, PY be tangent to the incircle of ABC, where X, Y lie on the incircle and do not lie on AC, AB. Denote $\angle BAC = \alpha$, $\angle CBA = \beta$, $\angle ACB = \gamma$.

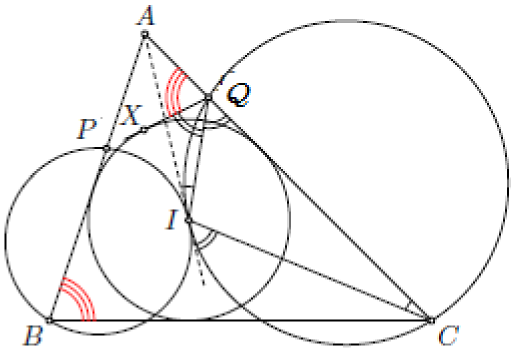
Since AI is tangent to the circumcircle of CQI we get $\angle QIA = \angle QCI = \frac{\gamma}{2}$. Thus

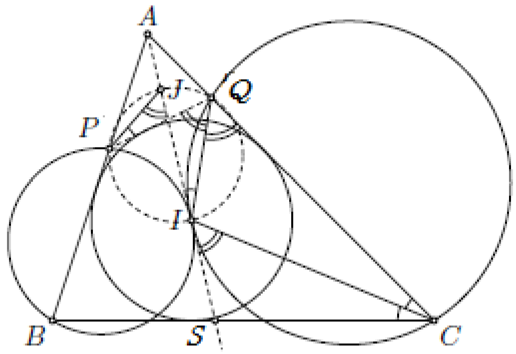
$$\angle IQC = \angle IAQ + \angle QIA = \frac{\alpha}{2} + \frac{\gamma}{2}.$$

By the definition of X we have $\angle IQC = \angle XQI$, therefore

$$\angle AQX = 180^\circ - \angle XQC = 180^\circ - \alpha - \gamma = \beta.$$

Similarly one can prove that $\angle APY = \gamma$. This means that Q, P, X, Y are collinear which leads us to the conclusion that $X = Y$ and QP is tangent to the incircle at X.





Solution 2. By the power of a point we have

$$AD \cdot AC = AI^2 = AP \cdot AB, \quad \text{which means that} \quad \frac{AQ}{AP} = \frac{AB}{AC}$$

and therefore triangles ADP, ABC are similar. Let J be the incenter of AQP. We obtain

$$\angle JPQ = \angle ICB = \angle QCI = \angle QIJ,$$

thus J, P, I, Q are concyclic. Let S be the intersection of AI and BC. It follows that

$$\angle IQP = \angle IJP = \angle SIC = \angle IQC.$$

This means that IQ is the angle bisector of $\angle CQP$, so QP is indeed tangent to the incircle of ABC.

Comment. The final angle chasing from the Solution 2 may simply be replaced by the observation that since J, P, I, Q are concyclic, then I is the A-excenter of triangle APQ.

Solution 3. Like before, notice that $AQ \cdot AC = AP \cdot AB = AI^2$. Consider the positive inversion Ψ with center A and power AI^2. This maps P to B (and vice-versa), Q to C

(and vice-versa), and keeps the incenter I fixed. The problem statement will follow from the fact that the image of the incircle of triangle ABC under Ψ is the so-called mixtilinear incircle of ABC, which is defined to be the circle tangent to the lines AB, AC, and the circumcircle of ABC. Indeed, since the image of the line QP is the circumcircle of ABC, and inversion preserves tangencies, this implies that QP is tangent to the incircle of ABC.

We justify the claim as follows: let γ be the incircle of ABC and let Γ_A be the A-mixtilinear incircle of ABC. Let K and L be the tangency points of γ with the sides AB and AC, and let U and V be the tangency points of Γ_A with the sides AB and AC, respectively. It is well-known that the incenter I is the midpoint of segment UV. In particular, since also $AI \perp UV$, this implies that $AU = AV = \frac{AI}{\cos\frac{A}{2}}$. Note that $AK = AL = AI \cdot \cos\frac{A}{2}$. Therefore, $AU \cdot AK = AV \cdot AL = AI^2$, which means that U and V are the images of K and L under Ψ. Since Γ_A is the unique circle simultaneously tangent to AB at U and to AC at V, it follows that the image of γ under Ψ must be precisely Γ_A, as claimed.

Solution by Achilleas Sinefakopoulos, Greece. From the power of a point theorem, we have

$$AP \cdot AB = AI^2 = AQ \cdot AC.$$

Hence $PBCQ$ is cyclic, and so, $\angle APQ = \angle BCA$. Let K be the circumcenter of $\triangle BIP$ and let L be the circumcenter of $\triangle QIC$. Then $\overline{KL}$ is perpendicular to $\overline{AI}$ at I.

Let N be the point of intersection of line $\overline{KL}$ with $\overline{AB}$.Then in the right triangle $\triangle NIA$, we have $\angle ANI = 90^\circ - \frac{\angle BAC}{2}$ and from the external angle theorem for triangle $\triangle BNI$, we have $\angle ANI = \frac{\angle ABC}{2} + \angle NIB$. Hence

$$\angle NIB = \angle ANI - \frac{\angle ABC}{2} = \left(90^\circ - \frac{\angle BAC}{2}\right) - \frac{\angle ABC}{2} = \frac{\angle BCA}{2}.$$

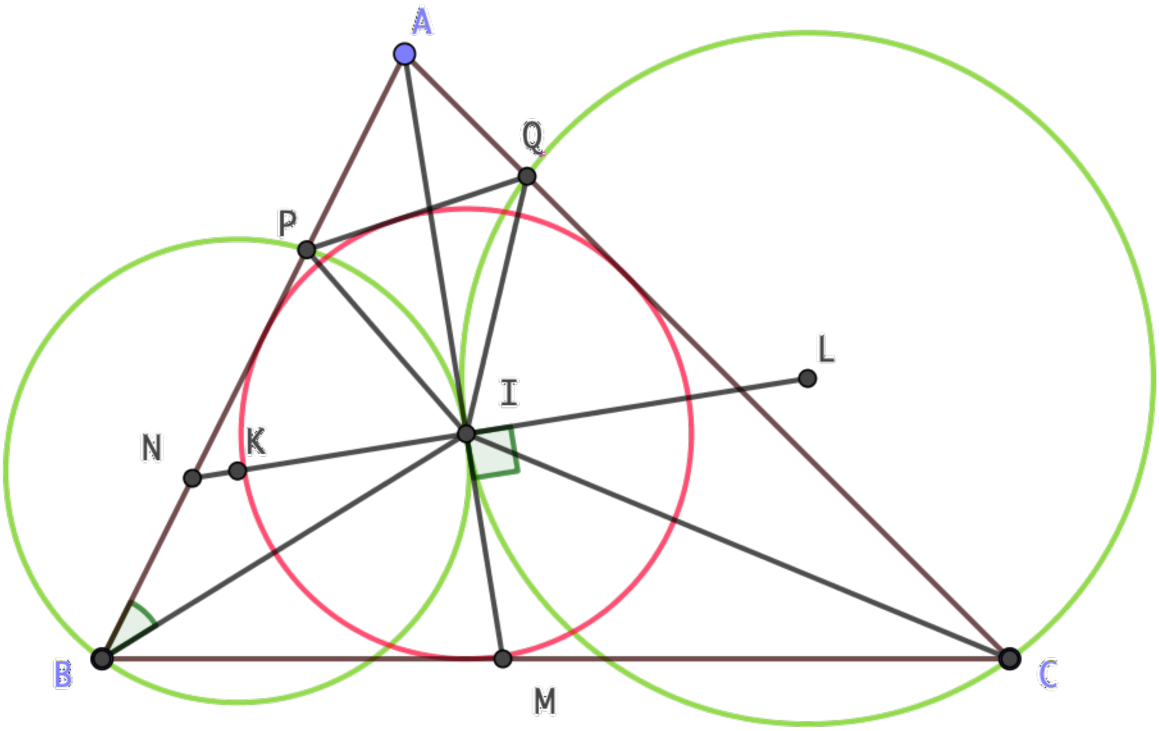
Since MI is tangent to the circumcircle of $\triangle BIP$ at I, we have

$$\angle BPI = \angle BIM = \angle NIM - \angle NIB = 90^\circ - \frac{\angle BCA}{2}.$$

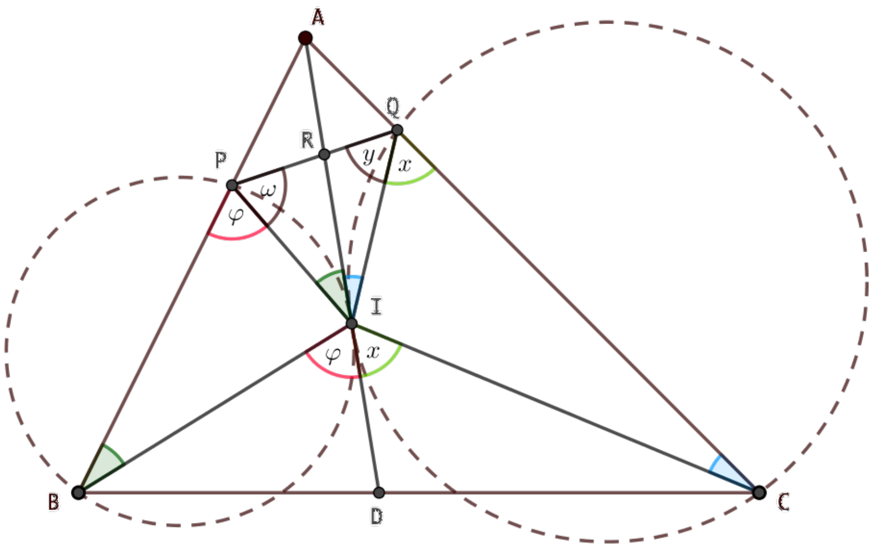
Also, since $\angle APQ = \angle BCA$, we have

$$\angle QPI = 180^\circ - \angle APQ - \angle BPI = 180^\circ - \angle BCA - \left(90^\circ - \frac{\angle BCA}{2}\right) = 90^\circ - \frac{\angle BCA}{2},$$

as well. Hence I lies on the angle bisector of $\angle BPQ$, and so it is equidistant from its sides $\overline{PQ}$ and $\overline{PB}$. Therefore, the distance of I from $\overline{PQ}$ equals the inradius of $\triangle ABC$, as desired.



Solution by Eirini Miliori (HEL2). Let D be the point of intersection of $\overline{AI}$ and $\overline{BC}$ and let R be the point of intersection of $\overline{AI}$ and $\overline{PQ}$. We have $\angle RIP = \angle PBI = \frac{\angle B}{2}$, $\angle RIQ = \angle ICQ = \frac{\angle C}{2}$, $\angle IQC = \angle DIC = x$ and $\angle BPI = \angle BID = \varphi$, since $\overline{AI}$ is tangent to both circles.



From the angle bisector theorem, we have

$$\frac{RQ}{RP} = \frac{AQ}{AP} \quad \text{and} \quad \frac{AC}{AB} = \frac{DC}{BD}.$$

Since $\overline{AI}$ is tangent to both circles at I, we have $AI^2 = AQ \cdot AC$ and $AI^2 = AP \cdot AB$. Therefore,

$$\frac{RQ}{RP} \cdot \frac{DC}{BD} = \frac{AQ \cdot AC}{AB \cdot AP} = 1. \tag{1}$$

From the sine law in triangles $\triangle QRI$ and $\triangle PRI$, it follows that $\dfrac{RQ}{\sin \frac{\angle C}{2}} = \dfrac{RI}{\sin y}$ and $\dfrac{RP}{\sin \frac{\angle B}{2}} = \dfrac{RI}{\sin \omega}$, respectively. Hence

$$\frac{RQ}{RP} \cdot \frac{\sin \frac{\angle B}{2}}{\sin \frac{\angle C}{2}} = \frac{\sin \omega}{\sin y}. \tag{2}$$

Similarly, from the sine law in triangles $\triangle IDC$ and $\triangle IDB$, it is $\dfrac{DC}{\sin x} = \dfrac{ID}{\sin \frac{\angle C}{2}}$ and $\dfrac{BD}{\sin \varphi} = \dfrac{ID}{\sin \frac{\angle B}{2}}$, and so

$$\frac{DC}{BD} \cdot \frac{\sin \varphi}{\sin x} = \frac{\sin \frac{\angle B}{2}}{\sin \frac{\angle C}{2}}. \tag{3}$$

By multiplying equations (2) with (3), we obtain $\dfrac{RQ}{RP} \cdot \dfrac{DC}{BD} \cdot \dfrac{\sin \varphi}{\sin x} = \dfrac{\sin \omega}{\sin y}$, which combined with (1) and cross-multiplying yields

$$\sin \varphi \cdot \sin y = \sin \omega \cdot \sin x. \tag{4}$$

Let $\theta = 90^\circ + \frac{\angle A}{2}$. Since I is the incenter of $\triangle ABC$, we have $x = 90^\circ + \frac{\angle A}{2} - \varphi = \theta - \phi$. Also, in triangle $\triangle PIQ$, we see that $\omega + y + \frac{\angle B}{2} + \frac{\angle C}{2} = 180^\circ$, and so $y = \theta - \omega$.

Therefore, equation (4) yields

$$\sin \varphi \cdot \sin(\theta - \omega) = \sin \omega \cdot \sin(\theta - \varphi),$$

or

$$\frac{1}{2}\left(\cos(\varphi - \theta + \omega) - \cos(\varphi + \theta - \omega)\right) = \frac{1}{2}\left(\cos(\omega - \theta + \varphi) - \cos(\omega + \theta - \varphi)\right),$$

which is equivalent to

$$\cos(\varphi + \theta - \omega) = \cos(\omega + \theta - \varphi).$$

So

$$\varphi + \theta - \omega = 2k \cdot 180^\circ \pm (\omega + \theta - \varphi), \quad (k \in \mathbb{Z}.)$$

If $\varphi + \theta - \omega = 2k \cdot 180^\circ + (\omega + \theta - \varphi)$, then $2(\varphi - \omega) = 2k \cdot 180^\circ$, with $|\varphi - \omega| < 180^\circ$ forcing $k = 0$ and $\varphi = \omega$. If $\varphi + \theta - \omega = 2k \cdot 180^\circ - (\omega + \theta - \varphi)$, then $2\theta = 2k \cdot 180^\circ$, which contradicts the fact that $0^\circ < \theta < 180^\circ$. Hence $\varphi = \omega$, and so PI is the angle bisector of $\angle QPB$.

Therefore the distance of I from $\overline{PQ}$ is the same with the distance of I from AB, which is equal to the inradius of $\triangle ABC$. Consequently, $\overline{PQ}$ is tangent to the incircle of $\triangle ABC$.

Problem 5 (Netherlands).

Let $n \geq 2$ be an integer, and let $a_1, a_2, \ldots, a_n$ be positive integers. Show that there exist positive integers $b_1, b_2, \ldots, b_n$ satisfying the following three conditions:

1. $a_i \leq b_i$ for $i = 1, 2, \ldots, n$;
2. the remainders of $b_1, b_2, \ldots, b_n$ on division by n are pairwise different; and
3. $b_1 + \cdots + b_n \leq n \left(\dfrac{n-1}{2} + \left\lfloor \dfrac{a_1 + \cdots + a_n}{n} \right\rfloor \right)$.

(Here, $\lfloor x \rfloor$ denotes the integer part of real number x, that is, the largest integer that does not exceed x.)

Solution 1. We define the b_i recursively by letting b_i be the smallest integer such that $b_i \geq a_i$ and such that b_i is not congruent to any of $b_1, \ldots, b_{i-1}$ modulo n. Then $b_i - a_i \leq i - 1$, since of the i consecutive integers $a_i, a_i + 1, \ldots, a_i + i - 1$, at most $i - 1$ are congruent to one of $b_1, \ldots, b_{i-1}$ modulo n. Since all b_i are distinct modulo n, we have $\sum_{i=1}^n b_i \equiv \sum_{i=1}^n (i-1) = \frac{1}{2}n(n-1)$ modulo n, so n divides $\sum_{i=1}^n b_i - \frac{1}{2}n(n-1)$. Moreover, we have $\sum_{i=1}^n b_i - \sum_{i=1}^n a_i \leq \sum_{i=1}^n (i-1) = \frac{1}{2}n(n-1)$, hence $\sum_{i=1}^n b_i - \frac{1}{2}n(n-1) \leq \sum_{i=1}^n$. As the left hand side is divisible by n, we have

$$\frac{1}{n}\left(\sum_{i=1}^n b_i - \frac{1}{2}n\,(n-1)\right) \leq \left\lfloor \frac{1}{n}\sum_{i=1}^n a_i \right\rfloor$$

which we can rewrite as

$$\sum_{i=1}^n b_i \leq n\left(\frac{n-1}{2} + \left\lfloor \frac{1}{n}\sum_{i=1}^n a_i \right\rfloor\right)$$

as required.

Solution 2. Note that the problem is invariant under each of the following operations:

- adding a multiple of n to some a_i (and the corresponding b_i);
- adding the same integer to all a_i (and all b_i);
- permuting the index set $1, 2, \ldots, n$.

We may therefore remove the restriction that our a_i and b_i be positive.

For each congruence class $\overline{k}$ modulo n ($\overline{k} = \overline{0}, \ldots, \overline{n-1}$), let $h(k)$ be the number of i such that a_i belongs to $\overline{k}$. We will now show that the problem is solved if we can find a $t \in \mathbb{Z}$ such that

$$\begin{array}{ccc} h(t) & \geq & 1, \\ h(t) + h(t+1) & \geq & 2, \\ h(t) + h(t+1) + h(t+2) & \geq & 3, \\ & \vdots & \end{array}$$

Indeed, these inequalities guarantee the existence of elements $a_{i_1} \in \overline{t}$, $a_{i_2} \in \overline{t} \cup \overline{t+1}$, $a_{i_3} \in \overline{t} \cup \overline{t+1} \cup \overline{t+2}$, et cetera, where all i_k are different. Subtracting appropriate multiples of n and reordering our elements, we may assume $a_1 = t$, $a_2 \in \{t, t+1\}$, $a_3 \in \{t, t+1, t+2\}$, et cetera. Finally subtracting t from the complete sequence, we may assume $a_1 = 0$, $a_2 \in \{0, 1\}$, $a_3 \in \{0, 1, 2\}$ et cetera. Now simply setting $b_i = i - 1$ for all i suffices, since $a_i \leq b_i$ for all i, the b_i are all different modulo n, and

$$\sum_{i=1}^{n} b_i = \frac{n(n-1)}{2} \leq \frac{n(n-1)}{2} + n\left[\frac{\sum_{i=1}^{n} a_i}{n}\right].$$

Put $x_i = h(i) - 1$ for all $i = 0, \ldots, n-1$. Note that $x_i \geq -1$, because $h(i) \geq 0$. If we have $x_i \geq 0$ for all $i = 0, \ldots, n-1$, then taking $t = 0$ completes the proof. Otherwise, we can pick some index j such that $x_j = -1$. Let $y_i = x_i$ where $i = 0, \ldots, j-1, j+1, \ldots, n-1$ and $y_j = 0$. For sequence $\{y_i\}$ we have

$$\sum_{i=0}^{n-1} y_i = \sum_{i=0}^{n-1} x_i + 1 = \sum_{i=0}^{n-1} h(i) - n + 1 = 1,$$

so from Raney's lemma there exists index k such that $\sum_{i=k}^{k+j} y_i > 0$ for all $j = 0, \ldots, n-1$ where $y_{n+j} = y_j$ for $j = 0, \ldots, k-1$. Taking $t = k$ we will have

$$\sum_{t=k}^{k+i} h(t) - (i+1) = \sum_{t=k}^{k+i} x(t) \geq \sum_{t=k}^{k+i} y(t) - 1 \geq 0,$$

for all $i = 0, \ldots, n-1$ and we are done.

Solution 3. Choose a random permutation $c_1, \ldots, c_n$ of the integers $1, 2, \ldots, n$. Let $b_i = a_i + f(c_i - a_i)$, where $f(x) \in \{0, \ldots, n-1\}$ denotes a remainder of x modulo n. Observe, that for such defined sequence the first two conditions hold. The expected value of $B := b_1 + \ldots + b_n$ is easily seen to be equal to $a_1 + \ldots + a_n + n(n-1)/2$. Indeed, for each i the random number $c_i - a_i$ has uniform distribution modulo n, thus the expected value of $f(c_i - a_i)$ is $(0 + \ldots + (n-1))/n = (n-1)/2$. Therefore we may find such c that $B \leq a_1 + \ldots + a_n + n(n-1)/2$. But $B - n(n-1)/2$ is divisible by n and therefore $B \leq n[(a_1 + \ldots + a_n)/n] + n(n-1)/2$ as needed.

Solution 4. We will prove the required statement for all sequences of non-negative integers a_i by induction on n.

Case $n = 1$ is obvious, just set $b_1 = a_1$.

Now suppose that the statement is true for some $n \geq 1$; we shall prove it for $n + 1$.

First note that, by subtracting a multiple of $n + 1$ to each a_i and possibly rearranging indices we can reduce the problem to the case where $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} < n + 1$.

Now, by the induction hypothesis there exists a sequence $d_1, d_2, \ldots, d_n$ which satisfies the properties required by the statement in relation to the numbers $a_1, \ldots, a_n$. Set $I = \{i | 1 \leq i \leq n \text{ and } d_i \bmod n \geq a_i\}$ and construct b_i, for $i = 1, \ldots, n+1$, as follows:

$$b_i = \begin{cases} d_i \bmod n, \text{ when } i \in I, \\ n + 1 + (d_i \bmod n), \text{ when } i \in \{1, \ldots, n\} \setminus I, \\ n, \text{ for } i = n + 1. \end{cases}$$

Now, $a_i \le d_i \bmod n \le b_i$ for $i \in I$, while for $i \notin I$ we have $a_i \le n \le b_i$. Thus the sequence $(b_i)_{i=1}^{n+1}$ satisfies the first condition from the problem statement.

By the induction hypothesis, the numbers $d_i \bmod n$ are distinct for $i \in \{1, \ldots, n\}$, so the values $b_i \bmod (n+1)$ are distinct elements of $\{0, \ldots, n-1\}$ for $i \in \{1, \ldots, n\}$. Since $b_{n+1} = n$, the second condition is also satisfied.

Denote $k = |I|$. We have

$$\sum_{i=1}^{n+1} b_i = \sum_{i=1}^{n} b_i + n = \sum_{i=1}^{n} d_i \bmod n + (n-k)(n+1) + n =$$

$$\frac{n(n+1)}{2} + (n-k)(n+1),$$

hence we need to show that

$$\frac{n(n+1)}{2} + (n-k)(n+1) \le \frac{n(n+1)}{2} + (n+1)\left\lceil \frac{\sum_{i=1}^{n+1} a_i}{n+1} \right\rceil;$$

equivalently, that

$$n - k \le \left\lceil \frac{\sum_{i=1}^{n+1} a_i}{n+1} \right\rceil.$$

Next, from the induction hypothesis we have

$$\frac{n(n-1)}{2} + n\left\lceil \frac{\sum_{i=1}^{n} a_i}{n} \right\rceil \ge \sum_{i=1}^{n} d_i = \sum_{i \in I} d_i + \sum_{i \notin I} d_i \ge$$

$$\sum_{i \in I} d_i \bmod n + \sum_{i \notin I} (n + d_i \bmod n) = \frac{n(n-1)}{2} + (n-k)n$$

or

$$n - k \le \left\lceil \frac{\sum_{i=1}^{n} a_i}{n} \right\rceil.$$

Thus, it's enough to show that

$$\frac{\sum_{i=1}^{n} a_i}{n} \le \frac{\sum_{i=1}^{n+1} a_i}{n+1}$$

because then

$$n - k \le \left\lceil \frac{\sum_{i=1}^{n} a_i}{n} \right\rceil \le \left\lceil \frac{\sum_{i=1}^{n+1} a_i}{n+1} \right\rceil.$$

But the required inequality is equivalent to $\sum_{i=1}^{n} a_i \le n a_{n+1}$, which is obvious.

Solution 5. We can assume that all $a_i \in \{0, 1, \ldots, n-1\}$, as we can deduct n from both a_i and b_i for arbitrary i without violating any of the three conditions from the problem statement. We shall also assume that $a_1 \le \ldots \le a_n$.

Now let us provide an algorithm for constructing $b_1, \ldots, b_n$.

We start at step 1 by choosing $f(1)$ to be the maximum i in $\{1,\ldots,n\}$ such that $a_i \leq n-1$, that is $f(1)=n$. We set $b_{f(1)} = n-1$.

Having performed steps 1 through j, at step $j+1$ we set $f(j+1)$ to be the maximum i in $\{1,\ldots,n\} \setminus \{f(1),\ldots,f(j)\}$ such that $a_i \leq n-j-1$, if such an index exists. If it does, we set $b_{f(j+1)} = n-j-1$. If there is no such index, then we define $T=j$ and assign to the terms b_i, where $i \notin f(\{1,\ldots,j\})$, the values $n, n+1\ldots, 2n-j-1$, in any order, thus concluding the run of our algorithm.

Notice that the sequence $(b_i)_{i=1}^n$ satisfies the first and second required conditions by construction. We wish to show that it also satisfies the third.

Notice that, since the values chosen for the b_i's are those from $n-T$ to $2n-T-1$, we have

$$\sum_{i=1}^{n} b_i = \frac{n(n-1)}{2} + (n-T)n.$$

It therefore suffices to show that

$$\left\lceil \frac{a_1+\ldots+a_n}{n} \right\rceil \geq n-T,$$

or (since the RHS is obviously an integer) $a_1+\ldots+a_n \geq (n-T)n$.

First, we show that there exists $1 \leq i \leq T$ such that $n-i = b_{f(i)} = a_{f(i)}$.

Indeed, this is true if $a_n = n-1$, so we may suppose $a_n < n-1$ and therefore $a_{n-1} \leq n-2$, so that $T \geq 2$. If $a_{n-1} = n-2$, we are done. If not, then $a_{n-1} < n-2$ and therefore $a_{n-2} \leq n-3$ and $T \geq 3$. Inductively, we actually obtain $T=n$ and necessarily $f(n)=1$ and $a_1 = b_1 = 0$, which gives the desired result.

Now let t be the largest such index i. We know that $n-t = b_{f(t)} = a_{f(t)}$ and therefore $a_1 \leq \ldots \leq a_{f(t)} \leq n-t$. If we have $a_1 = \ldots = a_{f(t)} = n-t$, then $T=t$ and we have $a_i \geq n-T$ for all i, hence $\sum_i a_i \geq n(n-T)$. Otherwise, $T>t$ and in fact one can show $T = t+f(t+1)$ by proceeding inductively and using the fact that t is the *last* time for which $a_{f(t)} = b_{f(t)}$.

Now we get that, since $a_{f(t+1)+1} \geq n-t$, then $\sum_i a_i \geq (n-t)(n-f(t+1)) = (n-T+f(t+1))(n-f(t+1)) = n(n-T)+nf(t+1)-f(t+1)(n-T+f(t+1)) = n(n-T)+tf(t+1) \geq n(n-T)$.

Greedy algorithm variant 1 (ISR). Consider the residues $0,\ldots,n-1$ modulo n arranged in a circle clockwise, and place each a_i on its corresponding residue; so that on each residue there is a stack of all a_is congruent to it modulo n, and the sum of the sizes of all stacks is exactly n. We iteratively flatten and spread the stacks forward, in such a way that the a_is are placed in the nearest available space on the circle clockwise (skipping over any already flattened residue or still standing stack). We may choose the order in which the stacks are flattened. Since the total amount of numbers equals the total number of spaces, there is always an available space and at the end all spaces are covered. The b_is are then defined by adding to each a_i the number of places it was moved forward, which clearly satifies (i) and (ii), and we must prove that they satisfy (iii) as well.

Suppose that we flatten a stack of k numbers at a residue i, causing it to overtake a stack of l numbers at residue $j \in (i, i+k)$ (we can allow j to be larger than n and identify it with its residue modulo n). Then in fact in fact in whichever order we would flatten the two stacks, the total number of forward steps would be the same, and the total sum of the corresponding b_t (such that $a_t \bmod n \in \{i,j\}$) would be the same. Moreover, we can merge the stacks to a single stack of $k+l$ numbers at residue i, by replacing each $a_t \equiv j \pmod n$ by $a'_t = a_t - (j-i)$, and this stack would be flattened forward into the same positions as the separate stacks would have been, so applying our algorithm to the new stacks will yield the same total sum of $\sum b_i$ – but the a_is are strictly decreased, so $\sum a_i$ is decreased, so $\left\lfloor \frac{\sum a_i}{n} \right\rfloor$ is not increased – so by merging the stacks, we can only make the inequality we wish to prove tighter.

Thus, as long as there is some stack that when flattened will overtake another stack, we may merge stacks and only make the inequality tighter. Since the amount of numbers equals the amount of places, the merging process terminates with stacks of sizes $k_1, \ldots, k_m$, such that the stack j, when flattened, will exactly cover the interval to the next stack. Clearly the numbers in each such stack were advanced by a total of $\sum_{t=1}^{k_j-1} = \frac{k_j(k_j-1)}{2}$, thus $\sum b_i = \sum a_i + \sum_j \frac{k_j(k_j-1)}{2}$. Writing $\sum a_i = n \cdot r + s$ with $0 \le s < n$, we must therefore show

$$s + \sum_j \frac{k_j(k_j-1)}{2} \le \frac{n(n-1)}{2}.$$

Ending 1. Observing that both sides of the last inequality are congruent modulo n (both are congruent to the sum of all different residues), and that $0 \le s < n$, the inequality is eqivalent to the simpler $\sum_j \frac{k_j(k_j-1)}{2} \le \frac{n(n-1)}{2}$. Since $x(x-1)$ is convex, and k_j are non-negative integers with $\sum_j k_j = n$, the left hand side is maximal when $k_{j'} = n$ and the rest are 0, and then eqaulity is achieved. (Alternatively it follows easily for any non-negative reals from AM-GM.)

Ending 2. If $m = 1$ (and $k_1 = n$), then all numbers are in a single stack and have the same residue, so $s = 0$ and equality is attained. If $m \ge 2$, then by convexity $\sum_j \frac{k_j(k_j-1)}{2}$ is maximal for $m = 2$ and $(k_1, k_2) = (n-1, 1)$, where it equals $\frac{(n-1)(n-2)}{2}$. Since we always have $s \le n-1$, we find

$$s + \sum_j \frac{k_j(k_j-1)}{2} \le (n-1) + \frac{(n-1)(n-2)}{2} = \frac{n(n-1)}{2}$$

as required.

Greedy algorithm variant 1' (ISR). We apply the same algorithm as in the previous solution. However, this time we note that we may merge stacks not only when they overlap after flattening, but also when they merely touch front-to-back: That is, we relax the condition $j \in (i, i+k)$ to $j \in (i, i+k]$; the argument for why such merges are allowed is exactly the same (But note that this is now sharp, as merging non-touching stacks can cause the sum of b_is to decrease).

We now observe that as long as there at least two stacks left, at least one will spread to touch (or overtake) the next stack, so we can perform merges until there is only one stack left. We are left with verifying that the inequality indeed holds for the case of only one stack which is spread forward, and this is indeed immediate (and in fact equality is achieved).

Greedy algorithm variant 2 (ISR). Let $c_i = a_i \bmod n$. Iteratively define $b_i = a_i + l_i$ greedily, write $d_i = c_i + l_i$, and observe that $l_i \le n-1$ (since all residues are present in $a_i, \dots, a_i + n - 1$), hence $0 \le d_i \le 2n-2$. Let $I = \{i \in I : d_i \ge n\}$, and note that $d_i = b_i \bmod n$ if $i \notin I$ and $d_i = (b_i \bmod n) + n$ if $i \in I$. Then we must show

$$\begin{aligned}
&\sum (a_i + l_i) = \sum b_i \le \frac{n(n-1)}{2} + n \left\lfloor \frac{\sum a_i}{n} \right\rfloor \\
\iff &\sum (c_i + l_i) \le \sum (b_i \bmod n) + n \left\lfloor \frac{\sum c_i}{n} \right\rfloor \\
\iff &n|I| \le n \left\lfloor \frac{\sum c_i}{n} \right\rfloor \iff |I| \le \left\lfloor \frac{\sum c_i}{n} \right\rfloor \iff |I| \le \frac{\sum c_i}{n}
\end{aligned}$$

Let $k = |I|$, and for each $0 \le m < n$ let $J_m = \{i : c_i \ge n - m\}$. We claim that there must be some m for which $|J_m| \ge m + k$ (clearly for such m, at least k of the sums d_j with $j \in J_m$ must exceed n, i.e. at least k of the elements of J_m must also be in I, so this m is a "witness" to the fact $|I| \ge k$). Once we find such an m, then we clearly have

$$\sum c_i \ge (n-m)|J_m| \ge (n-m)(k+m) = nk + m(n-(k+m)) \ge nk = n|I|$$

as required. We now construct such an m explicitly.

If $k = 0$, then clearly $m = n$ works (and also the original inequality is trivial). Otherwise, there are some d_is greater than n, and let $r + n = \max d_i$, and suppose $d_t = r + n$ and let $s = c_t$. Note that $r < s < r + n$ since $l_t < n$. Let $m \ge 0$ be the smallest number such that $n - m - 1$ is not in $\{d_1, \dots, d_t\}$, or equivalently m is the largest such that $[n-m, n) \subset \{d_1, \dots, d_t\}$. We claim that this m satisfies the required property. More specifically, we claim that $J'_m = \{i \le t : d_i \ge n - m\}$ contains exactly $m + k$ elements and is a subset of J_m.

Note that by the greediness of the algorithm, it is impossible that for $[c_i, d_i)$ to contain numbers congruent to $d_j \bmod n$ with $j > i$ (otherwise, the greedy choice would prefer d_j to d_i at stage i). We call this the *greedy property*. In particular, it follows that all i such that $d_i \in [s, d_t) = [c_t, d_t)$ must satisfy $i < t$. Additionally, $\{d_i\}$ is disjoint from $[n + r + 1, 2n)$ (by maximality of d_t), but does intersect every residue class, so it contains $[r+1, n)$ and in particular also $[s, n)$. By the greedy property the latter can only be attained by d_i with $i < t$, thus $[s, n) \subset \{d_1, \dots, d_t\}$, and in particular $n - m \le s$ (and in particular $m \ge 1$).

On the other hand $n - m > r$ (since $r \notin \{d_i\}$ at all), so $n - m - 1 \ge r$. It follows that there is a time $t' \ge t$ for which $d_{t'} \equiv n - m - 1 \pmod n$: If $n - m - 1 = r$ then this is true for $t' = t$ with $d_t = n + r = 2n - m - 1$; whereas if $n - m - 1 \in [r+1, n)$ then there is some t' for which $d_{t'} = n - m - 1$, and by the definition of m it satisfies $t' > t$.

Therefore for all $i < t \le t'$ for which $d_i \ge n - m$, necessarily also $c_i \ge n - m$, since otherwise $d_{t'} \in [c_i, d_i)$, in contradiction to the greedy property. This is also true for $i = t$, since $c_t = s \ge n - m$ as previously shown. Thus, $J'_m \subset J_m$ as claimed.

Finally, since by definition of m and greediness we have $[n-m, n) \cup \{d_i : i \in I\} \subset \{d_1, \dots, d_t\}$, we find that $\{d_j : j \in J'_m\} = [n-m, n) \cup \{d_i : i \in I\}$ and thus $|J'_m| = |[n-m, n)| + |I| = m + k$ as claimed.

Problem 6 (United Kingdom).

On a circle, Alina draws 2019 chords, the endpoints of which are all different. A point is considered *marked* if it is either

(i) one of the 4038 endpoints of a chord; or

(ii) an intersection point of at least two chords.

Alina labels each marked point. Of the 4038 points meeting criterion (i), Alina labels 2019 points with a 0 and the other 2019 points with a 1. She labels each point meeting criterion (ii) with an arbitrary integer (not necessarily positive).

Along each chord, Alina considers the segments connecting two consecutive marked points. (A chord with k marked points has $k-1$ such segments.) She labels each such segment in yellow with the sum of the labels of its two endpoints and in blue with the absolute value of their difference.

Alina finds that the $N+1$ yellow labels take each value $0, 1, \ldots, N$ exactly once. Show that at least one blue label is a multiple of 3.

(A *chord* is a line segment joining two different points on a circle.)

Solution 1. First we prove the following:

Lemma: *if we color all of the points white or black, then the number of white-black edges, which we denote E_{WB}, is equal modulo 2 to the number of white (or black) points on the circumference, which we denote C_W, resp. C_B.*

Observe that changing the colour of any interior point does not change the parity of E_{WB}, as each interior point has even degree, so it suffices to show the statement holds when all interior points are black. But then $E_{WB} = C_W$ so certainly the parities are equal.

Now returning to the original problem, assume that no two adjacent vertex labels differ by a multiple of three, and three-colour the vertices according to the residue class of the labels modulo 3. Let E_{01} denote the number of edges between 0-vertices and 1-vertices, and C_0 denote the number of 0-vertices on the boundary, and so on.

Then, consider the two-coloring obtained by combining the 1-vertices and 2-vertices. By applying the lemma, we see that $E_{01} + E_{02} \equiv C_0 \bmod 2$.

$$\text{Similarly } E_{01} + E_{12} \equiv C_1, \quad \text{and } E_{02} + E_{12} \equiv C_2, \quad \bmod 2.$$

Using the fact that $C_0 = C_1 = 2019$ and $C_2 = 0$, we deduce that either E_{02} and E_{12} are even and E_{01} is odd; or E_{02} and E_{12} are odd and E_{01} is even.

But if the edge labels are the first N non-negative integers, then $E_{01} = E_{12}$ unless $N \equiv 0$ modulo 3, in which case $E_{01} = E_{02}$. So however Alina chooses the vertex labels, it is not possible that the multiset of edge labels is $\{0, \ldots, N\}$.

Hence in fact two vertex labels must differ by a multiple of 3.

Solution 2. As before, colour vertices based on their label modulo 3.

Suppose this gives a valid 3-colouring of the graph with 2019 0s and 2019 1s on the circumference. Identify pairs of 0-labelled vertices and pairs of 1-labelled vertices on the circumference, with one 0 and one 1 left over. The resulting graph has even degrees except these two leaves. So the connected component $\mathcal{C}$ containing these leaves has an Eulerian path, and any other component has an Eulerian cycle.

Let E^*_{01} denote the number of edges between 0-vertices and 1-vertices in $\mathcal{C}$, and let E'_{01} denote the number of such edges in the other components, and so on. By studying whether a given vertex has label congruent to 0 modulo 3 or not as we go along the Eulerian path in $\mathcal{C}$, we find $E^*_{01} + E^*_{02}$ is odd, and similarly $E^*_{01} + E^*_{12}$ is odd. Since neither start nor end vertex is a 2-vertex, $E^*_{02} + E^*_{12}$ must be even.

Applying the same argument for the Eulerian cycle in each other component and adding up, we find that $E'_{01} + E'_{02}$, $E'_{01} + E'_{12}$, $E'_{02} + E'_{12}$ are all even. So, again we find $E_{01} + E_{02}$, $E_{01} + E_{12}$ are odd, and $E_{02} + E_{12}$ is even, and we finish as in the original solution.