Solutions of APMO 2015

Problem 1. Let ABC be a triangle, and let D be a point on side BC. A line through D intersects side AB at X and ray AC at Y. The circumcircle of triangle BXD intersects the circumcircle ω of triangle ABC again at point $Z \neq B$. The lines ZD and ZY intersect ω again at V and W, respectively. Prove that $AB = VW$.

Solution. Suppose XY intersects ω at points P and Q, where Q lies between X and Y. We will show that V and W are the reflections of A and B with respect to the perpendicular bisector of PQ. From this, it follows that $AVWB$ is an isosceles trapezoid and hence $AB = VW$.

First, note that

$$\angle BZD = \angle AXY = \angle APQ + \angle BAP = \angle APQ + \angle BZP,$$

so $\angle APQ = \angle PZV = \angle PQV$, and hence V is the reflection of A with respect to the perpendicular bisector of PQ.

Now, suppose W' is the reflection of B with respect to the perpendicular bisector of PQ, and let Z' be the intersection of YW' and ω. It suffices to show that B, X, D, Z' are concyclic. Note that

$$\angle YDC = \angle PDB = \angle PCB + \angle QPC = \angle W'PQ + \angle QPC = \angle W'PC = \angle YZ'C.$$

So D, C, Y, Z' are concyclic. Next, $\angle BZ'D = \angle CZ'B - \angle CZ'D = 180° - \angle BXD$ and due to the previous concyclicity we are done.

Alternative solution 1. Using cyclic quadrilaterals $BXDZ$ and $ABZV$ in turn, we have $\angle ZDY = \angle ZBA = \angle ZCY$. So $ZDCY$ is cyclic.

Using cyclic quadrilaterals $ABZC$ and $ZDCY$ in turn, we have $\angle AZB = \angle ACB = \angle WZV$ (or $180° - \angle WZV$ if Z lies between W and C).

So $AB = VW$ because they subtend equal (or supplementary) angles in ω. □

Alternative solution 2. Using cyclic quadrilaterals $BXDZ$ and $ABZV$ in turn, we have $\angle ZDY = \angle ZBA = \angle ZCY$. So $ZDCY$ is cyclic.

Using cyclic quadrilaterals $BXDZ$ and $ABZV$ in turn, we have $\angle DXA = \angle VZB = 180° - BAV$. So $XD \parallel AV$.

Using cyclic quadrilaterals $ZDCY$ and $BCWZ$ in turn, we have $\angle YDC = \angle YZC = \angle WBC$. So $XD \parallel BW$.

Hence $BW \parallel AV$ which implies that $AVWB$ is an isosceles trapezium with $AB = VW$. □

Problem 2. Let $S = \{2, 3, 4, \ldots\}$ denote the set of integers that are greater than or equal to 2. Does there exist a function $f : S \to S$ such that

$$f(a)f(b) = f(a^2b^2) \text{ for all } a, b \in S \text{ with } a \neq b?$$

Solution. We prove that there is no such function. For arbitrary elements a and b of S, choose an integer c that is greater than both of them. Since $bc > a$ and $c > b$, we have

$$f(a^4b^4c^4) = f(a^2)f(b^2c^2) = f(a^2)f(b)f(c).$$

Furthermore, since $ac > b$ and $c > a$, we have

$$f(a^4b^4c^4) = f(b^2)f(a^2c^2) = f(b^2)f(a)f(c).$$

Comparing these two equations, we find that for all elements a and b of S,

$$f(a^2)f(b) = f(b^2)f(a) \quad \Longrightarrow \quad \frac{f(a^2)}{f(a)} = \frac{f(b^2)}{f(b)}.$$

It follows that there exists a positive rational number k such that

$$f(a^2) = kf(a), \qquad \text{for all } a \in S. \tag{1}$$

Substituting this into the functional equation yields

$$f(ab) = \frac{f(a)f(b)}{k}, \qquad \text{for all } a, b \in S \text{ with } a \neq b. \tag{2}$$

Now combine the functional equation with equations (1) and (2) to obtain

$$f(a)f(a^2) = f(a^6) = \frac{f(a)f(a^5)}{k} = \frac{f(a)f(a)f(a^4)}{k^2} = \frac{f(a)f(a)f(a^2)}{k}, \qquad \text{for all } a \in S.$$

It follows that $f(a) = k$ for all $a \in S$. Substituting $a = 2$ and $b = 3$ into the functional equation yields $k = 1$, however $1 \notin S$ and hence we have no solutions.

Problem 3. A sequence of real numbers $a_0, a_1, \ldots$ is said to be *good* if the following three conditions hold.

(i) The value of a_0 is a positive integer.

(ii) For each non-negative integer i we have $a_{i+1} = 2a_i + 1$ or $a_{i+1} = \dfrac{a_i}{a_i + 2}$.

(iii) There exists a positive integer k such that $a_k = 2014$.

Find the smallest positive integer n such that there exists a good sequence $a_0, a_1, \ldots$ of real numbers with the property that $a_n = 2014$.

Answer: 60.

Solution. Note that

$$a_{i+1} + 1 = 2(a_i + 1) \text{ or } a_{i+1} + 1 = \frac{a_i + a_i + 2}{a_i + 2} = \frac{2(a_i + 1)}{a_i + 2}.$$

Hence

$$\frac{1}{a_{i+1} + 1} = \frac{1}{2} \cdot \frac{1}{a_i + 1} \text{ or } \frac{1}{a_{i+1} + 1} = \frac{a_i + 2}{2(a_i + 1)} = \frac{1}{2} \cdot \frac{1}{a_i + 1} + \frac{1}{2}.$$

Therefore,

$$\frac{1}{a_k + 1} = \frac{1}{2^k} \cdot \frac{1}{a_0 + 1} + \sum_{i=1}^{k} \frac{\varepsilon_i}{2^{k-i+1}}, \tag{1}$$

where $\varepsilon_i = 0$ or 1. Multiplying both sides by $2^k(a_k + 1)$ and putting $a_k = 2014$, we get

$$2^k = \frac{2015}{a_0 + 1} + 2015 \cdot \left(\sum_{i=1}^{k} \varepsilon_i \cdot 2^{i-1} \right),$$

where $\varepsilon_i = 0$ or 1. Since $\gcd(2, 2015) = 1$, we have $a_0 + 1 = 2015$ and $a_0 = 2014$. Therefore,

$$2^k - 1 = 2015 \cdot \left(\sum_{i=1}^{k} \varepsilon_i \cdot 2^{i-1} \right),$$

where $\varepsilon_i = 0$ or 1. We now need to find the smallest k such that $2015 | 2^k - 1$. Since $2015 = 5 \cdot 13 \cdot 31$, from the Fermat little theorem we obtain $5 | 2^4 - 1$, $13 | 2^{12} - 1$ and $31 | 2^{30} - 1$. We also have $\text{lcm}[4, 12, 30] = 60$, hence $5 | 2^{60} - 1$, $13 | 2^{60} - 1$ and $31 | 2^{60} - 1$, which gives $2015 | 2^{60} - 1$.

But $5 \nmid 2^{30} - 1$ and so $k = 60$ is the smallest positive integer such that $2015 | 2^k - 1$. To conclude, the smallest positive integer k such that $a_k = 2014$ is when $k = 60$.

Alternative solution 1. Clearly all members of the sequence are positive rational numbers. For each positive integer i, we have $a_i = \dfrac{a_{i+1} - 1}{2}$ or $a_i = \dfrac{2a_{i+1}}{1 - a_{i+1}}$. Since $a_i > 0$ we deduce that

$$a_i = \begin{cases} \dfrac{a_{i+1} - 1}{2} & \text{if } a_{i+1} > 1 \\ \dfrac{2a_{i+1}}{1 - a_{i+1}} & \text{if } a_{i+1} < 1. \end{cases}$$

Thus a_i is uniquely determined from a_{i+1}. Hence starting from $a_k = 2014$, we simply run the sequence backwards until we reach a positive integer. We compute as follows.

$$\tfrac{2014}{1}, \tfrac{2013}{2}, \tfrac{2011}{4}, \tfrac{2007}{8}, \tfrac{1999}{16}, \tfrac{1983}{32}, \tfrac{1951}{64}, \tfrac{1887}{128}, \tfrac{1759}{256}, \tfrac{1503}{512}, \tfrac{991}{1024}, \tfrac{1982}{33}, \tfrac{1949}{66}, \tfrac{1883}{132}, \tfrac{1751}{264}, \tfrac{1487}{528}, \tfrac{959}{1056}, \tfrac{1918}{97}, \tfrac{1821}{194}, \tfrac{1627}{388},$$

$$\tfrac{1239}{776}, \tfrac{463}{1552}, \tfrac{926}{1089}, \tfrac{1852}{163}, \tfrac{1689}{326}, \tfrac{1363}{652}, \tfrac{711}{1304}, \tfrac{1422}{593}, \tfrac{829}{1186}, \tfrac{1658}{357}, \tfrac{1301}{714}, \tfrac{587}{1428}, \tfrac{1174}{841}, \tfrac{333}{1682}, \tfrac{666}{1349}, \tfrac{1332}{683}, \tfrac{649}{1366}, \tfrac{1298}{717}, \tfrac{581}{1434}, \tfrac{1162}{853},$$

$$\tfrac{309}{1706}, \tfrac{618}{1397}, \tfrac{1236}{779}, \tfrac{457}{1558}, \tfrac{914}{1101}, \tfrac{1828}{187}, \tfrac{1641}{374}, \tfrac{1267}{748}, \tfrac{519}{1496}, \tfrac{1038}{977}, \tfrac{61}{1954}, \tfrac{122}{1893}, \tfrac{244}{1771}, \tfrac{488}{1527}, \tfrac{976}{1039}, \tfrac{1952}{63}, \tfrac{1889}{126}, \tfrac{1763}{252}, \tfrac{1511}{504}, \tfrac{1007}{1008}, \tfrac{2014}{1}.$$

There are 61 terms in the above list. Thus $k = 60$. $\square$

Alternative solution 1 is quite computationally intensive. Calculating the first few terms indicates some patterns that are easy to prove. This is shown in the next solution.

Alternative solution 2. Start with $a_k = \dfrac{m_0}{n_0}$ where $m_0 = 2014$ and $n_0 = 1$ as in alternative solution 1. By inverting the sequence as in alternative solution 1, we have $a_{k-i} = \dfrac{m_i}{n_i}$ for $i \geq 0$ where

$$(m_{i+1}, n_{i+1}) = \begin{cases} (m_i - n_i, 2n_i) & \text{if } m_i > n_i \\ (2m_i, n_i - m_i) & \text{if } m_i < n_i. \end{cases}$$

Easy inductions show that $m_i + n_i = 2015$, $1 \leq m_i, n_i \leq 2014$ and $\gcd(m_i, n_i) = 1$ for $i \geq 0$. Since $a_0 \in \mathbb{N}^+$ and $\gcd(m_k, n_k) = 1$, we require $n_k = 1$. An easy induction shows that $(m_i, n_i) \equiv (-2^i, 2^i) \pmod{2015}$ for $i = 0, 1, \ldots, k$.

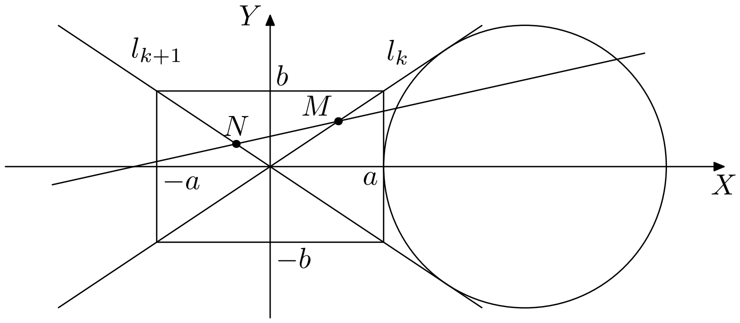
Thus $2^k \equiv 1 \pmod{2015}$. As in the official solution, the smallest such k is $k = 60$. This yields $n_k \equiv 1 \pmod{2015}$. But since $1 \leq n_k, m_k \leq 2014$, it follows that a_0 is an integer. $\square$

Problem 4. Let n be a positive integer. Consider $2n$ distinct lines on the plane, no two of which are parallel. Of the $2n$ lines, n are colored blue, the other n are colored red. Let $\mathcal{B}$ be the set of all points on the plane that lie on at least one blue line, and $\mathcal{R}$ the set of all points on the plane that lie on at least one red line. Prove that there exists a circle that intersects $\mathcal{B}$ in exactly $2n - 1$ points, and also intersects $\mathcal{R}$ in exactly $2n - 1$ points.

Solution. Consider a line ℓ on the plane and a point P on it such that ℓ is not parallel to any of the $2n$ lines. Rotate ℓ about P counterclockwise until it is parallel to one of the $2n$ lines. Take note of that line and keep rotating until all the $2n$ lines are met. The $2n$ lines are now ordered according to which line is met before or after. Say the lines are in order $\ell_1, \ldots, \ell_{2n}$. Clearly there must be $k \in \{1, \ldots, 2n - 1\}$ such that ℓ_k and ℓ_{k+1} are of different colors.

Now we set up a system of $X-$ and $Y-$ axes on the plane. Consider the two angular bisectors of ℓ_k and ℓ_{k+1}. If we rotate ℓ_{k+1} counterclockwise, the line will be parallel to one of the bisectors before the other. Let the bisector that is parallel to the rotation of ℓ_{k+1} first be the X–axis, and the other the Y–axis. From now on, we will be using the directed angle notation: for lines s and s', we define $\angle(s, s')$ to be a real number in $[0, \pi)$ denoting the angle in radians such that when s is rotated counterclockwise by $\angle(s, s')$ radian, it becomes parallel to s'. Using this notation, we notice that there is no $i = 1, \ldots, 2n$ such that $\angle(X, l_i)$ is between $\angle(X, \ell_k)$ and $\angle(X, \ell_{k+1})$.

Because the $2n$ lines are distinct, the set S of all the intersections between ℓ_i and ℓ_j $(i \neq j)$ is a finite set of points. Consider a rectangle with two opposite vertices lying on ℓ_k and the other two lying on ℓ_{k+1}. With respect to the origin (the intersection of ℓ_k and ℓ_{k+1}), we can enlarge the rectangle as much as we want, while all the vertices remain on the lines. Thus, there is one of these rectangles R which contains all the points in S in its interior. Since each side of R is parallel to either $X-$ or $Y-$ axis, R is a part of the four lines $x = \pm a$, $y = \pm b$. where $a, b > 0$.



Consider the circle $\mathcal{C}$ tangent to the right of the $x = a$ side of the rectangle, and to both ℓ_k and ℓ_{k+1}. We claim that this circle intersects $\mathcal{B}$ in exactly $2n-1$ points, and also intersects $\mathcal{R}$ in exactly $2n-1$ points. Since $\mathcal{C}$ is tangent to both ℓ_k and ℓ_{k+1} and the two lines have different colors, it is enough to show that $\mathcal{C}$ intersects with each of the other $2n-2$ lines in exactly 2 points. Note that no two lines intersect on the circle because all the intersections between lines are in S which is in the interior of R.

Consider any line L among these $2n-2$ lines. Let L intersect with ℓ_k and ℓ_{k+1} at the points M and N, respectively (M and N are not necessarily distinct). Notice that both M and N must be inside R. There are two cases:

(i) L intersects R on the $x = -a$ side once and another time on $x = a$ side;

(ii) L intersects $y = -b$ and $y = b$ sides.

However, if (ii) happens, $\angle(\ell_k, L)$ and $\angle(L, \ell_{k+1})$ would be both positive, and then $\angle(X, L)$ would be between $\angle(X, \ell_k)$ and $\angle(X, \ell_{k+1})$, a contradiction. Thus, only (i) can happen. Then L intersects $\mathcal{C}$ in exactly two points, and we are done.

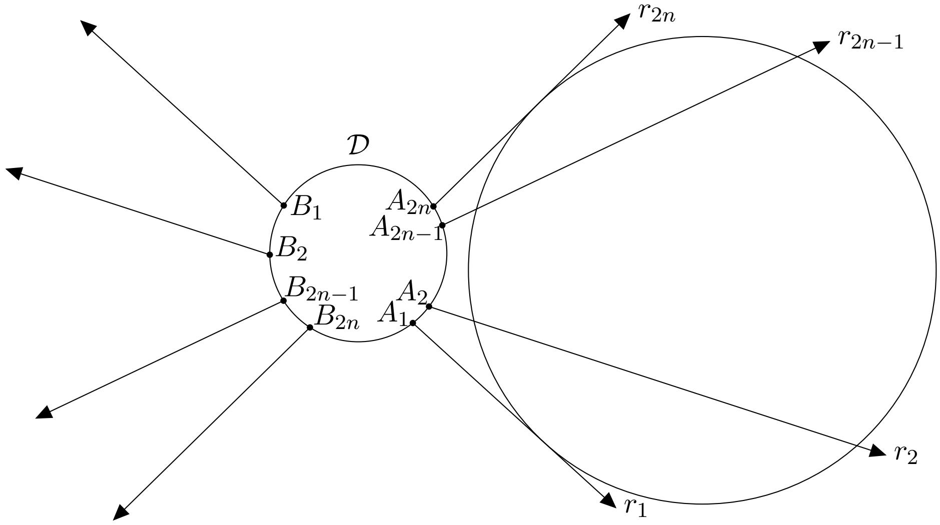
Alternative solution. By rotating the diagram we can ensure that no line is vertical. Let $\ell_1, \ell_2, \ldots, \ell_{2n}$ be the lines listed in order of increasing gradient. Then there is a k such that lines ℓ_k and ℓ_{k+1} are oppositely coloured. By rotating our coordinate system and cyclicly relabelling our lines we can ensure that $\ell_1, \ell_2, \ldots, \ell_{2n}$ are listed in order of increasing gradient, ℓ_1 and ℓ_{2n} are oppositely coloured, and no line is vertical.

Let $\mathcal{D}$ be a circle centred at the origin and of sufficiently large radius so that

- All intersection points of all pairs of lines lie strictly inside $\mathcal{D}$; and
- Each line ℓ_i intersects $\mathcal{D}$ in two points A_i and B_i say, such that A_i is on the right semicircle (the part of the circle in the positive x half plane) and B_i is on the left semicircle.

Note that the anticlockwise order of the points A_i, B_i around $\mathcal{D}$ is $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n$.

(If A_{i+1} occurred before A_i then rays r_i and r_{i+1} (as defined below) would intersect outside $\mathcal{D}$.)



For each i, let r_i be the ray that is the part of the line ℓ_i starting from point A_i and that extends to the right. Let $\mathcal{C}$ be any circle tangent to r_1 and r_{2n}, that lies entirely to the right of $\mathcal{D}$. Then $\mathcal{C}$ intersects each of $r_2, r_3, \ldots, r_{2n-1}$ twice and is tangent to r_1 and r_{2n}. Thus $\mathcal{C}$ has the required properties. $\square$

Problem 5. Determine all sequences $a_0, a_1, a_2, \ldots$ of positive integers with $a_0 \geq 2015$ such that for all integers $n \geq 1$:

(i) a_{n+2} is divisible by a_n;

(ii) $|s_{n+1} - (n+1)a_n| = 1$, where $s_{n+1} = a_{n+1} - a_n + a_{n-1} - \cdots + (-1)^{n+1}a_0$.

Answer: There are two families of answers:

(a) $a_n = c(n+2)n!$ for all $n \geq 1$ and $a_0 = c + 1$ for some integer $c \geq 2014$, and

(b) $a_n = c(n+2)n!$ for all $n \geq 1$ and $a_0 = c - 1$ for some integer $c \geq 2016$.

Solution. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of positive integers satisfying the given conditions. We can rewrite (ii) as $s_{n+1} = (n+1)a_n + h_n$, where $h_n \in \{-1, 1\}$. Substituting n with $n-1$ yields $s_n = na_{n-1} + h_{n-1}$, where $h_{n-1} \in \{-1, 1\}$. Note that $a_{n+1} = s_{n+1} + s_n$, therefore there exists $\delta_n \in \{-2, 0, 2\}$ such that

$$a_{n+1} = (n+1)a_n + na_{n-1} + \delta_n. \tag{1}$$

We also have $|s_2 - 2a_1| = 1$, which yields $a_0 = 3a_1 - a_2 \pm 1 \leq 3a_1$, and therefore $a_1 \geq \frac{a_0}{3} \geq 671$. Substituting $n = 2$ in (1), we find that $a_3 = 3a_2 + 2a_1 + \delta_2$. Since $a_1 | a_3$, we have $a_1 | 3a_2 + \delta_2$, and therefore $a_2 \geq 223$. Using (1), we obtain that $a_n \geq 223$ for all $n \geq 0$.

Lemma 1: For $n \geq 4$, we have $a_{n+2} = (n+1)(n+4)a_n$.

Proof. For $n \geq 3$ we have

$$a_n = na_{n-1} + (n-1)a_{n-2} + \delta_{n-1} > na_{n-1} + 3. \tag{2}$$

By applying (2) with n substituted by $n-1$ we have for $n \geq 4$,

$$a_n = na_{n-1} + (n-1)a_{n-2} + \delta_{n-1} < na_{n-1} + (a_{n-1} - 3) + \delta_{n-1} < (n+1)a_{n-1}. \tag{3}$$

Using (1) to write a_{n+2} in terms of a_n and a_{n-1} along with (2), we obtain that for $n \geq 3$,

$$\begin{aligned} a_{n+2} &= (n+3)(n+1)a_n + (n+2)na_{n-1} + (n+2)\delta_n + \delta_{n+1} \\ &< (n+3)(n+1)a_n + (n+2)na_{n-1} + 3(n+2) \\ &< (n^2 + 5n + 5)a_n. \end{aligned}$$

Also for $n \geq 4$,

$$\begin{aligned} a_{n+2} &= (n+3)(n+1)a_n + (n+2)na_{n-1} + (n+2)\delta_n + \delta_{n+1} \\ &> (n+3)(n+1)a_n + na_n \\ &= (n^2+5n+3)a_n. \end{aligned}$$

Since $a_n | a_{n+2}$, we obtain that $a_{n+2} = (n^2+5n+4)a_n = (n+1)(n+4)a_n$, as desired. $\square$

Lemma 2: For $n \geq 4$, we have $a_{n+1} = \dfrac{(n+1)(n+3)}{n+2}\,a_n$.

Proof. Using the recurrence $a_{n+3} = (n+3)a_{n+2} + (n+2)a_{n+1} + \delta_{n+2}$ and writing a_{n+3}, a_{n+2} in terms of a_{n+1}, a_n according to Lemma 1 we obtain

$$(n+2)(n+4)a_{n+1} = (n+3)(n+1)(n+4)a_n + \delta_{n+2}.$$

Hence $n+4 | \delta_{n+2}$, which yields $\delta_{n+2} = 0$ and $a_{n+1} = \dfrac{(n+1)(n+3)}{n+2}\,a_n$, as desired. $\square$

Suppose there exists $n \geq 1$ such that $a_{n+1} \neq \dfrac{(n+1)(n+3)}{n+2}\,a_n$. By Lemma 2, there exist a greatest integer $1 \leq m \leq 3$ with this property. Then $a_{m+2} = \dfrac{(m+2)(m+4)}{m+3}\,a_{m+1}$. If $\delta_{m+1} = 0$, we have $a_{m+1} = \dfrac{(m+1)(m+3)}{m+2}\,a_m$, which contradicts our choice of m. Thus $\delta_{m+1} \neq 0$.

Clearly $m+3 | a_{m+1}$. Write $a_{m+1} = (m+3)k$ and $a_{m+2} = (m+2)(m+4)k$. Then $(m+1)a_m + \delta_{m+1} = a_{m+2} - (m+2)a_{m+1} = (m+2)k$. So, $a_m | (m+2)k - \delta_{m+1}$. But a_m also divides $a_{m+2} = (m+2)(m+4)k$. Combining the two divisibility conditions, we obtain $a_m | (m+4)\delta_{m+1}$. Since $\delta_{m+1} \neq 0$, we have $a_m | 2m+8 \leq 14$, which contradicts the previous result that $a_n \geq 223$ for all nonnegative integers n.

So, $a_{n+1} = \dfrac{(n+1)(n+3)}{n+2}\,a_n$ for $n \geq 1$. Substituting $n = 1$ yields $3 | a_1$. Letting $a_1 = 3c$, we have by induction that $a_n = n!(n+2)c$ for $n \geq 1$. Since $|s_2 - 2a_1| = 1$, we then get $a_0 = c \pm 1$, yielding the two families of solutions. By noting that $(n+2)n! = n! + (n+1)!$, we have $s_{n+1} = c(n+2)! + (-1)^n(c - a_0)$. Hence both families of solutions satisfy the given conditions.