8^{TH} European Mathematical Cup

14^{th} December 2019 - 22^{th} December 2019

Junior Category

MLADI NADARENI MATEMATIČARI
Marin Getaldić

Problem 1. Every positive integer is marked with a number from the set $\{0, 1, 2\}$, according to the following rule:

if a positive integer k is marked with j, then the integer $k + j$ is marked with 0.

Let S denote the sum of marks of the first 2019 positive integers. Determine the maximum possible value of S.

(Ivan Novak)

Problem 2. Define a sequence $x_1, x_2, x_3, \ldots$ such that $x_1 = \sqrt{2}$ and

$$x_{n+1} = x_n + \frac{1}{x_n} \text{ for } n \in \mathbb{N}.$$

Prove that the following inequality holds:

$$\frac{x_1^2}{2x_1x_2 - 1} + \frac{x_2^2}{2x_2x_3 - 1} + \ldots + \frac{x_{2018}^2}{2x_{2018}x_{2019} - 1} + \frac{x_{2019}^2}{2x_{2019}x_{2020} - 1} > \frac{2019^2}{x_{2019}^2 + \frac{1}{x_{2019}^2}}.$$

(Ivan Novak)

Problem 3. Let ABC be a triangle with circumcircle ω. Let l_B and l_C be two lines through the points B and C, respectively, such that $l_B \parallel l_C$. The second intersections of l_B and l_C with ω are D and E, respectively. Assume that D and E are on the same side of BC as A. Let DA intersect l_C at F and let EA intersect l_B at G. If O, O_1 and O_2 are circumcenters of the triangles ABC, ADG and AEF, respectively, and P is the circumcenter of the triangle OO_1O_2, prove that $l_B \parallel OP \parallel l_C$.

(Stefan Lozanovski)

Problem 4. Let u be a positive rational number and m be a positive integer. Define a sequence $q_1, q_2, q_3, \ldots$ such that $q_1 = u$ and for $n \geqslant 2$:

$$\text{if } q_{n-1} = \frac{a}{b} \text{ for some relatively prime positive integers } a \text{ and } b, \text{ then } q_n = \frac{a + mb}{b + 1}.$$

Determine all positive integers m such that the sequence $q_1, q_2, q_3, \ldots$ is eventually periodic for any positive rational number u.

Remark: A sequence $x_1, x_2, x_3, \ldots$ is *eventually periodic* if there are positive integers c and t such that $x_n = x_{n+t}$ for all $n \geqslant c$.

(Petar Nizić-Nikolac)

Time: 240 minutes.

Each problem is worth 10 points.

The use of calculators or any other instruments except rulers and compasses is not permitted.

8^{TH} European Mathematical Cup

14^{th} December 2019 - 22^{th} December 2019

Senior Category

MLADI NADARENI MATEMATIČARI
Marin Getaldić

Problem 1. For positive integers a and b, let $\gcd(a, b)$ denote their greatest common divisor. Determine all pairs of positive integers (m, n) such that for any two positive integers x and y such that $x \mid m$ and $y \mid n$,

$$\gcd(x + y, mn) > 1.$$

(Ivan Novak)

Problem 2. Let n be a positive integer. A $n \times n$ board consisting of n^2 cells, each being a unit square coloured either black or white, is called *convex* if for every black coloured cell, both the cell directly to the left of it (if it exists) and the cell directly above it (if it exists) are also coloured black. We define the *beauty* of a board as the number of pairs of its cells (u, v) such that u is black, v is white and u and v are in the same row or column. Determine the maximum possible beauty of a convex $n \times n$ board.

(Ivan Novak)

Problem 3. In an acute triangle ABC with $|AB| \neq |AC|$, let I be the incenter and O the circumcenter. The incircle is tangent to $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ in D, E and F respectively. Prove that if the line parallel to EF passing through I, the line parallel to AO passing through D and the altitude from A are concurrent, then the point of concurrence is the orthocenter of the triangle ABC.

(Petar Nizić-Nikolac)

Problem 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) + f(yf(x) + f(y)) = f(x + 2f(y)) + xy$$

for all $x, y \in \mathbb{R}$.

(Adrian Beker)

Time: 240 minutes.

Each problem is worth 10 points.

The use of calculators or any other instruments except rulers and compasses is not permitted.