20-та Македонска математичка олимпијада

17-та JMMO

Library Olympiads

Mathematical Olympiads

Macedonian Mathematical Olympiads
Balkan Mathematical Olympiads
European Girl's Mathematical Olympiad
European Mathematical Cup
Mediterranean Mathematical Olympiad
Asian-Pacific Mathematical Olympiad

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Macedonia, Skopje 2019

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Foreword

This year in the Macedonia were held various competitions in the field of mathematics on all levels in primary and secondary school: school, municipality, regional, state competitions and Olympiads, as it is a tradition for many years in the past. Also, Republic of Macedonia was one of the participating countries on some world famous math competitions abroad.

In December 2018, the European Mathematical Cup was held in Skopje. Students from all over the country were competing in two categories Junior and Senior.

On April 14, 2019 the 26-th Macedonian Mathematical Olympiad, MMO 2019, was held in FON University, Skopje, for the students from secondary school. After all rigorous selection processes which raised from the complete system of the competitions in the Republic of Macedonia, the BMO team was formed. The 36-nd Balkan Mathematical Olympiad, BMO 2019, was held in Chisinau, Moldova in May 2019.

On May 29, 2019 the 23-th Junior Macedonian Mathematical Olympiad, JMMO 2019, was held in the Faculty of Mechanical Engineering, Skopje. There the Macedonian team of the best 6 contestants under 15,5 years, was elected. They were participants in the 23-th Junior Balkan Mathematical Olympiad, JBMO 2019, which was held in June 2019 in Agros, Republic of Cyprus.

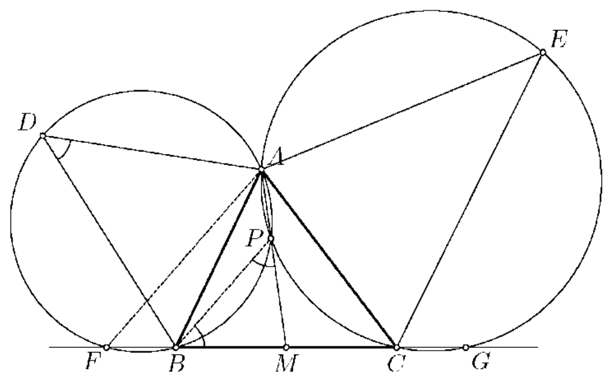
Then after the IMO team selection test, the IMO team was formed. This year the International Mathematical Olympiad, IMO 2019, will take place in Bath, the United Kingdom of Great Britain in July 2019.

The content of this book consists of the mathematical competitions that already took place in Macedonia, the Balkan region and wider abroad, as well as the solutions.

XXVI Macedonian Mathematical Olympiad
FON University, Skopje
14.04.2019, Skopje

Problem 1. Let ABC be an acute triangle, M be midpoint of the segment BC and the centers of the excircles with respect of M of the triangles AMB and AMC are D and E, respectively. The The circumcircle of the triangle ABD meets the line BC at the points B and F. The circumcircle of the triangle ACE meets the line BC at the points C and G. Prove that $BF = CG$.

Solution. (BMO shortlist) Obviously, we have $\measuredangle ADB = 90° - \frac{1}{2}\measuredangle AMB$ and $\measuredangle AEC = 90° - \frac{1}{2}\measuredangle AMC$.



Let the circumcircles of ΔADB and ΔAEC meet again the line AM at the points P and P', respectively. Let we notice that the point M is outside of the circumcircles of ΔADB and ΔAEC, since $\measuredangle ADB + \measuredangle AMB < 180°$ and $\measuredangle AEC + \measuredangle AMC < 180°$, so P and P' lie on the ray MA. Furthermore, $\measuredangle BPM = \measuredangle BDA = 90° - \frac{1}{2}\measuredangle PMB$, hence the triangle BPM is isosceles, so $MP = MB$. Analogously, $MP' = MC = MB$, so $P' \equiv P$.
Now, using the power of the point M, we obtain $MB \cdot MF = MP \cdot MA = MC \cdot MG$, i.e. $MF = MG = MA$, hence $BF = CG$.

Problem 2. Let n be a positive integer. If $r \equiv n(\mathrm{mod}\, 2)$ and $r \in \{0,1\}$, then find the number of the integer solutions of the system of equations

$$\begin{cases} x+y+z=r \\ |x|+|y|+|z|=n \end{cases}.$$

Solution. Let n be a even positive integer, that is $r = 0$. Then the problem can be reformulated as to find the number of integer solutions of the system of equations

$$\begin{cases} x+y+z=0 \\ |x|+|y|+|z|=n \end{cases} \quad .....(1)$$

Lemma. 1) At least one of the numbers x, y, z has absolute value $\frac{n}{2}$.

2) Each of x, y, z has absolute value $\le \frac{n}{2}$.

Proof. It is clear that one of the numbers x, y, z must be positive; otherwise we obtain contradiction with the first equation of the system of equations (1). Without loss the generality, we may assume $x > 0$.
Indeed, if $x > \frac{n}{2}$, from $x = -(y+z)$, and from $|y|+|z| \ge |y+z| > \frac{n}{2}$ we obtain contradiction with the second equation of the system of equations (1).

If $0<x<\frac{n}{2}$, then at leas one of the numbers y,z is smaller than 0. We consider two cases: Case 1. $y<0,z<0$, and Case 2. $yz<0$.

Case 1. $|y+z|=|y|+|z|$ and $y+z=-x$, so $|x|+|y|+|z|<\frac{n}{2}+\frac{n}{2}$, which is contradiction.

Case 2. Let $y<0<z$. In this case $x+z=-y$, that is $|y|=|x+z|=|x|+|z|$ from where we obtain

$$2|y|=|y|+|x+z|=|x|+|y|+|z|=n \text{ or } |y|=\frac{n}{2}.$$

The case when $x<0$ is analogues. This completes the lemma.

Continuation of the solution. Let only one of the numbers x,y,z be positive. Without loss of generality, let $x>0$, and then $x=\frac{n}{2}$ and $y+z=-\frac{n}{2}$. From the lemma, it follows that all the ordered triples

$$\left(\tfrac{n}{2},-\tfrac{n}{2},0\right),\left(\tfrac{n}{2},-\tfrac{n}{2}+1,-1\right),\left(\tfrac{n}{2},-\tfrac{n}{2}+2,-2\right),...,\left(\tfrac{n}{2},0,-\tfrac{n}{2}\right)$$

are solution of the system of equations (1), and those are $\frac{n}{2}+1$ solutions. Changing the position of $\frac{n}{2}$ (at the second and at the third coordinate) and applying the same discussion, we obtain $3\left(\frac{n}{2}+1\right)$ordered triples which are solution of the system of equations (1). Let any two of x,y,z are positive. Without loss of generality, let $x>0,y>0$. Then $z=-\frac{n}{2}$ and $x+y=\frac{n}{2}$. From the lemma, it follows that all the ordered triples

$$\left(1,\tfrac{n}{2}-1,-\tfrac{n}{2}\right),\left(2,\tfrac{n}{2}-2,-\tfrac{n}{2}\right),\left(3,\tfrac{n}{2}-3,-\tfrac{n}{2}\right),...,\left(\tfrac{n}{2}-1,1,-\tfrac{n}{2}\right)$$

are solution of the system of equations (1), and those are $\frac{n}{2}-1$ solutions. Changing the position of $-\frac{n}{2}$ (at the first and at the second coordinate) and applying the same discussion, we obtain $3\left(\frac{n}{2}-1\right)$ ordered triples which are solution of the system of equations (1). Finally, we obtain that the total number of solutions of the system of equations (1) is

$$3\left(\tfrac{n}{2}+1\right)+3\left(\tfrac{n}{2}-1\right)=3n.$$

Now, let n be a odd positive integer , that is $r=1$. Then, the system (1) can be written as

$$\begin{cases} x+y+z=1 \\ |x|+|y|+|z|=n \end{cases}.$$

In analogues way as the case when n is even, (using the appropriate lemma obtained when replacing $\frac{n}{2}$ with $\frac{n+1}{2}$), we obtain that the total number of solutions of the system of equations (1) is

$$3\left(\tfrac{n-1}{2}+1\right)+3\left(\tfrac{n-1}{2}\right)=3n.$$

Problem 3. Let ABC be an isosceles triangle ($AB=AC$) and let M be a midpoint of the segment BC. The point P is chosen such that $PB<PC$ and PA is parallel to BC. Let X and Y are point from the lines PB and PC, respectively, such that the point B is on the segment PX, C is on the segment PY and $\measuredangle PXM=\measuredangle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

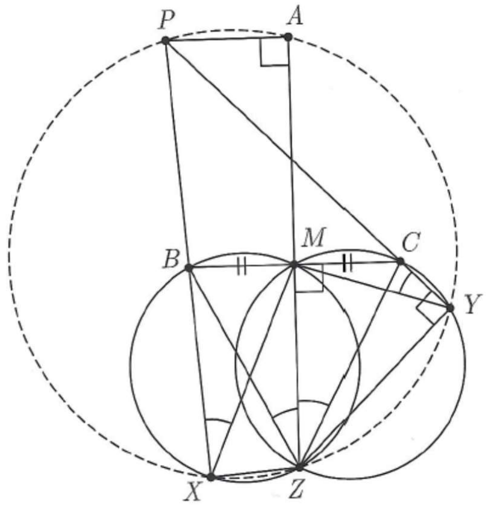
Solution (IMO Shortlist). Since $AB=AC$, AM is a axes of symmetry of the segment BC, we have $\measuredangle PAM=\measuredangle AMC=90°$.

Let Z be intersection point of the line AM and the normal of PC, passing through Y (notice that Z is on the ray AM after the point M). We have, $\measuredangle PAZ=\measuredangle PYZ=90°$. Hence, the points P,A,Y and Z are concyclic.

Since $\measuredangle CMZ=\measuredangle CYZ=90°$, the quadrilateral $CYZM$ is cyclic, so $\measuredangle CZM=\measuredangle CYM$. By the condition of the problem, $\measuredangle CYM=\measuredangle BXM$, and since ZM is axes of symmetry of the angle $\measuredangle BZC$, we have $\measuredangle CZM=\measuredangle BZM$. So, $\measuredangle BXM=\measuredangle BZM$. Now, we have that the points B,X,Z and M are concyclic, so $\measuredangle BXZ=180°-\measuredangle BMZ=90°$.

Finally, e obtain that $\measuredangle PXZ = \measuredangle PYZ = \measuredangle PAZ = 90°$, hence the points P, A, X, Y, Z are concyclic, i.e. the quadrilateral $APXY$ is cyclic.

Remark. The construction of the point Z, can be made in a different ways. One way is the point Z to be second intersection of the circle CMY and the line AM. Another way to introduce the point Z is as a second intersection point of the circumcircles of the triangles CMY and BMX.



Problem 4. Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that

$$n! + f(m)! \mid f(n)! + f(m!)$$

for all $m,n\in\mathbb{N}$.

Solution. (BMO Shortlist) Taking $m=n=1$ in (*) yields $1+f(1)! \mid f(1)! + f(1)$ and hence $1+f(1)! \mid f(1)-1$. Since, $|f(1)-1| < f(1)!+1$, that is $1+f(1)! \mid f(1)-1$ it follows that $f(1)-1=0$, i.e. $f(1)=1$.

For $m=1$ in (*) we have $n!+1 \mid f(n)!+1$, which implies $n! \le f(n)!$, i.e. $n \le f(n)$. On the other hand, taking $(m,n)=(1,p-1)$ for any prime number p and using Wilson's theorem we obtain $p \mid (p-1)!+1 \mid f(p-1)!+1$, implying $f(p-1) < p$. But $f(p-1) \ge p-1$ and from $f(p-1) < p$ we conclude that

$$f(p-1) = p-1.$$

Next, fix a positive integer m. For any prime number p, setting $n=p-1$ in (*) yields $(p-1)! + f(m)! \mid (p-1)! + f(m!)$, hence

$$(p-1)! + f(m)! \mid f(m!) - f(m)!,$$

For all prime numbers p. This implies $f(m!) = f(m)!$, for all $m\in\mathbb{N}$, so (*) can be rewritten as $n! + f(m)! \mid f(n)! + f(m)!$. This implies

$$n! + f(m)! \mid f(n)! - n!,$$

for all $m,n\in\mathbb{N}$. Fixing $n\in\mathbb{N}$ and taking m large enough, we conclude that $f(n)! = n!$, i.e. $f(n)=n$, for all $n\in\mathbb{N}$.

Problem 5. Let n be a given positive integer. Sisyphus performs a sequence on a board consisting of $n+1$ squares in a row, numbered from 0 to n, starting from left to right. At the beginning, n stones are put into square numbered 0, and the other squares are empty. At any turn, Sisyphus chooses any nonempty square with k stones, takes on of these stones and moves it to the right but at most k squares (the chosen stone should stay within the board). The goal of Sisyphus is to place all n stones at the square n. Prove that Sisyphus can not achieve his goal in less that $\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \ldots + \left\lceil \frac{n}{n} \right\rceil$ moves. Notation $\lceil x \rceil$ stands for the least integer not smaller than x.

Solution. (IMO Shortlist) The stones are indistinguishable, and all have the same origin and the same final position. So, at any position we can prescribe which stone from the chosen square to move. We do it in the following manner. Number the stone from 1 to n. At any turn, after choosing a square, Sisyphus moves the stone with the largest number from the square.

This way, when stone k is moved from some square, that square contains not more than k stones (since all their numbers are at most k). Therefore, stone k is moved by at most k squares at each turn. Since the total shift of the stone is exactly n, at least $\left\lceil \frac{n}{k} \right\rceil$ moves of stone k should have been made, for every $k = 1, 2, \ldots, n$.

By summing up over all $k = 1, 2, \ldots, n$, we get the required estimate.

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Junior Macedonian Mathematical Olympiad 2019

Faculty of Mechanical Engineering - Skopje

26.05.2019

Problem 1. Find all prime numbers which can be written as $1 + 2^p + 3^p + \ldots + p^p$ where p is prime number.

Solution. *First solution.* For $p = 2$ we have $1 + 2^2 = 5$, which is a prime number. Let $p > 2$, so p is an odd number. Then

$$1 + 2^p + 3^p + \ldots + p^p = (1^p + (p-1)^p) + (2^p + (p-2)^p) + \ldots + ((\tfrac{p-1}{2})^p + (\tfrac{p+1}{2})^p) + p^p$$

is divisible by p, because each number in the brackets is divisible by p. Indeed, from $k^p \equiv k^p \pmod{p}$, $p - k \equiv -k \pmod{p}$ and $(p-k)^p \equiv (-k)^p \equiv -k^p \pmod{p}$, it follows that $k^p + (p-k)^p \equiv k^p - k^p \equiv 0 \pmod{p}$. This means that for every odd prime number p, the number of the form $1 + 2^p + 3^p + \ldots + p^p$ is compound number.

Finally, the only solution of the problem is the number 5 which is obtained when $p = 2$.

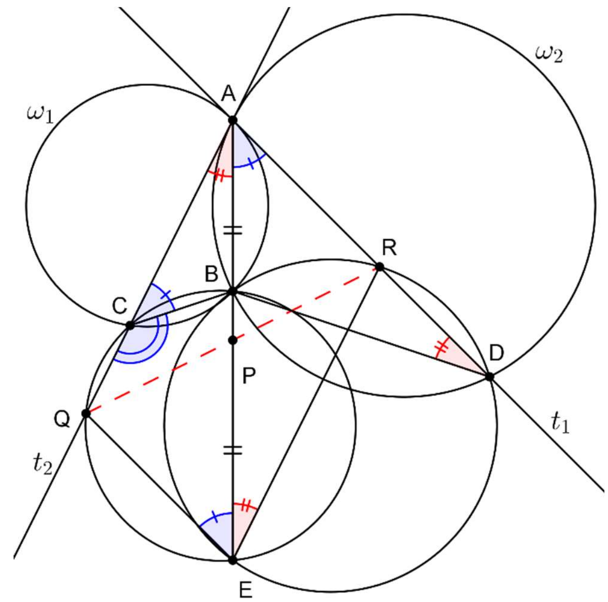
Second solution. From the Fermat's theorem we have $a^p \equiv a \pmod{p}$, for all positive integers a and every prime number p. Then

$$1 + 2^p + 3^p + \ldots + p^p \equiv 1 + 2 + \ldots + p = \tfrac{p(p+1)}{2} \pmod{p}.$$

If p is an odd number, then $p+1$ is even, hence $\frac{p(p+1)}{2} \equiv 0 \pmod{p}$, so in this case there is no such prime number with form as in the condition of the problem. For $p = 2$ we get $1 + 2^2 = 5$, so the only solution of the problem is the number 5. ■

Problem 2. The circles ω_1 and ω_2 meet at two points A and B. Let t_1 and t_2 are tangents of ω_1 and ω_2, respectively, passing through the point A. Let the second intersection of the circle ω_1 and t_2 is C, and the second intersection of ω_2 and t_1 is D.

The points P and E lie on the ray AB, such that B lies between A and P, P lies between A and E and $\overline{AE} = 2 \cdot \overline{AP}$. The circumcircle of $\triangle BCE$ intersects t_2 again at the point Q, and the circumcircle of $\triangle BDE$ intersects t_1 again at the point R. Prove that the points P, Q and R are colinear.



Solution. Using the property for angle between the tangent t_1 and the chord AB in the circle ω_1 we have $\angle BCA = \angle BAD$. From the cyclic quadrilateral $BCQE$ we obtain $\angle QEB + \angle QCB = 180°$. Hence,

$$\angle QEA \equiv \angle QEB = 180° - \angle QCB = \angle BCA \\ = \angle BAD \equiv \angle EAR,$$

so, we obtain $QE \parallel AR$.

By the property between the angle of the tangent t_2 and the chord AB in the circle ω_2 we have $\angle ADB = \angle BAC$.

From the cyclic quadrilateral $BRDE$ we obtain $\angle REB = \angle RDB$. Hence,

$$\angle REA \equiv \angle REB = \angle RDB \equiv \angle ADB = \angle BAC \\ \equiv \angle EAQ,$$

so we obtain $RE \parallel AQ$.

Hence, the quadrilateral $AQER$ is parallelogram. By $\overline{AE} = 2 \cdot \overline{AP}$, we have $\overline{AP} = \overline{PE}$, i.e. P is a midpoint of the diagonal AE in the parallelogram $AQER$. Since the property of the diagonals in a parallelogram, we have that P must be midpoint of the other diagonal QR, i.e. $P \in QR$. Hence, the points P, Q and R are colinear. ■

3. We consider coloring in the plane such that:

- Choose arbitrary positive integer m
- Let $K_1, K_2, ..., K_m$ are different circles with radii not equal to zero such that $K_i \subset K_j$ or $K_j \subset K_i$ for $i \neq j$,
- The points of the plane which are outside of some arbitrary chosen circle are differently colored with the respect of point inside in the circle.

Let we have 2019 point lying in the plane such that no three of them are collinear.
Find the maximal number of colors, such that the points are colored satisfying the conditions of the problem.

Solution. By the condition of the problem the maximal number of colors is less or equal than 2019.

We will prove that the number 2019 can be achieved. In this case it is enough to show that there exist circles $K_1, K_2, ..., K_{2019}$ defining different coloring of the points. Let we consider all the segments such that the given points are end points. They are $\frac{2019 \cdot 2018}{2}$ such segments. Then we construct axes of symmetry of these segments. We choose point such is not lying on any axes of symmetry. Clearly, the distances between this point and all other points are different real numbers (the chosen point does not lie on any of the axes of symmetry) and with these distances we form strictly increasing sequence $0 < s_1 < s_2 < ... < s_{2019}$. Then, we can find numbers $r_i, i = 1, 2, ..., 2019$ such that $s_1 < r_1 < s_2 < r_2 < ... < s_{2019} < r_{2019}$. Finally, we construct 2019 concentric circles with centre in some chosen point and radii $r_i, 1 \leq i \leq 2019$. Clearly, these circles are defining coloring, in which one every chosen point is differently colored. ■

Problem 4. Let the real numbers a, b and c are such that

$$(a+b)(b+c)(c+a) = abc \text{ и } (a^9+b^9)(b^9+c^9)(c^9+a^9) = (abc)^9.$$

Prove that at least one of the numbers a, b, c is zero.

Solution. For an arbitrary real numbers x and y hold $x^2 - xy + y^2 \geq xy$ and $x^6 - x^3y^3 + y^6 \geq x^3y^3$, so

$$x^9+y^9=(x+y)(x^2-xy+y^2)(x^6-x^3y^3+y^6)\geq(x+y)xyx^3y^3=(x+y)x^4y^4.$$

The equality holds if and only if $x=y$. Using the upper inequalities and the conditions of the problem, we obtain

$$(abc)^9=(a^9+b^9)(b^9+c^9)(c^9+a^9)\geq(a+b)(b+c)(c+a)a^4b^4b^4c^4c^4a^4=a^9b^9c^9.$$

If $a\neq 0, b\neq 0, c\neq 0$ the equality holds when $a=b=c$. Having this of mind and the equation $(a+b)(b+c)(c+a)=abc$ we have that $8a^3=a^3$, i.e. $a=0$, which is a contradiction. Finally, at least one of the numbers a,b,c is zero.■

Problem 5. Let $p_1,p_2,...,p_k$ be different prime numbers. Find the number of positive integers of the form

$$p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k},\ \alpha_i\in\mathbb{N} \text{ for which } \alpha_1\alpha_2...\alpha_k=p_1p_2...p_k.$$

Solution. It is clear that from $p_1p_2...p_k=\alpha_1\alpha_2...\alpha_k$ it follows that for every $i\in\{1,2,...,k\}$, $p_i\,|\,\alpha_s$ holds, for some $s\in\{1,2,...,k\}$. Having this, we will construct ordered $k-$tuple $(\alpha_1,...,\alpha_k)$ of positive integers for which the equation $p_1p_2...p_k=\alpha_1\alpha_2...\alpha_k$ holds. We take $k-$tuple $(1,1,...,1)$. We multiply one arbitrary term from this $k-$tuple with p_1. Then, we multiply one term from the obtained $k-$tuple with p_2, which means that for p_2, we have k possible ways. Continuing this procedure, we can obtain that for all prime numbers p_3 ,..., p_k we have k possible ways. So, we have that there exist k^k ordered $k-$tuples $(\alpha_1,...,\alpha_k)$ such that $p_1p_2...p_k=\alpha_1\alpha_2...\alpha_k$. Clearly, all of these $k-$tuples determines a number of form $p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}$, $\alpha_i\in\mathbb{N}$ and different $k-$tuples determine different numbers like in the statement of the problem. Indeed, if $(\alpha_1,...,\alpha_k)\neq(\beta_1,...,\beta_k)$, then $\alpha_i\neq\beta_i$ for at least one $i\in\{1,2,...,k\}$, so $p_i^{\alpha_i}\neq p_i^{\beta_i}$, hence $p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}\neq p_1^{\beta_1}p_2^{\beta_2}...p_k^{\beta_k}$.

Conversely, every number of form $p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}$, $\alpha_i\in\mathbb{N}$ such that $\alpha_1\alpha_2...\alpha_k=p_1p_2...p_k$ determines ordered $k-$tuple $(\alpha_1,...,\alpha_k)$ of positive integers such that $p_1p_2...p_k=\alpha_1\alpha_2...\alpha_k$.

Finally, from the previous considerations we can conclude the that number of positive integers of form $p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}$, $\alpha_i\in\mathbb{N}$ such that $\alpha_1\alpha_2...\alpha_k=p_1p_2...p_k$ is equal to k^k. ■

36th Balkan Mathematical Olympiad Chisinau, Republic of Moldova April 30-May 5, 2019

Problem 1. Let P be the set of all prime numbers. Find all functions $f:\mathrm{P}\to\mathrm{P}$ such that

$$f(p)^{f(q)}+q^p=f(q)^{f(p)}+p^q$$

holds for all $p,q\in\mathrm{P}$.

Problem 2. Let a,b,c be real numbers, such that $0\leq a\leq b\leq c$ and $a+b+c=ab+bc+ca>0$. Prove that $\sqrt{bc}(a+1)\geq 2$. Find all triples (a,b,c) for which equality holds.

Problem 3. Let ABC be an acute triangle. Let X and Y be two distinct interior points of the segment BC such that $\angle CAX=\angle YAB$. Suppose that:

1) K and S are the feet of perpendiculars from B to the lines AX and AY respectively;
2) T and L are the feet of perpendiculars from C to the lines AX and AY respectively.

Prove that KL and ST intersect on the line BC.

Problem 4. A grid consists of all points of the form (m,n) where m and n are integers with $|m|\leq 2019$, $|n|\leq 2019$ and $|m|+|n|<4038$. We call the points (m,n) of the grid with either $|m|=2019$ or $|n|=2019$ the boundary points. The four lines $x=\pm 2019$ and $y=\pm 2019$ are called boundary lines. Two points in the grid are called neighbours if the distance between them is equal to 1.

Anna and Bob play a game on this grid.

Anna starts with a token at the point $(0,0)$. They take turns, with Bob playing first.

1) On each of his turns, Bob deletes at most two boundary points on each boundary line.
2) On each of her turns, Anna makes exactly three steps, where a step consists of moving her token from its current point to any neighbouring point, which has not been deleted.

As soon as Anna places her token on some boundary point which has not been deleted, the game is over and Anna wins.

Does Anna have a winning strategy?

Solutions.

Problem 1. Let P be the set of all prime numbers. Find all functions $f:\mathrm{P}\to\mathrm{P}$ such that
$f(p)^{f(q)}+q^p=f(q)^{f(p)}+p^q$
holds for all $p,q\in\mathrm{P}$.

Solution. Obviously, the identical function $f(p)=p$ for all $p\in\mathrm{P}$ is a solution. We will show that this is the only one.

First we will show that $f(2)=2$. Taking $q=2$ and p any odd prime number, we have

$$f(p)^{f(2)}+2^p=f(2)^{f(p)}+p^2.$$

Assume that $f(2)\neq 2$. It follows that $f(2)$ is odd and so $f(p)=2$ for any odd prime number p. Taking any two different odd prime numbers p,q we have

$$2^2+q^p=2^2+p^q \Rightarrow p^q=q^p \Rightarrow p=q,$$

contradiction. Hence, $f(2)=2$.

So for any odd prime number p we have

$$f(p)^2+2^p=2^{f(p)}+p^2.$$

Copy this relation as

$$2^p-p^2=2^{f(p)}-f(p)^2. \qquad (1)$$

Let T be the set of all positive integers greater than 2, i.e. $T=\{3,4,5,...\}$. The function $g:T\to\mathbb{Z}$, $g(n)=2^n-n^2$, is strictly increasing, i.e.

$$g(n+1)-g(n)=2^n-2n-1>0 \qquad (2)$$

for all $n\in T$. We show this by induction. Indeed, for $n=3$ it is true, $2^3-2\cdot 3-1>0$.

Assume that $2^k-2k-1>0$. It follows that for $n=k+1$ we have

$$2^{k+1}-2(k+1)-1=(2^k-2k-1)+(2^k-2)>0,$$

for any $k\geq 3$. Therefore, (2) is true for all $n\in T$.

As consequence, (1) holds if and only if $f(p)=p$ for all odd prime numbers p, as well as for $p=2$.

Therefore, the only function that satisfies the given relation is $f(p)=p$, for all $p\in\mathrm{P}$.

Problem 2. Let a,b,c be real numbers, such that $0\leq a\leq b\leq c$ and $a+b+c=ab+bc+ca>0$.
Prove that $\sqrt{bc}(a+1)\geq 2$. Find all triples (a,b,c) for which equality holds.

Solution. Let $a+b+c=ab+bc+ca=k$. Since $(a+b+c)^2\geq 3(ab+bc+ca)$, we get that $k^2\geq 3k$. Since, $k>0$, we obtain that $k\geq 3$.

We have $bc\geq ca\geq ab$, so from the above relation we deduce that $bc\geq 1$.

By AM-GM, $b+c\geq 2\sqrt{bc}$ and consequently $b+c\geq 2$. The equality holds iff $b=c$.

The constraint gives us

$$a=\frac{b+c-bc}{b+c-1}=1-\frac{bc-1}{b+c-1}\geq 1-\frac{bc-1}{2\sqrt{bc}-1}=\frac{\sqrt{bc}(2-\sqrt{bc})}{2\sqrt{bc}-1}.$$

For $\sqrt{bc}=2$, condition $a\geq 0$ gives $\sqrt{bc}(a+1)\geq 2$ with equality iff $a=0$ and $b=c=2$.

For $\sqrt{bc}<2$, taking into account the estimation for a, we get

$$a\sqrt{bc}\geq \frac{bc(2-\sqrt{bc})}{2\sqrt{bc}-1}=\frac{bc}{2\sqrt{bc}-1}(2-\sqrt{bc}).$$

Since $\frac{bc}{2\sqrt{bc}-1}\geq 1$, with equality for $bc=1$, we get $\sqrt{bc}(a+1)\geq 2$ with equality iff $a=b=c=1$.

For $\sqrt{bc}>2$ we have $\sqrt{bc}(a+1)>2(a+1)\geq 2$.

The proof is complete.

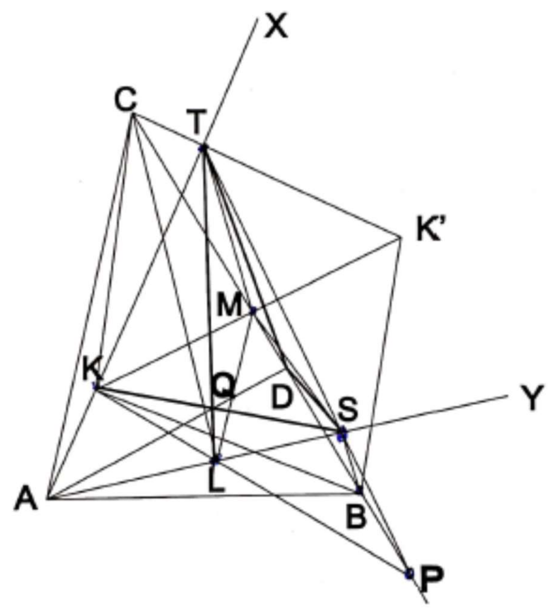
The equality holds iff $a=b=c=1$ or $a=0$ and $b=c=2$.

Problem 3. Let ABC be an acute triangle. Let X and Y be two distinct interior points of the segment BC such that $\angle CAX=\angle YAB$. Suppose that:

1) K and S are the feet of perpendiculars from B to the lines AX and AY respectively;
2) T and L are the feet of perpendiculars from C to the lines AX and AY respectively.

Prove that KL and ST intersect on the line BC.

Solution.



Denote $\phi=\angle XAB=\angle YAC$, $\alpha=\angle CAX=\angle BAY$. Then, because the quadrilaterals $ABSK$ and $ACTL$ are cyclic, we have

$$\angle BSK+\angle BAK=180^\circ=\angle BSK+\phi=\angle LAC+\angle LTC=\angle LTC+\phi,$$

so due to 90-degree angles formed, we have $\angle KSL=\angle KTL$. Thus, $KLST$ is cyclic.

Consider M to be the midpoint of BC and K' to be symmetric point of K with respect to M. Then, $BKCK'$ is a parallelogram, and so $BK\,||\,CK'$. But $BK\,||\,CT$, because they are both perpendicular to AX. So, K' lies on CT and, as $\angle KTK'=90^\circ$ and M is the midpoint of KK', $MK=MT$. In a similar way, we have that $MS=ML$. Thus the center of $(KLST)$ is M.

Consider D to be the foot of altitude from A to BC. Then, D belongs in both $(ABKS)$ and $(ACLT)$. So,

$$\angle ADT+\angle ACT=180^\circ=\angle ABS+\angle ADS=\angle ADT+90^\circ-\alpha=\angle ADS+90^\circ-\alpha,$$

And AD is the bisector of $\angle SDT$.

Because DM is perpendicular to AD, DM is the external bisector of this angle, and, as $MS = MT$, it follows that $DMST$ is cyclic. In a similar way, we have that $DMLK$ is also cyclic.
So, we have that ST, KL and DM are radical axes of these three circles, $(KLST)$, $(DMST)$, $(DMKL)$. These lines are, therefore, concurrent, and we have proved the desired result.

Second solution. We continue after proving that M is the center of $(KLST)$. If D is the foot of perpendicular from A to BC, then $ASDKB$ is cyclic, as well as $ATDLC$. The radical axes of those two circles and $(KLST)$ are concurrent, thus KS and LT intersect on point $Q \in AD$. So, if P is the intersection point of KL and TS, due to Brokard's theorem, AQ is perpendicular to MP. This is, of course, equivalent to proving that P belongs on BC.

Problem 4. A grid consists of all points of the form (m,n) where m and n are integers with $|m| \leq 2019$, $|n| \leq 2019$ and $|m|+|n| < 4038$. We call the points (m,n) of the grid with either $|m| = 2019$ or $|n| = 2019$ the boundary points. The four lines $x = \pm 2019$ and $y = \pm 2019$ are called boundary lines. Two points in the grid are called neighbours if the distance between them is equal to 1.

Anna and Bob play a game on this grid.

Anna starts with a token at the point $(0,0)$. They take turns, with Bob playing first.

1) On each of his turns, Bob deletes at most two boundary points on each boundary line.

2) On each of her turns, Anna makes exactly three steps, where a step consists of moving her token from its current point to any neighbouring point, which has not been deleted.

As soon as Anna places her token on some boundary point which has not been deleted, the game is over and Anna wins.

Does Anna have a winning strategy?

Solution. Anna does not have a winning strategy. We will provide a winning strategy for Bob. It is enough to describe his strategy for the deletions on the line $y = 2019$.

Bob starts by deleting $(0,2019)$ and $(-1,2019)$. Once Anna completes her step, he deletes the next two available points on the left if Anna decreased her x-coordinate, the next two available point to the left and the next available point to the right if Anna did not change her x-coordinate. The only exception to the above rule is on the very first time Anna decreases x by exactly 1. In that step, Bob deletes the next available point to the left and next available point to the right.

Bob's strategy guarantees the following: If Anna makes a sequence of steps reaching (x, y), with $x > 0$ and the exact opposite sequence of moves in the horizontal direction reaching (x, y) then Bob deletes at least as many points to the left of $(0,2019)$ in the first sequence than points to the right of $(0,2019)$ in the second sequence.

So we may assume for contradiction that Anna wins by placing her token at $(k,2019)$ for some $k > 0$. Define $\Delta = 3m - (2x + y)$ where m is the total number of points deleted by Bob to the right of $(0,2019)$, and (x, y) is the position of Anna's token.

For each sequence of steps performed first by Anna and then by Bob, Δ does not decrease. This can be seen by looking at the following table exhibiting the changes in $3m$ and $2x + y$. We have extended the cases where $2x + y < 0$.

Step	(0,3)	(1,2)	(-1,2)	(2,1)	(0,1)	(3,0)	(1,0)	(2,-1)	(1,-2)
m	1	2	0 (or 1)	2	1	2	2	2	2
$3m$	3	6	0 (or 3)	6	3	6	6	6	6
$2x+y$	3	4	0	5	1	6	2	3	0

The table also shows that if in this sequence of steps Anna changes y by $+1$ or -2, then Δ is increased by 1.Also, if Anna changes y by $+2$ or -1 then the first time this happens Δ is increased by 2. (This also holds if her move is $(0,-1)$ or $(-2,-1)$ which are not shown in the table.)

Since Anna wins by placing her token at $(k,2019)$ we must have $m \leq k-1$ and $k \leq 2018$. So at that exact moment we have:

$\Delta = 3m - (2k + 2019) = k - 2022 \le -4$.

So in her last turn she must have decreased Δ by at least 4. So her last step must have been $(1,2)$ or $(2,1)$ which give a decrease of 4 and 5 respectively. (It could not be $(3,0)$ because the she must have already won. Also she could not have done just one or two moves in her last turn since this is not enough for the required decrease in Δ.)

If her last step was $(1,2)$ then just before doing it we had $y = 2017$ and $\Delta = 0$. This means that in one of her steps the total change in y was not 0 mod 3. However in that case we have seen that $\Delta > 0$, a contradiction.

If her last step was $(2,1)$ then just before doing it we had $y = 2018$ and $\Delta = 0$ or $\Delta = 1$. So she must have made at least two steps with the change of y being $+1$ or -2 or at least one step with the change of y being $+2$ or -1. In both cases, consulting the table, we get an increase of at least 2 in Δ, a contradiction.

Note 1: If Anna is allowed to make at most three moves at each step, then she actually has a winning strategy.

Note 2: If 2019 is replaced by $N > 1$ then Bob has a winning strategy if and only if $3 \mid N$.

Junior Balkan Mathematical Olimpiad

Saturday, June 22, 2019

Problem 1. Find all prime numbers p for which there exist non-negative integers x, y and z such that the number

$$x^p + y^p + z^p - x - y - z$$

is a product of exactly three distinct prime numbers.

Solution. For $p = 2$, we take $x = y = 4$ and $z = 3$. Then $x^p + y^p + z^p - x - y - z = 30 = 2 \cdot 3 \cdot 5$. For $p = 3$, we can take $x = 3$ and $y = 2$ and $z = 1$. Then again $x^p + y^p + z^p - x - y - z = 30 = 2 \cdot 3 \cdot 5$. For $p = 5$, we can take $x = 2$ and $y = 1$ and $z = 1$. Again $x^p + y^p + z^p - x - y - z = 30 = 2 \cdot 3 \cdot 5$.

Assume now that $p \ge 7$. Working modulo 2 and modulo 3 we see that the expression is divisible by both 2 and 3. Moreover, by Fermat`s Little Theorem, we have

$$x^p + y^p + z^p - x - y - z \equiv x + y + z - x - y - z = 0 \bmod p.$$

Therefore, by the given conditions, we have to solve the equation

$$x^p + y^p + z^p - x - y - z = 6p.$$

If one of the numbers x, y and z is bigger or equal to 2, let`s say $x \ge 2$, then

$$6p \ge x^p - x = x\left(x^{p-1} - 1\right) \ge 2\left(2^{p-1} - 1\right) = 2^p - 2.$$

It is easy to check by induction that $2^p - 2 > 6p$ for all primes $p \ge 7$. This contradiction shows that there are no more values of p which satisfy the required property.

Remark. There are couple of other ways to prove that $2^p - 2 > 6p$ for all primes $p \ge 7$. For example, we can use the Binomial Theorem as follows:

$$2^p - 2 \ge 1 + p + \frac{p(p-1)}{2} + \frac{p(p-1)(p-2)}{6} - 2 \ge 1 + p + 3p + 5p - 2 > 6p.$$

We can also use Bernoulli`s Inequality as follows:

$$2^p - 2 = 8(1+1)^{p-3} - 2 \ge 8\left(1 + (p-3)\right) - 2 = 8p - 18 > 6p.$$

The last inequality is true for $p \ge 11$. For $p = 7$ we can see directly that $2^p - 2 > 6p$.

Problem 2. Let a, b be two distinct real numbers and let c be a positive real number such that

$$a^4 - 2019a = b^4 - 2019b = c.$$

Prove that $-\sqrt{c} < ab < 0$.

Solution 1. Firstly, we can see that

$$2019(a-b)=a^4-b^4=(a-b)(a+b)(a^2+b^2).$$

Since $a\neq b$, we get $(a+b)(a^2+b^2)=2019$, so $a+b\neq 0$. Thus

$$\begin{aligned}2c&=a^4-2019a+b^4-2019b=\\&=a^4+b^4-2019(a+b)=\\&=a^4+b^4-(a+b)^2(a^2+b^2)=\\&=-2ab(a^2+ab+b^2).\end{aligned}$$

Hence $ab(a^2+ab+b^2)=-c<0$. Note that

$$a^2+ab+b^2=\tfrac{1}{2}\left(a^2+b^2+(a+b)^2\right)\geq 0,$$

thus $ab<0$. Finally, $a^2+ab+b^2=(a+b)^2-ab>-ab$ (the equality does not occur since $a+b\neq 0$). So

$$-c=ab(a^2+ab+b^2)<-(ab)^2\Rightarrow(ab)^2<c\Rightarrow-\sqrt{c}<ab<\sqrt{c}.$$

Therefore, we have $-\sqrt{c}<ab<0$.

Solution 2. By Descartes` Rule of Signs, the polynomial $p(x)=x^4-2019x-c$ has exactly one positive root and exactly one negative root. So a,b must be its two real roots. Since one of them is positive and the other is negative, then $ab<0$. Let $r\pm is$ be the two non-real roots of $p(x)$.

By Vieta, we have

$$ab(r^2+s^2)=-c, \tag{1}$$

$$a+b+2r=0, \tag{2}$$

$$ab+2ar+2br+r^2+s^2=0. \tag{3}$$

Using (2) and (3), we have

$$r^2+s^2=-2r(a+b)-ab=(a+b)^2-ab\geq-ab. \tag{4}$$

If in the last inequality we actually have an equality, then $a+b=0$. Then (2) gives $r=0$ and (3) gives $s^2=-ab$. Thus the roots of $p(x)$ are $a,-a,ia,-ia.$ This would give that $p(x)=x^4+a^4$, a contradiction.
So the inequality in (4) is strict and now from (1) we get

$$c=-(r^2+s^2)ab>(ab)^2,$$

which gives that $ab>-\sqrt{c}$.

Problem 3. Acute triangle ABC is such that $AB<AC$. The perpendicular bisector of side BC intersects lines AB and AC at points P and Q, respectively. Let H be the orthocenter of triangle ABC, and let M and N be the midpoints of segments BC and PQ, respectively. Prove that lines HM and AN meet on the circumcircle of ABC.

Solution. We have

$$\measuredangle APQ=\measuredangle BPM=90^\circ-\measuredangle MBP=90^\circ-\measuredangle CBA=\measuredangle HCB,$$

and

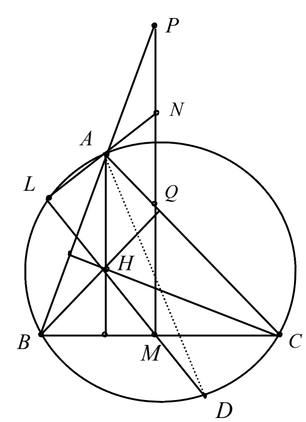
$$\measuredangle AQP=\measuredangle MQC=90^\circ-\measuredangle QCM=90^\circ-\measuredangle ACB=\measuredangle CBH.$$

From this two equalities, we see that the triangles APQ and HCB are similar. Moreover, since M and N are the midpoints of the segments BC and PQ respectively, then the triangles AQN and HBM are also similar. Therefore, we have $\measuredangle ANQ=\measuredangle HMB$.

Let L be the intersection of AN and HM. We have

$$\measuredangle MLN=180^\circ-\measuredangle LNM-\measuredangle NML=180^\circ-\measuredangle LMB-\measuredangle NML=180^\circ-\measuredangle NMB=90^\circ.$$

Now, let D be the point on the circumcircle of ABC diametrically opposite to A. It is known that D is also the reflection of point H over the point M. Therefore, we have that D belongs on HM and that $\measuredangle DLA=\measuredangle MLA=\measuredangle MLN=90^\circ$. But, as DA is the diameter of the circumcircle



of ABC, the condition that $\measuredangle DLA = 90^\circ$ is enough to conclude that L belongs on the circumcircle of ABC.

Remark. There is spiral similarity mapping AQP to HBC. Since the similarity maps AN to HM, it also maps AH to NM, and since these two lines are parallel, the centre of the similarity is $L = AN \cap HM$. Since the similarity maps BC to QP, its centre belongs on the circumcircle of BCX, where $X = BQ \cap PC$. But X is the reflection of A on QM and so it must belong on the circumcircle of ABC. Hence so must L.

Problem 4. A 5×100 table is divided into 500 unit square cells, where n of them are coloured black and the rest are coloured white. Two unit square cells are called *adjacent* if they share a common side. Each of the unit square cells has at most two adjacent black unit square cells. Find the largest possible value of n.

Solution 1. If we color all cells along all edges of the table together with the entire middle row except the second and the last-but-one cell, the condition is satisfied and there are 302 black cells. The figure below exhibits this coloring for 5×8 case.

We can cover the table by one fragment like the first one on the figure below, 24 fragments like the middle one, and one fragment like the third one.

a	b	a	b			h	i	h	i		
c	a	b			f	g	h	i			m
	c			f	g	f	g			m	
c	d	e			f	g	j	k			m
d	e	d	e			j	k	j	k		

In each fragment, among the cells with the same letter, there are at most two colored black, so the total number of colored cells is at most $(5+24\cdot 6+1)\cdot 2+2=302$.

Solution 2. Consider the cells adjacent to all cells of the second and fourth row. Counting multiplicity, each cell in the first and the fifth row is counted once, each cell in the third row twice, while each cell in the second and fourth row is also counted twice apart from their first and last cells which are counted only once.

So there are 204 cells counted once and 296 cells counted twice. Those cells contain, counting multiplicity, at most 400 black cells. Suppose a of the cells have multiplicity one and b of them have multiplicity 2. Then $a+2b\le 400$ and $a\le 204$. Thus

$$2a+2b\le 400+a\le 604,$$

and so $a+b\le 302$ as required.

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Asian Pacific mathematical olympiad

12.03.2019

Problems and Solutions

1. Let $\mathbb{Z}^+$ be the set of positive integers. Determine all functions $f:\mathbb{Z}^+\to\mathbb{Z}^+$ such that $a^2+f(a)f(b)$ is divisible by $f(a)+b$ for all positive integers a and b.

Answer: The answer is $f(n)=n$ for all positive integers n.

Clearly, $f(n)=n$ for all $n\in\mathbb{Z}^+$ satisfies the original relation. We show some possible approaches to prove that this is the only possible function.

Solution. First we perform the following substitutions on the original relation:

1. With $a=b=1$, we find that $f(1)+1\,|\,f(1)^2+1$, which implies $f(1)=1$.
2. With $a=1$, we find that $b+1\,|\,f(b)+1$. In particular, $b\le f(b)$ for all $b\in\mathbb{Z}^+$.
3. With $b=1$, we find that $f(a)+1\,|\,a^2+f(a)$, and thus $f(a)+1\,|\,a^2-1$. In particular, $f(a)\le a^2-2$ for all $a\ge 2$.

Now, let p be any odd prime. Substituting $a=p$ and $b=f(p)$ in the original relation, we find that $2f(p)\,|\,p^2+f(p)f(f(p))$. Therefore, $f(p)\,|\,p^2$. Hence the possible values of $f(p)$ are $1, p$ and p^2. By (2) above, $f(p)\ge p$ and by (3) above $f(p)\le p^2-2$. So $f(p)=p$ for all primes p.

Substituting $a=p$ into the original relation, we find that $b+p\,|\,p2+pf(b)$. However, since $(b+p)(f(b)+p-b)=p^2-b^2+bf(b)+pf(b)$, we have $b+p\,|\,bf(b)-b^2$. Thus, for any fixed b this holds for arbitrarily large primes p and therefore we must have $bf(b)-b^2=0$, or $f(b)=b$, as desired. ■

Solution 2. As above, we have relations (1)-(3). In (2) and (3), for $b=2$ we have $3\,|\,f(2)+1$ and $f(2)+1\,|\,3$. These imply $f(2)=2$.

Now, using $a=2$ we get $2+b\,|\,4+2f(b)$. Let $f(b)=x$. We have

$$1+x\equiv 0 \pmod{b+1}$$
$$4+2x\equiv 0 \pmod{b+2}.$$

From the first equation $x\equiv b \pmod{b+1}$ so $x=b+(b+1)t$ for some integer $t\ge 0$. Then

$$0\equiv 4+2x\equiv 4+2\big(b+(b+1)t\big)\equiv 4+2(-2-t)\equiv -2t \pmod{b+2}.$$

Also $t\le b-2$ because $1+x\,|\,b^2-1$ by (3).

If $b+2$ is odd, then $t\equiv 0 \pmod{b+2}$. Then $t=0$, which implies $f(b)=b$.

If $b+2$ is even, then $t\equiv 0 \pmod{(b+2)/2}$. Then $t=0$ or $t=(b+2)/2$. But if $t\ne 0$, then by definition $(b+4)/2=(1+t)=(x+1)/(b+1)$ and since $x+1\,|\,b^2-1$, then $(b+4)/2$ divides $b-1$. Therefore $b+4\,|\,10$ and the only possibility is $b=6$. So for even $b, b\ne 6$ we have $f(b)=b$.

Finally, by (2) and (3), for $b=6$ we have $7\,|\,f(6)+1$ and $f(6)+1\,|\,35$. This means $f(6)=6$ or $f(6)=34$. The later is discarded as, for $a=5, b=6$, we have by the original equation that $11\,|\,5(5+f(6))$. Therefore $f(n)=n$ for every positive integer n. ■

Solution 3. We proceed by induction. As in Solution 1, we have $f(1)=1$. Suppose that $f(n-1)=n-1$ for some integer $n\ge 2$.

With the substitution $a=n$ and $b=n-1$ in the original relation we obtain that $f(n)+n-1\,|\,n^2+f(n)(n-1)$. Since $f(n)+n-1\,|\,(n-1)(f(n)+n-1)$, then $f(n)+n-1\,|\,2n-1$.

With the substitution $a=n-1$ and $b=n$ in the original relation we obtain that

$$2n-1\,|\,(n-1)2+(n-1)f(n)=(n-1)(n-1+f(n)).$$

Since $(2n-1, n-1)=1$, we deduce that $2n-1\,|\,f(n)+n-1$.

Therefore, $f(n)+n-1=2n-1$, which implies the desired $f(n)=n$.■

2. Let m be a fixed positive integer. The infinite sequence $\{a_n\}_{n\geq 1}$ is defined in the following way: a_1 is a positive integer, and for every integer $n\geq 1$ we have

$$a_{n+1}=\begin{cases} a_n^2+2^m & \text{if } a_n<2^m \\ \dfrac{a_n}{2} & \text{if } a_n\geq 2^m. \end{cases}$$

For each m, determine all possible values of a_1 such that every term in the sequence is an integer.

Answer: The only value of m for which valid values of a_1 exist is $m=2$. In that case, the only solutions are $a_1=2^l$ for $l\geq 1$.

Solution. Suppose that for integers m and a_1 all the terms of the sequence are integers. For each $i\geq 1$, write the ith term of the sequence as $a_i=b_i 2^{c_i}$ where b_i is the largest odd divisor of a_i (the "odd part" of a_i) and c_i is a nonnegative integer.

Lemma 1. The sequence $b_1,b_2,\ldots$ is bounded above by 2^m.

Proof. Suppose this is not the case and take an index i for which $b_i>2^m$ and for which c_i is minimal. Since $a_i\geq b_i>2^m$, we are in the second case of the recursion. Therefore, $a_{i+1}=a_i/2$ and thus $b_{i+1}=b_i>2^m$ and $c_{i+1}=c_i-1<c_i$. This contradicts the minimality of c_i.

Lemma 2. The sequence $b_1,b_2,\ldots$ is nondecreasing.

Proof. If $a_i\geq 2^m$, then $a_{i+1}=a_i/2$ and thus $b_{i+1}=b_i$. On the other hand, if $a_i<2^m$, then

$$a_{i+1}=a_i^2+2^m=b_i^2 2^{2c_i}+2^m,$$

and we have the following cases:

- If $2c_i>m$, then $a_{i+1}=2^m(b_i^2 2^{2c_i-m}+1)$, so $b_{i+1}=b_i^2 2^{2c_i-m}+1>b_i$,
- If $2c_i<m$, then $a_{i+1}=2^{2c_i}(b_i^2+2^{m-2c_i})$, so $b_{i+1}=b_i^2+2^{m-2c_i}>b_i$,
- If $2c_i=m$, then $a_{i+1}=2^{m+1}\cdot\dfrac{b_i^2+1}{2}$, so $b_{i+1}=\dfrac{b_i^2+1}{2}\geq b_i$ since $b_i^2+1\equiv 2 \pmod 4$.

By combining these two lemmas we obtain that the sequence $b_1,b_2,\ldots$ is eventually constant. Fix an index j such that $b_k=b_j$ for all $k\geq j$. Since a_n descends to $a_n/2$ whenever $a_n\geq 2^m$, there are infinitely many terms which are smaller than 2^m. Thus, we can choose an $i>j$ such that $a_i<2^m$. From the proof of Lemma 2, $a_i<2^m$ and $b_{i+1}=b_i$ can happen simultaneously only when $2c_i=m$ and $b_{i+1}=b_i=1$. By Lemma 2, the sequence $b_1,b_2,\ldots$ is constantly 1 and thus $a_1,a_2,\ldots$ are all powers of two. Tracing the sequence starting from $a_i=2^{c_i}=2^{m/2}<2^m$,

$$2^{m/2}\to 2^{m+1}\to 2^m\to 2^{m-1}\to 2^{2m-2}+2^m,$$

Note that this last term is a power of two if and only if $2m-2=m$. This implies that m must be equal to 2. When $m=2$ and $a_1=2^l$ for $l\geq 1$ the sequence eventually cycles through $2,8,4,2,\ldots$. When $m=2$ and $a_1=1$ the sequence fails as the first terms are $1,5,5/2$.■

Solution 2. Let m be a positive integer and suppose that $\{a_n\}$ consists only of positive integers. Call a number *small* if it is smaller than $2m$ and *large* otherwise. By the recursion, after a small number we have a large one and after a large one we successively divide by 2 until we get a small one.

First, we note that $\{a_n\}$ is bounded. Indeed, $a1$ turns into a small number after a finite number of steps. After this point, each small number is smaller than 2^m, so each large number is smaller than $2^{2m}+2^m$. Now, since $\{a_n\}$ is bounded and consists only of positive integers, it is eventually periodic. We focus only on the cycle.

Any small number a_n in the cycle can be writen as $a/2$ for a large, so $a_n \geq 2^{m-1}$, then $a_{n+1} \geq 2^{2m-2}+2^m = 2^{m-2}\left(4+2^m\right)$, so we have to divide a_{n+1} at least $m-1$ times by 2 until we get a small number. This means that $a_{n+m} = \left(a_n^2+2^m\right)/2^{m-1}$, so $2^{m-1} \mid a_n^2$, and therefore $2^{\lceil (m-1)/2 \rceil} \mid a_n$ for any small number a_n in the cycle. On the other hand, $a_n \leq 2^m-1$, so $a_{n+1} \leq 2^{2m}-2^{m+1}+1+2^m \leq 2^m\left(2^m-1\right)$, so we have to divide a_{n+1} at most m times by two until we get a small number. This means that after a_n, the next small number is either $N = a_{m+n} = (a_n^2/2^{m-1})+2$ or $a_{m+n+1} = N/2/$. Ina any case $2^{\lceil (m-1)/2 \rceil}$ divides N.

If m is odd, then $x^2 \equiv -2\left(\bmod\ 2^{\lceil (m-1)/2 \rceil}\right)$ has a solution $x = a_n/2^{(m-1)/2}$. If $(m-1)/2 \geq 2 \Leftrightarrow m \geq 5$ then $x^2 \equiv -2(\bmod 4)$, which has no solution. So if m is odd, then $m \leq 3$.

If m is even, then $2^{m-1} \mid a_n^2 \Rightarrow 2^{\lceil (m-1)/2 \rceil} \mid a_n \Leftrightarrow 2^{m/2} \mid a_n$. Then if $a_n = 2^{m/2}x$, $2x^2 \equiv -2(\bmod\ 2^{m/2}) \Leftrightarrow x^2 \equiv -1(\bmod\ 2^{(m/2)-1})$ which is not possible for $m \geq 6$. So if m is even, then $m \leq 4$.

The cases $m = 1, 2, 3, 4$ are handed manually, checking the possible small numbers in the cycle, which have to be in the interval $[2^{m-1}, 2^m)$ and be divisible by $2^{\lceil (m-1)/2 \rceil}$:

- For $m=1$, the only small number is 1, which leads to 5, then 5/2.
- For $m=2$, the only eligible small number is 2, which gives the cycle (2, 8, 4). The only way to get to 2 is by dividing 4 by 2, so the starting numbers greater than 2 are all numbers that lead to 4, which are the powers of 2.
- For $m=3$, the eligible small numbers are 4 and 6; we then obtain 4, 24, 12, 6, 44, 22, 11, 11/2.
- For $m=4$, the eligible small numbers are 8 and 12; we then obtain 8, 80, 40, 20, 10, ... or 12, 160, 80, 40, 20, 10, ..., but in either case 10 is not an elegible small number.

3. Let ABC be a scalene triangle with circumcircle Γ. Let M be the midpoint of BC. A variable point P is selected in the line segment AM. The circumcircles of triangles BPM and CPM intersect Γ again at points D and E, respectively. The lines DP and EP intersect (a second time) the circumcircles to triangles CPM and BPM at X and Y, respectively. Prove that as P varies, the circumcircle of $\triangle AXY$ passes through a fixed point T distinct from A.

Solution. Let N be the radical center of the circumcircles of triangles ABC, BMP and CMP. The pairwise radical axes of these circles are BD, CE and PM, and hence they concur at N. Now, note that in directed angles:

$$\angle MCE = \angle MPE = \angle MPY = \angle MBY.$$

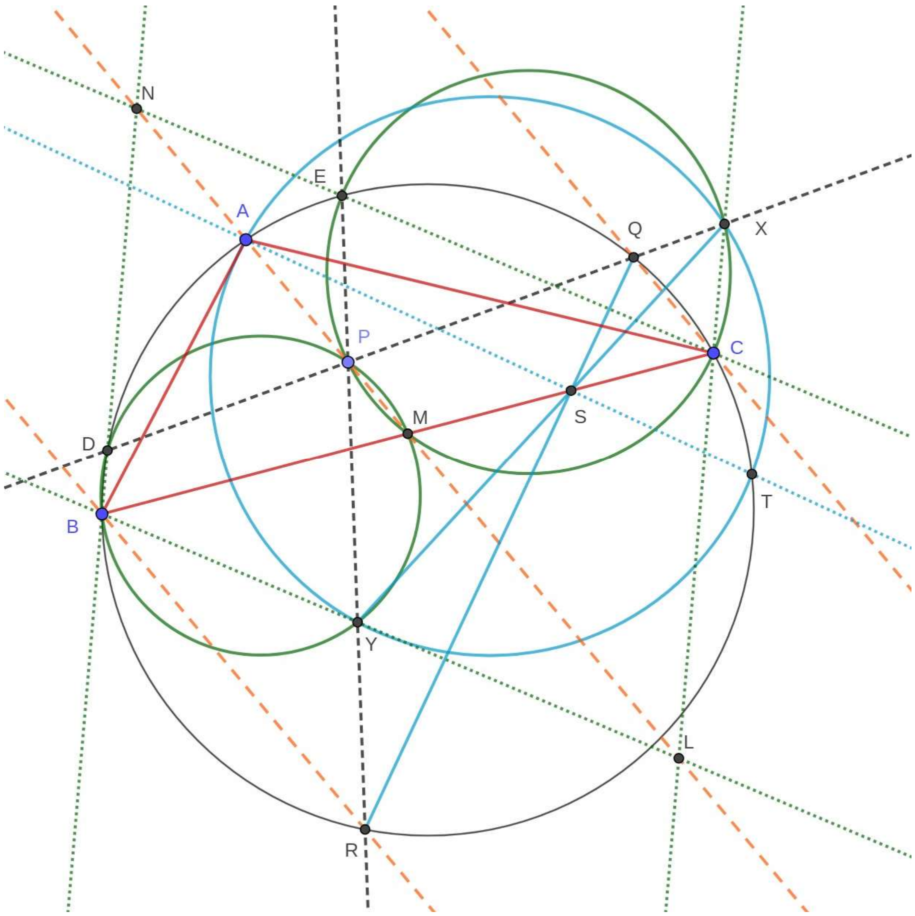
It follows that BY is parallel to CE, and analogously that CX is parallel to BD. Then, if L is the intersection of BY and CX, it follows that $BNCL$ is a parallelogram. Since $BM = MC$ we deduce that L is the reflection of N with respect to M, and therefore $L \in AM$. Using power of a point from L to the circumcircles of triangles BPM and CPM, we have

$$LY \cdot LB = LP \cdot LM = LX \cdot LC.$$

Hence, $BYXC$ is cyclic. Using the cyclic quadrilateral we find in directed angles:

$$\angle LXY = \angle LBC = \angle BCN = \angle NDE.$$

Since $CX \parallel BN$, it follows that $XY \parallel DE$.



Let Q and R be two points in Γ such that CQ, BR, and AM are all parallel. Then in directed angles:

$$\angle QDB = \angle QCB = \angle AMB = \angle PMB = \angle PDB.$$

Then D, P, Q are collinear. Analogously E, P, R are collinear. From here we get $\angle PRQ = \angle PDE = \angle PXY$, since XY and DE are parallel. Therefore $QRYX$ is cyclic. Let S be the radical center of the circumcircle of triangle ABC and the circles $BCYX$ and $QRYX$. This point lies in the lines BC, QR and XY because these are the radical axes of the circles. Let T be the second intersection of AS with Γ. By power of a point from S to the circumcircle of ABC and the circle $BCXY$ we have

$$SX \cdot SY = SB \cdot SC = ST \cdot SA.$$

Therefore T is in the circumcircle of triangle AXY. Since Q and R are fixed regardless of the choice of P, then S is also fixed, since it is the intersection of QR and BC. This implies T is also fixed, and therefore, the circumcircle of triangle AXY goes through $T \neq A$ for any choice of P. Now we show an alternative way to prove that $BCXY$ and $QRXT$ are cyclic. ■

Solution 2. Let the lines DP and EP meet the circumcircle of ABC again at Q and R, respectively. Then $\angle DQC \angle DBC = \angle DPM$, so $QC \parallel PM$. Similarly, $RB \parallel PM$.

Now, $\angle QCB = \angle PMB = \angle PXC = \angle(QX, CX)$, which is half of the arc QC in the circumcircle ω_C of QXC. So ω_C is tangent to BS; analogously, ω_B, the circumcicle of RYB, is also tangent to BC. Since $BR \parallel CQ$, the inscribed trapezoid $BRQC$ is isosceles, and by symmetry QR is also tangent to both circles, and the common perpendicular bisector of BR and CQ passes through the centers of ω_B and ω_C. Since $MB = MC$ and $PM \parallel BR \parallel CQ$, the line PM is the radical axis of ω_B and ω_C.

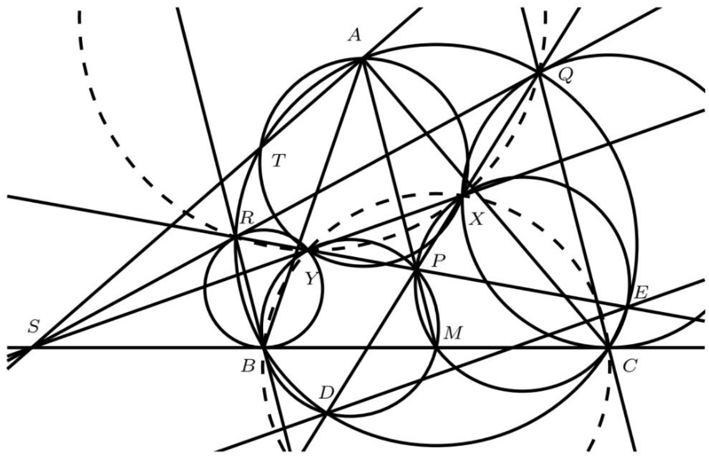
However, PM is also the radical axis of the circumcircles γ_B of PMB and γ_C of PMC. Let CX and PM meet at Z. Let $p(K,\omega)$ denote the power of a point K with respect to a circumference ω. We have

$$p(Z,\gamma_B)=p(Z,\gamma_C)=ZX\cdot ZC=p(Z,\omega_B)=p(Z,\omega_C).$$

Point Z is thus the radical center of γ_B, γ_C, ω_B, ω_C. Thus, the radical axes BY, CX, PM meet at Z. From here,

$$ZY\cdot ZB=ZC\cdot ZX\Rightarrow BCXY \text{ cyclic}$$
$$PY\cdot PR=PX\cdot PQ\Rightarrow QRXT \text{ cyclic.}$$

We may now finish as in Solution 1. ■



4. Consider a 2018 × 2019 board with integers in each unit square. Two unit squares are said to be neighbours if they share a common edge. In each turn, you choose some unit squares. Then for each chosen unit square the average of all its neighbours is calculated.

Finally, after these calculations are done, the number in each chosen unit square is replaced by the corresponding average. Is it always possible to make the numbers in all squares become the same after finitely many turns?

Answer: No

Solution. Let n be a positive integer relatively prime to 2 and 3. We may study the whole process modulo n by replacing divisions by 2, 3, 4 with multiplications by the corresponding inverses modulo n. If at some point the original process makes all the numbers equal, then the process modulo n will also have all the numbers equal. Our aim is to choose n and an initial configuration modulo n for which no process modulo n reaches a board with all numbers equal modulo n. We split this goal into two lemmas.

Lemma 1. There is a 2×3 board that stays constant modulo 5 and whose entries are not all equal.

Proof. Here is one such a board:

3	1	3
0	2	0

The fact that the board remains constant regardless of the choice of squares can be checked square by square.

Lemma 2. If there is an $r\times s$ board with $r\geq 2, s\geq 2$, that stays constant modulo 5, then there is also a $kr\times ls$ board with the same property.

Proof. We prove by a case by case analysis that repeateadly reflecting the $r\times s$ with respect to an edge preserves the property:

- If a cell had 4 neighbors, after reflections it still has the same neighbors.
- If a cell with a had 3 neighbors b,c,d, we have by hypothesis that $a\equiv 3^{-1}(b+c+d)\equiv$ $2(b+c+d)(\bmod 5)$. A reflection may add a as a neighbor of the cell and now

$$4^{-1}(a+b+c+d)\equiv 4(a+b+c+d)\equiv 4a+2a\equiv a(\text{mod } 5)$$

- If a cell with a had 2 neighbors b,c, we have by hypothesis that $a\equiv 2^{-1}(b+c)\equiv 3(b+c)(mod\,5)$. If the reflections add one a as neighbor, now
$$3^{-1}(a+b+c)\equiv 2(3(b+c)+b+c)\equiv 8(b+c)\equiv 3(b+c)\equiv a(\text{mod } 5).$$
- If a cell with a had 2 neighbors b,c, we have by hypothesis that $a\equiv 2^{-1}(b+c)\ (\text{mod } 5)$. If the reflections add two a's as neighbors, now
$4^{-1}(2a+b+c)\equiv(2^{-1}a+2^{-1}a)\equiv a(\text{mod } 5)$.

In the three cases, any cell is still preserved modulo 5 after an operation. Hence we can fill in the $kr\times ls$ board by $k\times l$ copies by reflection.

Since 2|2018 and 3|2019, we can get through reflections the following board:

3	1	3	3	1	3	
0	2	0	0	2	0	...
0	2	0	0	2	0	
3	1	3	3	1	3	

...

By the lemmas above, the board is invariant modulo 5, so the answer is no. ■

5. Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that
$$f(x^2+f(y))=f(f(x))+f(y^2)+2f(xy)$$
for all real number x and y.

Answer: The possible functions are $f(x)=0$ for all x and $f(x)=x^2$ for all x.

Solution. By substituting $x=y=0$ in the given equation of the problem, we obtain that $f(0)=0$. Also, by substituting $y=0$, we get $f(x^2)=f(f(x))$ for any x.

Furthermore, by letting $y=1$ and simplifying, we get
$$2f(x)=f(x^2+f(1))-f(x^2)-f(1),$$
from which it follows that $f(-x)=f(x)$ must hold for every x.

Suppose now that $f(a)=f(b)$ holds for some pair of numbers a,b. Then, by letting $y=a$ and $y=b$ in the given equation, comparing the two resulting identities and using the fact that $f(a^2)=f(f(a))=f(f(b))=f(b^2)$ also holds under the assumption, we get the fact that
$$f(a)=f(b)\Rightarrow f(ax)=f(bx)\qquad \text{for any real number } x.$$
Consequently, if for some $a\neq 0, f(a)=0$, then we see that, for any x,
$$f(x)=f\left(a\cdot\frac{x}{a}\right)=f\left(0\cdot\frac{x}{a}\right)=f(0)=0,$$
which gives a trivial solution to the problem.

In the sequel, we shall try to find a non-trivial solution for the problem. So, let us assume from now on that if $a\neq 0$ then $f(a)\neq 0$ must hold. We first note that since $f(f(x))=f(x^2)$ for all x, the right-hand side of the given equation equals $f(x^2)+f(y^2)+2f(xy)$, which is invariant if we interchange x and y. Therefore, we have
$$f(x^2)+f(y^2)+2f(xy)=f(x^2+f(y))=f(y^2+f(x)),\ \text{for every pair } x,y. \qquad (2)$$

Next, let us show that for any $x, f(x)\geq 0$ must hold. Suppose, on the contrary, $f(s)=-t^2$ holds for some pair s,t of non-zero real numbers. By setting $x=s, y=t$ in the right hand side of (2), we get $f\left(s^2+f(t)\right)=f\left(t^2+f(s)\right)=f(0)=0,$ so $f(t)=-s^2$.

We also have $f\left(t^2\right)=f\left(-t^2\right)=f\left(f(s)\right)=f\left(s^2\right)$. By applying (2) with $x=\sqrt{s^2+t^2}$ and $y=s$, we obtain

$$f\left(s^2+t^2\right)+2f\left(s\cdot\sqrt{s^2+t^2}\right)=0,$$

and similarly, by applying (2) with $x=\sqrt{s^2+t^2}$ and $y=t$, we obtain

$$f\left(s^2+t^2\right)+2f\left(t\cdot\sqrt{s^2+t^2}\right)=0.$$

Consequently, we obtain

$$f\left(t\cdot\sqrt{s^2+t^2}\right)=f\left(s\cdot\sqrt{s^2+t^2}\right)$$

By applying (1) with $a=s\cdot\sqrt{s^2+t^2}$, $b=t\cdot\sqrt{s^2+t^2}$ and $x=1/\sqrt{s^2+t^2}$, we obtain $f(s)=f(t)=-s^2$, from which it follows that

$$0=f\left(s^2+f(s)\right)=f\left(s^2\right)+f\left(s^2\right)+2f\left(s^2\right)=4f\left(s^2\right),$$

a contradiction to the fact $s^2>0$. Thus we conclude that for all $x\neq 0, f(x)>0$ must be satisfied.

Now, we show the following fact

$$k>0, f(k)=1\Leftrightarrow k=1. \tag{3}$$

Let $k>0$ for which $f(k)=1$. We have $f\left(k^2\right)=f\left(f(k)\right)=f(1)$, so by (1), $f(1/k)=f(k)=1$, so we may assume $k\geq 1$. By applying (2) with $x=\sqrt{k^2-1}$ and $y=k$, and using $f(x)\geq 0$, we get

$$f\left(k^2-1+f(k)\right)=f\left(k^2-1\right)+f(k^2)+2f(k\sqrt{k^2-1})\geq f(k^2-1)+f(k^2).$$

This simplifies to $0\geq f\left(k^2-1\right)\geq 0$, so $k^2-1=0$ and thus $k=1$.

Next we focus on showing $f(1)=1$. If $f(1)=m\leq 1$, then we may proceed as above by setting $x=\sqrt{1-m}$ and $y=1$ to get $m=1$. If $f(1)=m\geq 1$, now we note that $f(m)=f\left(f(1)\right)=f\left(1^2\right)=f(1)=m\leq m^2$. We may then proceed as above with $x=\sqrt{m^2-m}$ and $y=1$ to show $m^2=m$ and thus $m=1$.

We are now ready to finish. Let $x>0$ and $m=f(x)$. Since $f\left(f(x)\right)=f\left(x^2\right)$, then $f\left(x^2\right)=f(m)$. But by (1), $f\left(m/x^2\right)=1$. Therefore $m=x^2$. For $x<0$, we have $f(x)=f(-x)=f\left(x^2\right)$ as well. Therefore, for all $x, f(x)=x^2$.■

Solution 2. After proving that $f(x)>0$ for $x\neq 0$ as in the previous solution, we may also proceed as follows. We claim that f is injective on the positive real numbers. Suppose that $a>b>0$ satisfy $f(a)=f(b)$. Then by setting $x=1/b$ in (1) we have $f(a/b)=f(1)$. Now, by induction on n and iteratively setting $x=a/b$ in (1) we get $f\left((a/b)^n\right)=1$ for any positive integer n.

Now, let $m=f(1)$ and n be a positive integer such that $(a/b)^n>m$. By setting $x=\sqrt{(a/b)^n-m}$ and $y=1$ in (2) we obtain that

$$f\left(\left(a/b\right)^n - m + f(1)\right) = f\left(\left(a/b\right)^n - m\right) + f(1^2) + 2f\left(\sqrt{\left(a/b\right)^n - m}\right) \geq f\left(\left(a/b\right)^n - m\right) + f(1).$$

Since $f\left(\left(a/b\right)^n\right) = f(1)$, this last equation simplifies to $f\left(\left(a/b\right)^n - m\right) \leq 0$ and thus $m = \left(a/b\right)^n$. But this is impossible since m is constant and $a/b > 1$. Thus, f is injective on the positive real numbers. Since $f(f(x)) = f(x^2)$, we obtain that $f(x) = x^2$ for any real value x. ■

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7th European mathematical cup

08 December 2018 – 16 December 2018

MLADI NADARENI MATEMATIČARI Marin Getaldić

Category Junior

Problem 1. Let a,b,c be non-zero real numbers such that

$$a^2 + b + c = \frac{1}{a}$$
$$b^2 + c + a = \frac{1}{b}.$$
$$c^2 + a + b = \frac{1}{c}$$

Prove that at least two of a,b,c are equal.

Problem 2. Find all pairs (x, y) of positive integers such that

$$xy \mid x^2 + 2y - 1.$$

Problem 3. Let ABC be an accute triangle with $|AB|<|AC|$ and orthocenter H. The circle with center A and radius $|AC|$ intersects the circumcircle of $\triangle ABC$ at point D and the circle with center of A and radius $|AB|$ intersects the segment $\overline{AD}$ at point K. The line through K parallel to CD intersects BC at point L. If M is the midpoint of $\overline{BC}$ and N is the foot of the perpendicular from H to AL, prove that the line MN bisects the segment $\overline{AH}$.

Problem 4. Let n be a positive integer. Ana and Banana are playing the following game:

First, Ana arranges $2n$ coups in a row on a table, each facing upside-down. She then places a ball under a cup and makes a hole in the table under some other cup. Banana then gives a finite sequence of commands to Ana, where each command consists two adjecent cups in the row.

Her goal is to achieve that the ball has fallen into the hole during the game. Assuming Banana has no information about the position of the hole and the position of the ball at any point, what is the smallest number of commands she has to giv in order to achieve her goal?

Category Senior

Problem 1. A partition of a positive integer is even if all its elements are even numbers. Similarly, a partition is odd if all its elements are odd. Determine all positive integers n such that the number of even partitions of n is equal to the number of odd partitions of n.

Remark: A partition of a positive integer n is a non-decreasing sequence of positive integers whose sum of elements equals n. For example, $(2,3,4)$, $(1,2,2,2,2)$ and (9) are partitions of 9.

Problem 2. Let ABC be a triangle with $|AB|<|AC|$. Let k be the circumcircle of ΔABC and let O be the center of k. Point M is the midpoint of the arc $\widehat{BC}$ of k not containing A. Let D be the second intersection of the perpendicular line from M

Problem 3. For which real numbers $k>1$ does there exist a bounded set of positive real numbers S with at least 3 elements such that

$k(a-b)\in S$

for all $a,b\in S$ with $a>b$?

Remark: A set of positive real numbers S is bounded if there exists a positive real number M such that $x<M$ for all $x\in S$.

Problem 4. Let x,y,m,n be integers greather than 1 such that

$$\underbrace{x^{x^{x^{\cdot^{\cdot^{x}}}}}}_{m\text{ times}}=\underbrace{y^{y^{y^{\cdot^{\cdot^{y}}}}}}_{n\text{ times}}$$

Does it follows that $m=n$?

Remark: This is a tetration operation, so we can also write ${}^m x={}^n y$ for the initial condition.

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EGMO 2019, Kyiv, Ukraine

Tuesday, April 9, 2019

Problem 1. Find all triples (a,b,c) of real numbers such that $ab+bc+ca=1$ and

$$a^2b+c=b^2c+a=c^2a+b.$$

Problem 2. Let n be a positive integer. Dominoes are placed on a $2n\times 2n$ board in such a way that every cell of the board is adjacent to exactly one cell covered by a domino. For each n, determine the largest number of dominoes that can be placed in this way.

(A *domino* is a tile of size 2×1 or 1×2. Dominoes are placed on the board in such a way that each domino covers exactly two cells of the board, and dominoes do not overlap. Two cells are said to be *adjacent* if they are different and share a common side.)

Problem 3. Let ABC be a triangle such that $\measuredangle CAB>\measuredangle ABC$, and let I be its incentre. Let D be the point on segment BC such that $\measuredangle CAD=\measuredangle ABC$. Let ω be the circle tangent to AC at A and passing through I. Let X be the second point of intersection of ω and the circumcircle of ABC. Prove that the angle bisectors of $\measuredangle DAB$ and $\measuredangle CXB$ intersect at a point on line BC.

EGMO 2019, Kyiv, Ukraine

Wednesday, April 10 2019

Problem 4. Let ABC be a triangle with incentre I. The circle through B tangent to AI at I meets side AB again at P. The circle through C tangent to AI at I meets side AC again at Q. Prove that PQ is tangent to the incircle of ABC.

Problem 5. Let $n\geq 2$ be an integer, and let $a_1,a_2,\dots a_n$ be positive integers. Show that there exist positive integers $b_1,b_2,\dots b_n$ satisfying the following three conditions:

(A) $a_i \le b_i$ for $i = 1, 2, \ldots, n$;

(B) the remainders of $b_1, b_2, \ldots b_n$ on division by n are pairwise different; and

(C) $b_1 + \ldots + b_n \le n\left(\frac{n-1}{2} + \left\lfloor \frac{a_1 + \ldots + a_n}{n} \right\rfloor\right)$.

(Here, $\lfloor x \rfloor$ denotes the integer part of real number x, that is, the largest integer that does not exceed x.)

Problem 6. On a circle, Alina draws 2019 chords, the endpoints of which are all different. Points are considered *marked* if they are either
(i) one of the 4038 endpoints of a chord; or
(ii) an intersection point of at least two chords.
Alina labels each marked point. Of the 4038 points meeting criterion (i), Alina labels 2019 points with a 0 and the other 2019 points with a 1. She labels each point meeting criterion (ii) with an arbitrary integer (not necessarily positive).

Along each chord, Alina considers the segments connecting two consecutive marked points. (A chord with k marked points has $k-1$ such segments.) She labels each such segment in yellow with the sum of the labels of its two endpoints and in blue with the absolute value of their difference.

Alina finds that the $N+1$ yellow labels take each value $0, 1, \ldots, N$ exactly once. Show that at least one blue label is a multiple of 3.

(A *chord* is a line segment joining two different points on a circle.)

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XXII Mediterranean Mathematical Olympiad 2019

Problem 1

In the triangle *ABC*, in which $\angle A = 60^\circ$, $D \in (BC)$ is such that AD is the internal bisector of angle $\angle A$. Let it be r_B, r_C and r, respectively, the inradiuses of the triangles *ABD*, *ADC* and *ABC*. Show that

$$\frac{1}{r_B} + \frac{1}{r_C} = 2\left(\frac{1}{r} + \frac{1}{b} + \frac{1}{c}\right),$$

where b and c are the lengths of the sides *AC* and *AB* of the triangle *ABC*.

Problem 2

Let $m_1 < m_2 < \ldots < m_s$ be a sequence of $s \ge 2$ positive integers, none of which can be written as the sum of (two or more) distinct other numbers in the sequence. For every integer r with $1 \le r < s$ prove that

$r \cdot m_r + m_s \ge (r+1)(s-1)$.

Problem 3

Prove that there exist infinitely many positive integers x, y, z for which the sum of the digits in the decimal representation of $4x^4 + y^4 - z^2 + 4xyz$ is at most 2.

Problem 4

Let P be an interior point to an equilateral triangle of altitude one. If x, y, z are the distances from P to the sides of the triangle, then prove that

$x^2 + y^2 + z^2 \ge x^3 + y^3 + z^3 + 6xyz$.

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20-та Македонска
математичка олимпијада

17-та JMMO