FMC 2019
Category-SENIOR
SOLUTIONS AND MARKING SCHEMES

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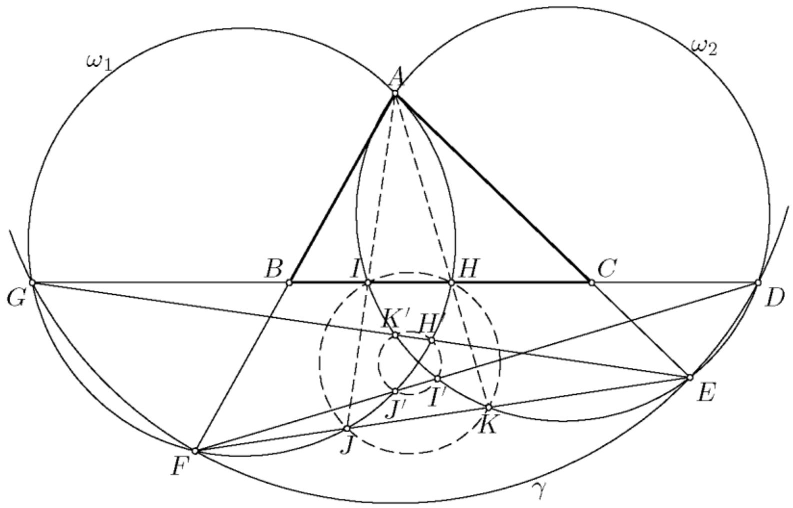
Problem 1. Let ABC be an acute triangle, for which $AB < AC < BC$ and let D be an arbitrary point on the line BC, after C. The circle $\omega(A, AD)$, intersects the half-lines AC, AB, CB in the points E, F, G, correspondingly. The circumscribed circle ω_1 of the triangle AFG intersects the lines FE, BC, GE, DF, again in the points J, H, H', J'. The circumscribed circle ω_2 of the triangle ADE intersects the lines FE, BC, GE, DF, again in the points I, K, K', I'. Prove that the quadrilaterals $HIJK$ and $H'I'J'K'$ are cyclic and the centers of their circumscribed circles coincide.

Solution. From $\measuredangle FAH = \measuredangle FGH = \measuredangle FGD = \frac{1}{2}\measuredangle FAD = 90° - \measuredangle AFD$, we conclude that $AH \perp DF$.



Similar, $\measuredangle DAI = 180° - \measuredangle DEI = 180° - \measuredangle DEF = \measuredangle DGF = \frac{1}{2}\measuredangle DAF$, so we have $AI \perp DF$. From this, we conclude that the points A, H, I are collinear. By analogy, we can conclude that the following triples of points (A, K, J), (A, H', I'), (A, K', J') are collinear.

The quadrilateral $HIJK$ is cyclic because $\measuredangle AIK = \measuredangle ADK = \measuredangle AGH = \measuredangle AJH$. By analogy, we conclude that the quadrilateral $H'I'J'K'$ is cyclic.

Finally, because $\measuredangle H'JH = \measuredangle H'GH = \measuredangle EGD = \measuredangle EFD = \measuredangle JFJ' = \measuredangle JHJ'$, the quadrilateral $HJJ'H'$ is isosceles trapezoid, where $HJ \parallel H'J'$, so the normal symmetrals of HJ and $H'J'$ are coinciding. By analogy, the normal symmetrals of IK and $I'K'$ are coinciding. From this, the centers of the circumscribed circles around $HIJK$ and $H'I'J'K'$ are coinciding.

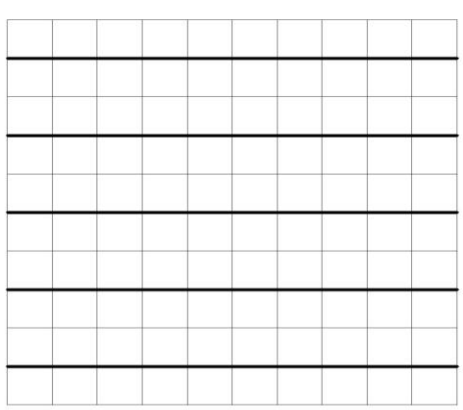
Problem 2. Let $n>2$ be a positive integer. A grasshopper is moving along the sides of an $n\times n$ square net, which is divided on n^2 unit squares. It moves so that
a) in every 1×1 unit square of the net, it passes only through one side
b) when it passes one side of 1×1 unit square of the net, it jumps on a vertex on another arbitrary 1×1 unit square of the net, which does not have a side on which the grasshopper moved along.
The grasshopper moves until the conditions can be fulfilled.

What is the shortest and the longest path that the grasshopper can go through if it moves according to the condition of the problem? Calculate its length and draw it on the net.

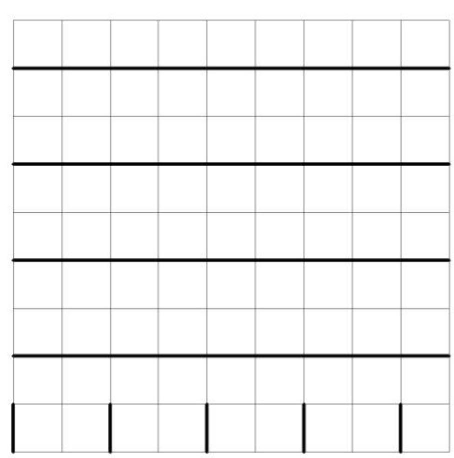
Solution. Let`s denote with $x$ the number of all sides on which the grasshopper moved along, and let`s denote with y the number of sides on which the grasshopper moved along and which are on the outside border of the square net. Then, every of the sides denoted with y is a side of exactly one unit square, i.e. $0\le y\le 4n-4$.

Every of the remaining $x-y$ sides on which the grasshopper moved along, are sides of exactly two 1×1 unit squares. According to this, we have $y+2(x-y)=n^2$, where n^2 is the number of all 1×1 unit squares in the net. So, $x=\frac{n^2+y}{2}$. Since, $0\le y\le 4n-4$, the shortest path that the grasshopper can move along is $\left[\frac{n^2+1}{2}\right]$, and the longest path is $\left[\frac{n^2+4n-4}{2}\right]$.

The strategy that it should use to go through the shortest and the longest path when n is even and odd is given in the drawings below.
The shortest path:



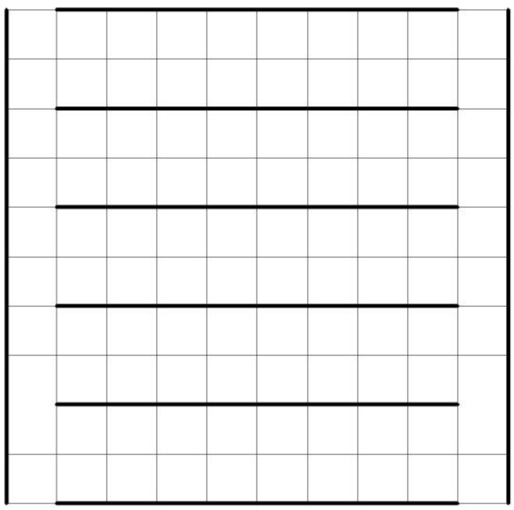
n is even



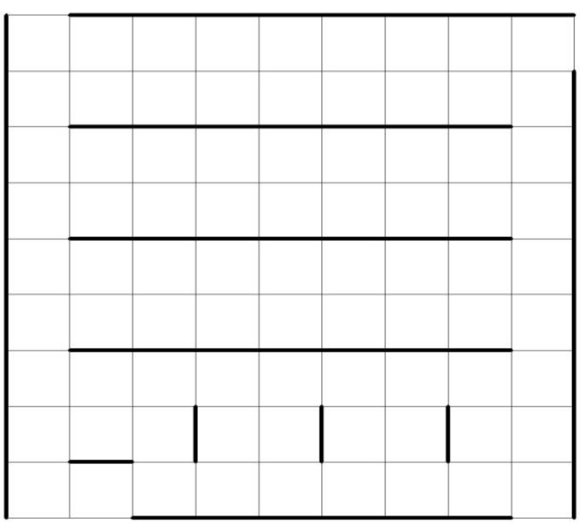
n is odd

The longest path:

n is even



n is odd



Problem 3. Determine all functions $f:(0,\infty)\to\mathbb{R}$ such that for every $x,y>0$

$$(x+\tfrac{1}{x})f(y)=f(xy)+f(\tfrac{y}{x}). \qquad (1)$$

Solution. We fix a real number $a>1$ and we introduce a new variable t. From (1) for $f(t)$, $f(t^2)$, $f(at)$ and $f(a^2t^2)$ we get the following system of equations:

$x=y=t$:	$(t+\frac{1}{t})f(t)=f(t^2)+f(1)$	(2.1)
$x=\frac{t}{a}, y=at$:	$(\frac{t}{a}+\frac{a}{t})f(at)=f(t^2)+f(a^2)$	(2.2)
$x=a^2t, y=t$:	$(a^2t+\frac{1}{a^2t})f(t)=f(a^2t^2)+f(\frac{1}{a^2})$	(2.3)
$x=y=at$:	$(at+\frac{1}{at})f(at)=f(a^2t^2)+f(1)$	(2.4)

If we eliminate $f(t^2)$ from (2.1) and (2.2), and if we eliminate $f(a^2t^2)$ from (2.3) and (2.4) we get

$$(t+\tfrac{1}{t})f(t)-(\tfrac{t}{a}+\tfrac{a}{t})f(at)=f(1)-f(a^2) \text{ and } (a^2t+\tfrac{1}{a^2t})f(t)-(at+\tfrac{1}{at})f(at)=f(\tfrac{1}{a^2})-f(1).$$

Now from the last two equations we express $f(t)$ and we get

$$((t+\tfrac{1}{t})(at+\tfrac{1}{at})-(\tfrac{t}{a}+\tfrac{a}{t})(a^2t+\tfrac{1}{a^2t}))f(t)=(at+\tfrac{1}{at})(f(1)-f(a^2))-(\tfrac{t}{a}+\tfrac{a}{t})(f(\tfrac{1}{a^2})-f(1))$$

i.e.

$$(a+\tfrac{1}{a}-(a^3+\tfrac{1}{a^3}))f(t)=(at+\tfrac{1}{at})(f(1)-f(a^2))-(\tfrac{t}{a}+\tfrac{a}{t})(f(\tfrac{1}{a^2})-f(1))$$

because $a+\frac{1}{a}-(a^3+\frac{1}{a^3})<0$, if we divide with the fix number $a+\frac{1}{a}-(a^3+\frac{1}{a^3})$ in the last equation, we get

$$f(t)=C_1t+\frac{C_2}{t}, \qquad (3)$$

where the numbers C_1 and C_2 can be expressed with $a, f(1), f(a^2)$ and $f(\frac{1}{a^2})$, i.e. they do not depend from t. The functions (3) satisfy the equation (1). Indeed,

$$(x+\tfrac{1}{x})f(y)=(x+\tfrac{1}{x})(C_1y+\tfrac{C_2}{y})=(C_1xy+\tfrac{C_2}{xy})+(C_1\tfrac{y}{x}+C_2\tfrac{x}{y})=f(xy)+f(\tfrac{y}{x}).$$

Problem 4. On two sheets of paper are written more than one positive integers. On the first paper n numbers are written and on the second paper m numbers are written. If one number is written on any of the papers then on the first paper is written also the sum of that number and 13, and on the second paper the difference of that number and 23. Calculate the value of $\frac{m}{n}$.

Solution. We will show that the value of the quotient $\frac{m}{n}$ is constant and that it is exactly $\frac{m}{n}=\frac{13}{23}$. Let`s consider the numbers from the first sheet as elements of the set A, while the numbers written on the second sheet are elements of the set B. We construct the sets A and B such that they satisfy the conditions of the problem, i.e. $|A|=23, |B|=13$ and if $k \in A \cup B$, then $k+13 \in A$ or $k-23 \in B$. We have,

$A=\{t, t+13, t+2\cdot 13, t+2\cdot 13, ..., t+22\cdot 13\}$ and

$B=\{t-23, t-2\cdot 23, t-3\cdot 23, ..., t-13\cdot 23\}$, for some $t \in \mathbb{N}$ and $t>13\cdot 23$.

From the construction it is clear that the sets A and B are disjoint sets. More over, if $a \in B$ and $a \neq t-13\cdot 23$, then $a-23 \in B$. If $a=t-13\cdot 23 \in B$, then $a+13 \in A$. If $b \in A$ and $b \neq t$, then $b+13 \in A$. If $b=t \in A$, then $b-23 \in B$. So, the sets A and B satisfy the conditions of the problem.

We will prove that $\frac{m}{n}=\frac{13}{23}$, for all A and B which satisfy the conditions of the problem. We construct a graph in the following way: the vertices of the graph represent the numbers from the union of the sets A and B. We draw directed edge from the vertex u towards the vertex v, if $u=v+13$, $u \in A$ or $u=v-23$, $u \in B$. From the condition of the problem, at least one edge goes out of every vertex of the graph and in every vertex one edge goes in. Indeed, if $u \in A$ or $u \in B$, then every $x=u-13$ or $x=u+23$ is determined (x may not be a vertex from the graph). It is clear that the total number of the edges which goes in is equal to the total number of edges that goes out of the vertices, so for every vertex there is exactly one edge that goes in and one that goes out. According to this, the graph is consisted of disjoint cycles. Let we consider an arbitrary cycle. We start with the vertex k: let it has a edges to the vertices from the set A and let it has b edges to the vertices from the set B. Then $k+13a-23b=k$ (since an edge from the first type corresponds to adding 13, while edge from the second type corresponds to subtracting 23). So, $13a=23b$, i.e. $\frac{b}{a}=\frac{13}{23}$. Because this ratio is the same for all disjoint cycles, we have that $\frac{m}{n}=\frac{13}{23}$.

Marking Scheme (Seniors)

Problem 1.

-Proving that the triples mentioned in the official solution are collinear **4 marks**

-proving that $AH \perp DF$ 1 mark

-proving that $AI \perp DH$ 1 mark

-concluding that A, H, I are collinear 1 mark

-Proving that the quadrilaterals $HIJK$ and $H'I'J'K'$ are cyclic **2 marks**

-proving only one of mentioned quadrilaterals is cyclic 1 mark

-Proving that the circumcenters coincide **4 marks**

-proving that the bisectors of HJ and $H'J'$ are coinciding 2 marks

-proving that the bisectors of IK and $I'K'$ are coinciding 1 mark

Remark. Proving only one of the above statements for bisectors worth 2 marks.

Problem 2.

-Noting that $0 \le y \le 4n-4$ or equivalent statement **2 marks**

-Obtaining the shortest and the longest paths **4 marks**

- Drawing the strategy for shortest and longest paths **4 marks**

Remark. For each situation student gets 1 mark.

Problem 3.

-Giving the equations (2.1)-(2.4) and eliminating $f(t^2)$ from (2.1) and (2.2), and eliminating $f(a^2t^2)$ from (2.3) and (2.4) **5 marks**

Remark. Each single equation from (2.1) to (2.4) worth 1 mark.

-Obtaining the solution given by equation (3) **4 marks**

-Checking that the obtained solution is satisfying the original equation (1) **1 mark**

General Remark. For guessing the solution without any justification the student obtains 1 mark.

Problem 4.

-Constructing the sets A and B or equivalent construction **4 marks**

-Proving that the sets A and B are disjoint **1 mark**

-Proving that $\frac{m}{n} = \frac{13}{23}$ **5 marks**

General Remark. Stating the correct result without any justification worth 1 mark.