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CONTENTS

23. **SOLVING TASKS FROM LINEAR PROGRAMMING USING GEOGEBRA**
Elena Karamazova Gelova [1], Mirjana Kocaleva Vitanova [2], Sonja Mančevska [3]

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About the geometric interprations of the basic interactions and some consequences

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Abstract

Our space-time consists of three 3-dimensional spaces: space S, space rotations SR and time T. The basic spaces are S and SR, while in case of constraints of these two spaces, the body will be time displaced, of change the speed of time as in case of gravitational field. In recent papers [15, 14] it is given a (possible) geometric description of basic interactions in the nature, by using the non-commutativity of translations and rotations in two groups of rotations and translations together: in the affine group, and the group which is locally isomorphic to $SO(4)$, or $Spin(4)$. In this paper our attention is mainly to the classical electromagnetic interactions and gravitational interactions. The electromagnetic interaction can be interpreted such that both charged particles are mutually rotated for an angle, while the gravitational interaction can be interpreted such that the gravitational body with mass M radially translates each point for length GM/c^2. Using these two interpretations, in this paper we prove that the mass is observed enlarged for coefficient $1/\sqrt{1-\frac{v^2}{c^2}}$, while the charge is observed unchanged according to the observer who moves with velocity $\mathbf{v}$. These two results are ad hoc used in physics, but now we have a deduction of them.

1 Introduction

The space and time were subject of interest from the old civilizations up to the present time. They were separated many centuries ago. Even in the Newton theory they are still separated and the time flow was considered as uniform phenomena in the universe. Remarkable approach in understanding the space and time was done by the well known scientist and philosopher Roger Boscovich (1711-1787), who was not well understood at that time. He made distinction between the real space-time and the space-time according to our observations (ref. [1]). More than one century before the Special Relativity, he wrote that there does not exist an absolute space in rest, i.e. about the relativity among the moving systems.

According to the Theory of Relativity there does not exist strong separation between the space and time, which is evident from the Lorentz transformations. This idea was generalized in the recent refs. ([2, 3, 4]) for the space, time and rotations. For each small body besides its 3 spatial coordinates, can be jointed also 3 degrees of freedom about its rotation in the space and also 3 degrees of freedom for the velocity of the considered body. These 3+3 degrees of freedom are of the same level and importance as the basic 3 spatial coordinates. There are three 3-dimensional sets: space S which is homeomorphic to S^3, spatial rotations SR which is also homeomorphic to S^3 and time T which is homeomorphic to $\mathbb{R}^3$. The space SR is homeomorphic to S^3 if it is considered as the group of quaternions with module 1, which is locally isomorphic to $SO(3,\mathbb{R})$. Each two of these sets may interfere analogously to the space and time in the Special Relativity.

The group of Lorentz transformations $O^{\uparrow}_{+}(1,3)$ is isomorphic to $SO(3,\mathbb{C})$, and if we consider this complex group as a group of real 6×6 matrices, this group is the required Lie group which connects the spaces S and T. The group which connects the spaces SR and T is the same group of transformations. This Lie group of transformations has Lie algebra which is determined by the matrices of type

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}, \tag{1}$$

where X and Y are antisymmetric 3×3 matrices. This Lie group will be denoted by G_{t}, because it connects the space T (temporal space) with the other two spaces. In ref. [4] the Lorentz transformations are converted as transformations in $S\times T$, given by 6×6 matrices.

The Lie group which connects the spaces S and SR has Lie algebra which consists of matrices of type

$$\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}, \tag{2}$$

where X and Y are antisymmetric 3×3 matrices. This Lie group is generated by the following 3 matrices of translation $\tau_{(\alpha,0,0)}$, $\tau_{(0,\alpha,0)}$ and $\tau_{(0,0,\alpha)}$ along the x, y, and z axes, where

$$\tau_{(\alpha,0,0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & 0 & 0 & \sin\alpha \\ 0 & 0 & \cos\alpha & 0 & -\sin\alpha & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \sin\alpha & 0 & \cos\alpha & 0 \\ 0 & -\sin\alpha & 0 & 0 & 0 & \cos\alpha \end{bmatrix},$$

while the other two matrices $\tau_{(0,\alpha,0)}$ and $\tau_{(0,0,\alpha)}$ are obtained by the cyclic permutation $(1,2,3,4,5,6)\mapsto(2,3,1,5,6,1)$ and also by the following 3 matrices of rotation $\rho_{(\alpha,0,0)}$, $\rho_{(0,\alpha,0)}$ and $\rho_{(0,0,\alpha)}$ around the x, y, and z axes, where

$$\rho_{(\alpha,0,0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 & 0 & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix},$$

while the other two matrices $\rho_{(0,\alpha,0)}$ and $\rho_{(0,0,\alpha)}$ are obtained by the same cyclic permutation. This group will be denoted by G_{s} as a space group, which connects the spaces S and SR. The group G_{s} is isomorphic to the group Spin(4) ([5]). We denote by $\mathcal{A}$ the affine group of translations and rotations in the Euclidean space, and as a set of 6×6 matrices it can be proved that its Lie algebra has the form

$$\begin{bmatrix} X & Y \\ 0 & X \end{bmatrix}, \tag{3}$$

where X and Y are antisymmetric 3×3 matrices.

The elements of the space S are measured in meters, while the elements of spatial rotations SR are measured in radians, and so there exists a local constant as a coefficient of proportionality between these two spaces, which is called *radius of range* R. The elementary particles, galaxies and the universe, have their own radii of range. While the velocity of light c connects the space and time, the radius of range connects the space and space rotations.

The multi-dimensional time was investigated also by another authors, for example in refs. [6, 7, 8, 9, 10, 11, 12, 13].

2 Exchanging among S, SR and T

The are 4 basic exchanges among the spaces S, SR and T. If their elements are denoted by s, r and t respectively, then the basic 4 exchanges are [14, 15]:

$$1.\ r \to s, \quad 2.\ s \to r, \quad 3.\ r \to t, \quad 4.\ s \to t.$$

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Figure 1: Basic four exchanges among the spaces S, SR and T.

In general $x \mapsto y$ means that if $x \in X$ is constrained to occur, then it will be converted into $y \in Y$. According to 1. and 2., when one of them is constrained then it converts into the other space from $S \times SR$. The case 3. says that when the rotation r is constrained, then it makes some changes in the time, for example in speed of time. But, if the rotation r is constrained, then it tries first to convert into $s \in S$, and if it is not admitted, then it converts into $t \in T$. So, the case 3.

must be of composite type $s \to r \to t$. Analogously, the case 4. must be of type $r \to s \to t$. But, the cases $t \mapsto s$ and $t \mapsto r$ are not admitted, because none may constrain the time. Also it is clear that the composite cases $s \to r \to s$ are not admitted. We give some simple examples given in [14, 15].

The first case $r \to s$ means that when the rotation is not permitted, then the particles will be translated, i.e. will be displaced. It occurs for example, when the spinning bodies are moving in a circle, or a spinning football ball moves on an arc in the air, instead of parabolic trajectory. In both cases, each point of the spinning body intends to rotate according to its trajectory in the space. But it is not completely admitted, because the body is solid. As a consequence, each point of the spinning body intends to be displaced (or translated) in the space and it moves according to the sum of all such small displacements. In general some of these displacements are also constrained, and in such a motion we obtain also changes of type 4. This is commented in many details in [16]. This displacement in the previous papers were called induced spin motions, or simply spin motions. These spin motions are non-inertial motions. For example, if a football ball stops to rotate around its non-constant axis, then it will continue to move according to the well known parabolic trajectory. Moreover, in [16] it is explained the variation of the length of the day with a period of 6 months. It is important to mention that if the spin motion is constrained, then it becomes inertial motion.

The second case $s \to r$ means that if the space displacement is not admitted completely or partially, then it induces a spatial rotation. For example, let a rigid body moves with a velocity $\mathbf{v}$. If one point S of the rigid body is constrained to move, then the body will start to rotate around the point S. So the rotation is induced in this case. Analogously to the spin motion, in this case we also have both cases 2. and 3.

Assume that a non-rotating body initially rests with respect to the Earth on almost infinity distance. Assume that this body freely falls toward the Earth under the Earth's gravitation. When the body comes at the surface of the Earth, it is not permitted to be displaced further. So, this constraint will cause time displacement, such that the time will be slower. Indeed, if the velocity at the surface is equal to v, then the constraint for the space displacement will induce slower time for coefficient $\lambda = \sqrt{1 - \frac{v^2}{c^2}}$. This is the case 4 ($s \to t$). Since $v \approx \sqrt{2GM/R}$, the time on the surface of the Earth is slower for coefficient $\lambda \approx \sqrt{1 - \frac{2GM}{Rc^2}} \approx 1 - \frac{GM}{Rc^2}$, which is also well known from the General Relativity up to approximation of c^{-2}. This example is important to mention in order to emphasize that the speed of time in gravitational field is slower because of the existence of the acceleration toward the center of the planet, but not conversely. Indeed, the gravitational acceleration is not a consequence since the speed of time is not constant close to the planet.

It is also interesting to mention if someone intends to construct a time-travel machine, it is necessary to use the cases 3. and 4.

3 The induced 4 cases and the basic interactions in the space.

Each of these four cases induces an interaction in the space. In general, it leads to global classification of the basic interactions in the nature. Let O_1 and O_2 be the centers of the bodies and X is close to O_2 which belongs to the second body, such that $\overrightarrow{O_2X} = (a, b, c)$ (Fig. 2), and let τ be a translation for arbitrary small vector $\overrightarrow{O_2X} = (a, b, c)$. The operator rotor below should be done with respect to the coordinates a, b and c. We consider a solid body, such that the vector τ can not be constrained. Hence we have the following 4 cases (see also Table 1):

1^*. The electro-weak interaction is a consequence of non-commutativity between τ and one rotation in the space group G_s. The rotation is partially constrained in G_{s}, and as a consequence it appears an induced translation in G_{s}, which occurs as electro-weak interaction.

2^*. The strong interaction is a consequence of non-commutativity between τ and one translation in the space group G_s. The translation is partially constrained in G_{s} and as a consequence it appears an induced angle or rotation, which leads to displacement observed as acceleration.

3^*. The electromagnetic interaction is a consequence of non-commutativity between τ and one rotation in the affine group $\mathcal{A}$. The rotation is partially constrained in $\mathcal{A}$, and as a consequence it appears an induced translation in the affine group $\mathcal{A}$, and further it leads to electromagnetic interaction.

4^*. The gravitational interaction is a consequence of non-commutativity between τ and one "radial translation" in the affine group $\mathcal{A}$. The translation is partially constrained in $\mathcal{A}$, and as a consequence it appears an induced angle of rotation, which leads further to gravitational interaction.

Group of trans.	rotation	translation
G_{s}	electro-weak int.	strong int. & gravity-weak int.
$\mathcal{A}$	electromagnetic int.	gravitational interaction

Table 1: Global scheme of the basic interactions.

In all interactions, beside the acceleration $\mathbf{a}$ it appears also an angular velocity $\mathbf{w}$. In case of the electromagnetic and gravitational interaction these two 3-dimensional vectors are parts of an antisymmetric tensor field among the inertial systems, which is analogous to the tensor of the electromagnetic field. So, these two interactions are called *temporal interactions.* In case of weak and strong interactions, we have again two vector fields $\mathbf{a}$ and $v_0\mathbf{w}$ instead of $c\mathbf{w}$, where v_0 is a local parameter analogous to the radius of range R. Indeed, the velocity c is characteristic only for electromagnetic and gravitational interactions. Instead of an antisymmetric tensor, in case of weak and strong interaction we have the following theorem ([15]):

Theorem 1. *In case of weak and strong interaction, i.e. in the Lie group* G_{s}*, it holds* $\mathbf{a} = v_0\mathbf{w}$ *or* $\mathbf{a} = -v_0\mathbf{w}$.

So, these two interactions (weak and strong) are called *spatial interactions*.

We will give a short presentations of the interactions classified in the following way: i) strong interaction, ii) electromagnetic and electro-weak interaction, and iii) gravitational and gravity-weak interaction. The following results were obtained in [15] and some of them also in [14].

3.1 Strong interaction

We need to present the observation from the center of arbitrary particle with radius of range R. In [15] is shown that in case of the polar coordinates only the distance r is changed and it is observed as $R\sin\frac{r}{R}$. Moreover, the metric is given by

$$(ds)^2=\Big(\cos\frac{r}{R}\Big)^2(dr)^2+\Big(\frac{R}{r}\sin\frac{r}{R}\Big)^2r^2[(d\phi)^2+\sin^2\phi(d\theta)^2]=$$
$$=R^2\Big[\Big(d(\sin\frac{r}{R})\Big)^2+\Big(\sin\frac{r}{R}\Big)^2((d\phi)^2+\sin^2\phi(d\theta)^2)\Big]. \tag{4}$$

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Figure 2: The strong interaction is a consequence of non-commutativity of translations for the vectors $\mathbf{r}$ and (a,b,c) in $S\times SR$.

Let us consider two nucleons with centers at O_1 and O_2 with radii of range R_1 and R_2 respectively, and $\mathbf{r}=\overrightarrow{O_1O_2}$ (Fig.2). The non-commutativity of the translations obtains by the angle of two translations: translation for vector $\overrightarrow{O_1X}$ observed by O_1, and then translation by vector $-\mathbf{r}$ observed by the point X, or almost the same by O_2. In G_s the angle as a result of two translations is analogous to rotation as a consequence of two rotations, i.e. the vector product of two vectors and we use the coefficient 1/2 analogous to the Thomas precession. The endpoint of translation is a point Y which is close to O_1, where almost there is no rotation between O_1 and Y. We use the notations $k_1=\frac{R_1}{r}\sin\frac{r}{R_1}$, $k_2=\frac{R_2}{r}\sin\frac{r}{R_2}$, and $k_1^*=\cos\frac{r}{R_1}$. Without loss of generality we assume that the vector $\mathbf{r}$ is parallel to the z-axis, i.e. $x=y=0$, and as a consequence of the metric, the vector $\overrightarrow{O_1X}$ is observed from O_1 as $k_1\mathbf{r}+(k_1a,k_1b,k_1^*c)$, while the vector $-\mathbf{r}$ from X is observed as $-k_2\mathbf{r}$. The normalization should be done with respect to the distance r. Using the form of matrices given in section 1, the angle φ is given by

$$\vec{\varphi}=\frac{-1}{2}\Big[\frac{k_1\mathbf{r}+(k_1a,k_1b,k_1^*c)}{r}\times\frac{-k_2\mathbf{r}}{r}\Big]=-k_1k_2\frac{\mathbf{r}\times(a,b,c)}{2r^2}.$$

Further we obtain

$$\mathrm{rot}\vec{\varphi}=-k_1k_2\frac{\mathbf{r}}{r^2},$$
$$\frac{1}{2}\mathrm{rot}\vec{\varphi}=-\frac{1}{2}\sin\frac{r}{R_1}\sin\frac{r}{R_2}\frac{R_1R_2}{r^2}\frac{\mathbf{r}}{r^2}. \tag{5}$$

Assume that the rotation is not admitted. The relative acceleration between the two bodies is given by

$$\mathbf{a}_{\rm rel} = -\frac{v_0^2}{2}\sin\frac{r}{R_1}\sin\frac{r}{R_2}\frac{R_1R_2}{r^2}\frac{\mathbf{r}}{r^2}. \tag{6}$$

Let us denote by $\mathbf{a}_1$ and $\mathbf{a}_2$ the accelerations of the first and the second body, then

$$\mathbf{a}_{\rm rel} = \mathbf{a}_2 - \mathbf{a}_1, \quad \mathbf{a}_1 = -\frac{m_2}{m_1+m_2}\mathbf{a}_{\rm rel}, \quad \mathbf{a}_2 = \frac{m_1}{m_1+m_2}\mathbf{a}_{\rm rel}.$$

The forces toward the first and toward the second body are opposite

$$\mathbf{f}_1 = m_1\mathbf{a}_1 = -\frac{m_1m_2}{m_1+m_2}\mathbf{a}_{\rm rel}, \quad \mathbf{f}_2 = m_2\mathbf{a}_2 = \frac{m_1m_2}{m_1+m_2}\mathbf{a}_{\rm rel}.$$

If the space displacement is not admitted. Then it appears a relative rotation of the two bodies which is given by

$$\mathbf{w}_{\rm rel} = -\frac{v_0}{2}\sin\frac{r}{R_1}\sin\frac{r}{R_2}\frac{R_1R_2}{r^2}\frac{\mathbf{r}}{r^2}. \tag{7}$$

Moreover the two bodies obtain opposite angular momentums

$$\mathbf{L}_1 = I_1\mathbf{w}_1 = -\frac{I_1I_2}{I_1+I_2}\mathbf{w}_{\rm rel}, \quad \mathbf{L}_2 = I_2\mathbf{w}_2 = \frac{I_1I_2}{I_1+I_2}\mathbf{w}_{\rm rel},$$

where I_1 and I_2 are the moments of inertia of the two bodies.

The previous formulas can be applied more generally, for example the first body can be a galaxy with radius of range R_1. If the second body is any star from the galaxy, then we put $R_2 = \infty$ and the previous formulas can be applied. As a consequence it is obtained ([15]) that it is not necessary to introduce dark matter, because the unknown force is just the strong force toward the center of the galaxy. The radius of range for the Milky Way is 17 kpc, which is twice longer than the distance from the Sun to the center of the galaxy. In case of the nucleons, the radius of range is about 1.41 fm.

3.2 Electromagnetic and electro-weak interaction

Let us consider two charged bodies with charges e_1 and e_2 and centers at O_1 and O_2, and let the second body has mass m_2. In the papers [14, 15] it is axiomatically assumed that the first body rotates the second body for angle

$$\vec{\theta} = \frac{e_1e_2}{4\pi\epsilon_0 r^2 m_2 c^2}\mathbf{r}, \tag{8}$$

around the radial direction $\mathbf{r} = (x, y, z) = \overrightarrow{O_1O_2}$. The angle θ has a physical interpretation as a potential, similar to the gravitational potential. Let τ be a translation for a small vector (a, b, c). We choose the coordinate system such that the angle of rotation is $(0, 0, \theta)$, i.e $x = y = 0$. Then the non-commutativity between the rotation for angle $\vec{\theta}$ and translation for vector (a,b,c) in the group $\mathcal{A}$ leads to translation of the point O_2 for vector $\overrightarrow{O_2O_2'} = (a(\cos\theta-1)-b\sin\theta, a\sin\theta+b(\cos\theta-1),0)$. Indeed, it obtains by the following procedure: First rotation for angle θ, then translation τ, then rotation for angle $-\theta$ and then translation τ. This translation leads to the angle

$$\overrightarrow{O_2O_1O_2'} = (a(\cos\theta-1)-b\sin\theta, a\sin\theta+b(\cos\theta-1),0)/r.$$

The unadmitted translation leads to the Coulomb's acceleration/force

$$\mathbf{a} = \frac{c^2}{2}\mathrm{rot}\,\Big(\frac{1}{r}(a(\cos\theta-1)-b\sin\theta, a\sin\theta+b(\cos\theta-1),0)\Big) = (\sin\theta)c^2\frac{\mathbf{r}}{r^2},$$

$$\mathbf{a} = \frac{e_1e_2}{4\pi\epsilon_0 r^3 m_2}\mathbf{r}, \quad \mathbf{f} = \frac{e_1e_2}{4\pi\epsilon_0 r^3}\mathbf{r}, \tag{9}$$

and so the electric field caused by the first charged body is

$$\mathbf{E} = \frac{e_1}{4\pi\epsilon_0 r^3}\mathbf{r}. \tag{10}$$

The induced angular velocities also appear.

Further let obtain the electro-weak interaction. Let us consider two charged particles with centers at O_1 and O_2, radii of range R_1 and R_2 and the coefficients k_1 and k_1^* have the same meaning as previously, and let (a,b,c) be a small vector of translation. Only the charged particles cause rotation. The rotations remain unchanged in both cases as in case of electromagnetic interaction, but there is change in the vector (a,b,c). Without loss of generality assume that the vector $\mathbf{r}$ is parallel to the z-axis, i.e. $x=y=0$. Then the vector (a,b,c) from the first particle is observed as (k_1a, k_1b, k_1^*c). Although the basic group is G_s, the calculations are analogous as in $\mathcal{A}$. In this case we have translation for vector

$$\overrightarrow{O_2O_2'} = (k_1a(\cos\theta-1)-k_1b\sin\theta, k_1a\sin\theta+k_1b(\cos\theta-1),0)$$

and it corresponds to angle

$$\angle\overrightarrow{O_2O_1O_2'} = (k_1a(\cos\theta-1)-k_1b\sin\theta, k_1a\sin\theta k_1b(\cos\theta-1),0)/r.$$

The unadmitted translation leads to the acceleration/force of the second body toward the first body

$$\mathbf{a} = \frac{v_0^2}{2}\mathrm{rot}\,\Big(\frac{1}{r}(k_1a(\cos\theta-1)-k_1b\sin\theta, k_1a\sin\theta+k_1b(\cos\theta-1),0)\Big),$$

$$\mathbf{a} = \frac{v_0^2}{c^2}\Big(\frac{R_1}{r}\sin\frac{r}{R_1}\Big)\frac{e_1e_2}{4\pi\epsilon_0 r^3 m_2}\mathbf{r}, \tag{11}$$

$$\mathbf{f} = \frac{v_0^2}{c^2}\Big(\frac{R_1}{r}\sin\frac{r}{R_1}\Big)\frac{e_1e_2}{4\pi\epsilon_0 r^3}\mathbf{r}. \tag{12}$$

Symmetrically, the force of the first charged body toward the second charged body is given by

$$\mathbf{f} = -\frac{v_0^2}{c^2}\Big(\frac{R_2}{r}\sin\frac{r}{R_2}\Big)\frac{e_1 e_2}{4\pi\epsilon_0 r^3}\mathbf{r}. \tag{13}$$

If $R_1 \neq R_2$, then these two forces are not opposite, and the symmetry is broken now. In a special case, when the mutual distance r between the two charged bodies is very close to 0 and v_0 is close to c, i.e. in case of high energies, then the weak interaction leads to the electromagnetic interaction.

Analogously to the strong interaction, the following angular velocity appears now

$$\mathbf{w} = \frac{v_0}{c^2}\Big(\frac{R_1}{r}\sin\frac{r}{R_1}\Big)\frac{e_1 e_2}{4\pi\epsilon_0 r^3 m_2}\mathbf{r}. \tag{14}$$

The electro-weak and electromagnetic interaction can not occur simultaneously, but when the distance between the particles increases, the electro-weak interaction transforms into electromagnetic interaction.

3.3 Gravitational and gravity-weak interaction.

While the charged bodies mutually rotate for an angle θ determined by the axis of their centers, in case of gravitation we have radial translation from the center of the gravitational body. In case of gravitation, the translation refers to all points, while in case of charged bodies both bodies must be charged, i.e. the un-charged bodies are not rotated. This axiomatic gravitational translation has magnitude MG/c^2, where M is the mass of the gravitational body.

This gravitational translation combines with translation for a small vector $\tau = (a, b, c)$, and the non-commutativity leads to the gravitational acceleration. Assume that a point X has a radius vector $\mathbf{r} = (x, y, z)$, while the point-mass is at $(0, 0, 0)$. If we apply first translation for vector τ and then gravitational translation, we obtain

$$X(x,y,z) \to (x+a, y+b, z+c) \to Y\Big((x+a, y+b, z+c)\Big(1 + \frac{GM}{r'c^2}\Big)\Big),$$

where $r' = r$ if we neglect the small lengths a, b, c. If we apply gravitational translation and then translation for vector τ, we obtain

$$X(x,y,z) \to (x,y,z)(1 + \frac{GM}{rc^2}) \to Y'\Big((x,y,z)\Big(1 + \frac{GM}{rc^2}\Big) + (a,b,c)\Big).$$

The non-commutativity of both translations gives an oriented angle

$$\overrightarrow{\angle YOY'} = \frac{\overrightarrow{OY} \times \overrightarrow{OY'}}{|OY| \cdot |OY'|} = -\frac{GM}{rc^2}\frac{(yc - zb, za - xc, xb - ya)}{r(r + GM/c^2)}.$$

Half of this angle is not admitted and it induces acceleration given by

$$\mathbf{g} = \frac{c^2}{2}\mathrm{rot}\overrightarrow{\angle BOB'} = -\mathbf{r}\frac{GM}{r^2(r + GM/c^2)} \approx -\mathbf{r}\frac{GM}{r^3}. \tag{15}$$

The other half which is admitted induces the known precessions in gravitational field for moving test body.

Analogously to the electro-weak, we have also gravity-weak interaction. It is close to gravitation via the group $\mathcal{A}$, and the observer will be a particle with radius of range R_1. Without loss of generality assume that the z-axis is parallel to the vector $\mathbf{r} = (x, y, z)$. Then the coordinates x, y, z, a and b should be multiplied by $k_1 = \frac{R_1}{r}\sin\frac{r}{R_1}$, while c should be multiplied by $k_1^* = \cos\frac{r}{R_1}$. It should be replaced into the coordinates of Y and Y', while the vectors $\overrightarrow{OY}$ and $\overrightarrow{OY'}$ should be divided by $(r + \frac{GM}{c^2})$. The calculation shows that the required acceleration is

$$\mathbf{g} = \frac{v_0^2}{2}\mathrm{rot}\angle\overrightarrow{YOY'} = -\frac{v_0^2}{c^2}\Big(\frac{R_1}{r}\sin\frac{r}{R_1}\Big)^2\frac{GM}{r^2(r+GM/c^2)}\mathbf{r}. \tag{16}$$

Analogously to the strong interaction, the we have the following angular velocity

$$\mathbf{w} = -\frac{v_0}{c^2}\Big(\frac{R_1}{r}\sin\frac{r}{R_1}\Big)^2\frac{GM}{r^2(r+GM/c^2)}\mathbf{r} \approx -\frac{v_0}{c^2}\Big(\frac{R_1}{r}\sin\frac{r}{R_1}\Big)^2\frac{GM}{r^3}\mathbf{r}. \tag{17}$$

In case of gravity-weak interaction the symmetry is also broken. The gravity-weak interaction is much smaller than the gravitational interaction and it is unknown.

The gravity-weak and gravitational interaction can not occur simultaneously, but when the distance between the particles increases, the gravity-weak interaction transforms into gravitational interaction.

4 Some results and comments

It is interesting that R.Boscovich in his ref. [1] considered also 4 basic cases between the space and time which are related to one point and analogous to them also 4 cases which are related for several points. He also comments which combinations, i.e. compositions among these cases are possible and which are not possible. There is an interesting analogy between his comments and the previous results.

The strong, weak and electromagnetic interactions are studied in the Standard Model. It is based on the Klein-Gordon equation and Dirac equation, from the Relativistic quantum theory [17]. While the non-relativistic theory starts from the formula $E = p^2/(2m)$, the Klein-Gordon equations starts from the corresponding relativistic formula $E^2 = p^2c^2 + m^2c^4$. The best experimental results in this direction are obtained by the Quantum Electrodynamics. Probably the reason is that the electromagnetic interactions belongs to temporal interactions, and the relativistic approach is convenient. According to this viewpoint the strong and weak interactions are not convenient to be researched by the same equation. For example, we can start from ([15]) $\mathbf{a} = \pm v_0\mathbf{w}$ (Theorem 1). Hence it follows that $a^2 = v_0^2w^2$ and by multiplication with m^2d^2 where m is the mass of the particle and d is a distance, we obtain $E^2 = L^2w^2$ where

$L = mdv_0$ is the angular momentum. The equation is $(E - Lw)(E + Lw) = 0$, and the sign $\pm$ in eq. $E = \pm Lw$ depends of the sign of the spin of the particle. It is analogous to the energy of the photon $E = \hbar w = h\nu$. Using wave theory the standard physical theory gives much attention of the interaction between the photon and the matter, and there Quantum Electrodynamics gives the best results. The results in section 3 are complementary, because in this paper we start from geometry and ignore the quantum and wave theory. The most important assumption is that we use the elementary particles as solid bodies, and the introduced small vector (a, b, c) can not be constrained. So we may conclude that if we consider the particles from wave theory, known theory should be applied, while if we consider as solid body, this model gives description of the interactions. However, the particles have dual nature.

To the end of this paper we consider some consequences from the previous results. Indeed, we start from the Lorentz invariance of the interactions. First we given some preliminaries which come from the metric in the 3-dimensional time. In the 3-dimensional time we distinguish two cases: i) the metric in the 6-dimensional space-time in $\mathbf{r}_s \times \mathbf{r}_t$, where the motion with velocity interpretes simply by rotating for an imaginary angle [3, 2], and ii) in case of active motion. In case i) the distance from a moving inertial coordinate system observes the length

$$|\Delta \mathbf{r}_s| = |\Delta \mathbf{r}| \sqrt{1 + \frac{\frac{v^2}{c^2} \sin^2 \psi}{1 - \frac{v^2}{c^2}}}, \tag{18}$$

where ψ is the angle between $\mathbf{v}$ and $\Delta \mathbf{r}$. In case of active motion, all lengths, i.e. in each directions should be multiplied by $\sqrt{1 - v^2/c^2}$. The last step (active motion) is a consequence since the time in moving systems changes, which is a subjective observation. Note that in case of both cases i) and ii) we obtain the same observation known from the Special Relativity. But since the active motion includes a subjectivity, we will use only the case i). According to this observation, each length in the direction of motion ($\psi = 0$) remains unchanged, while each direction which is orthogonal to the direction of motion ($\psi = \pi/2$), observes that the length is enlarged for coefficient $1/\sqrt{1 - \frac{v^2}{c^2}}$.

Now let us return to the Lorentz invariance. In case of the strong and weak interactions, the Lorentz covariance need not to be considered because the interactions are inside the group G_s. Let us start with the gravitational interaction. Assume that the observer is moving with velocity v with respect to the system in which the mass is in rest. In this case it is convenient to choose motion in a direction, such that the mass and the point which is translated for vector GM/c^2 are simultaneous. It occurs when the velocity is orthogonal to these two points. The Lorentz invariance means that the coefficients $\frac{GM}{c^2} : r$ must be preserved for the local observer and for the moving observer. It means that

$$\frac{GM}{c^2} : r = \frac{GM'}{c^2} : r'$$

where the mass M and the distance r are observed as M' and r' according to the moving observer. Since $r' = r/\sqrt{1-\frac{v^2}{c^2}}$, we obtain that $M' = M/\sqrt{1-\frac{v^2}{c^2}}$. This is known result, but accepted ad hoc without proof. It is suggested since the kinetic energy $mv^2/2$ can be written more precisely as $m(1/\sqrt{1-\frac{v^2}{c^2}}-1)$. Now, when we have a precise definition of the mass, we have also a precise proof.

Future let us consider two charged bodies with charges e_1 and e_2 which mutually rest on a distance r, and let the mass in rest of the second body be m_2. Then the second body is rotated for an angle (8). Let us choose an observer who moves with velocity v. Since the rotation is in the plane which is orthogonal to the axis which connects the two bodies, the observer should move in this direction. Otherwise he observes the angle as a part of ellipse. Let he observes the charges as e_1' and e_2'. Since the angle θ must be the same for both observers, we obtain

$$\frac{e_1e_2}{4\pi\epsilon_0 r m_2 c^2} = \frac{e_1'e_2'}{4\pi\epsilon_0 r' m_2 c^2}.$$

Since $r' = r$ according to (18), we obtain $e_1e_2 = e_1'e_2'$. If $e_1' = ke_1$, and hence also $e_2' = ke_2$, we obtain $k^2 = 1$, and hence $k = 1$ because $k > 0$. Thus we come to the known conclusion that the charge is observed unchanged in all inertial coordinate systems. It is accepted ad hoc without proof, according to the experiments.

Theorem 2. *The mass is observed enlarged for coefficient* $1/\sqrt{1-\frac{v^2}{c^2}}$ *while the charges is observed unchanged according to the observer who moves with velocity* $\mathbf{v}$.

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Cantor's Intersection Theorem in $(3, 1, \nabla)$-G-metrizable spaces

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Abstract
In this article we prove some properties of $(3, 1, \nabla)$-G-metrizable spaces and establish analogous Cantor's intersection theorem for those kinds of spaces.

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Keywords.$(3, 1, \nabla)$-G-metrizable space, Cantor's intersection theorem

1 Introduction

In metric spaces the Cantor's intersection theorem is well known fact and characterizes the metric completeness. It states that for an arbitrary complete metric space X, a sequence of nonempty, nested and closed subsets of X whose diameters tend to 0, has a single point intersection, and vice versa. Different type of generalisations have been obtained considering generalization of the spaces or the notion of decreasing sets.

Generalizations of metric spaces have been considered in lot of papers by many authors: Menger [14], Aleksandrov, Nemytskii [1], Mamuzić [15], Gähler [12], Nedev, Choban [18, 19, 20], Kopperman [13], Dhage, Mustafa, Sims [5, 16]. The notion of an (n, m, ρ)-metric, $n > m$, generalizing the usual notion of a pseudometric (the case $n = 2, m = 1$), and the notion of an $(n + 1)$-metric (as in [14] and [12]) was introduced in [6]. Connections between some of the topologies induced by a $(3, 1, \rho)$-metric and topologies induced by a pseudo-o-metric, o-metric and symmetric (as in [19]), are given in [7]. Some other characterizations of $(3, j, \rho)$-metrizable topological spaces, $j \in \{1, 2\}$, are given in [3, 4, 8, 9].

In this article we will consider only $(3, 1, \nabla)$-metrics, i.e. $(3, 1, \rho)$-metrics where $\rho = \nabla = \{(x, x, y) | x, y \in M\}$. The concept of subbasis that forms a topological space has also been considered by Gähler in [12]. Similarly we define the topology $\tau(G, d)$ for a $(3, 1, \rho)$-metric d generated by the subbasis of all ϵ-balls with center at (x, y), defined as in [3]. Here we gained some properties of $(3, 1, \nabla)$-G-metrizable spaces which combined with some assumptions enabled us to prove a variant of Cantor's intersection theorem in these kinds of spaces.

2 Preliminaries

We use basic definitions for $(3,1,\rho)$-metric spaces and $(3,1)$-metric spaces, as in [3].
Let $M \neq \varnothing$ and $M^{(3)} = M^3/\alpha$, where α is the equivalence relation on M^3 defined by:

$$(x,y,z)\alpha(u,v,w) \Leftrightarrow \pi(u,v,w) = (x,y,z),$$

where π is a permutation.

Note 2.1. *We will use the same notation (x,y,z) for the class in $M^{(3)}$ containing the triplet (x,y,z).*

Let ρ be a subset of $M^{(3)}$. We consider the following conditions for such set.

$(E0)$ $(x,x,x) \in \rho$, for any $x \in M$; and

$(E1)$ $(a,y,z),(x,a,z),(x,y,a) \in \rho \implies (x,y,z) \in \rho$, for any $x,y,z,a, \in M$.

Definition 2.2. *If ρ satisfies $(E0)$ and $(E1)$ we say that ρ is a $(3,1)$-equivalence.*

We give the following trivial examples.

Example 2.3. *The set $\Delta = \{(x,x,x)|x \in M\}$ is a $(3,1)$-equivalence on M.*

Example 2.4. *The set $\nabla = \{(x,x,y)|x,y \in M\}$ is a $(3,1)$-equivalence on M.*

Let $d : M^{(3)} \to \mathbb{R}_0^+$. We consider the following conditions for such a map.

$(M0)$ $d(x,x,x) = 0$, for any $x \in M$; and

$(M1)$ $d(x,y,z) \leq d(a,x,z) + d(x,a,z) + d(x,y,a)$, for any $x,y,z,a \in M$.

Lemma 2.5. *Let $d : M^{(3)} \to \mathbb{R}_0^+$ and $\rho_d = \{(x,y,z) \in M^{(3)}|d(x,y,z) = 0\}$. If d satisfies $(M0)$ and $(M1)$, then ρ_d is a $(3,1)$-equivalence.*

Proof. It follows directly from the previous definition. □

Definition 2.6. *Let $d : M^{(3)} \to \mathbb{R}_0^+$ and $\rho = \rho_d$.*
i) If d satisfies $(M0)$ and $(M1)$ we say that d is a $(3,1,\rho)$-metric on M, and the pair (M,d) is said to be a $(3,1,\rho)$-metric space.
ii) If d is a $(3,1,\Delta)$-metric on M, we say that d is a $(3,1)$-metric on M, and the pair (M,d) is said to be a $(3,1)$-metric space.

Again we state certain examples.

Example 2.7. *Let M be a nonempty set. The map $d : M^{(3)} \to \mathbb{R}_0^+$ defined by:*

$$d(x,y,z) = \begin{cases} 0 & ,(x,y,z) \in \Delta \\ 1 & ,\textit{otherwise} \end{cases}$$

is a $(3,1)$-metric on M (the discrete 3-metric).

Proof. Follows directly from Definition 2.6. $\square$

Example 2.8. *Let $D : M^2 \to \mathbb{R}_0^+$ be a metric on M. The map $d : M^{(3)} \to \mathbb{R}_0^+$ defined by:*

$$d(x, y, z) = \frac{D(x, y) + D(x, z) + D(y, z)}{2},$$

is a $(3, 1)$-metric on M.

Proof. Follows directly from Definition 2.6. $\square$

2.1 Topological framework

In this subsection we define a suitable framework in which we are considering our results. We define all the necessary topological notions in the sequel.

Definition 2.9. *Let (M, d) be a $(3, 1, \rho)$ metric space and $A \subseteq M, A \neq \emptyset$. We say that A is bounded if there is an $M > 0$ such that $d(x, y, z) \leq M$, for all $x, y, z \in M$.*
If A is bounded, we define the diameter of A as

$$\operatorname{diam} A = \sup\{d(x, y, z) | x, y, z \in M\}.$$

If A is not bounded, we write $\operatorname{diam} A = \infty$.

Definition 2.10. *We say that a sequence $(x_n)_{n=1}^{\infty}$ in a $(3, 1, \rho)$-metric space (M, d) is G-convergent if there is an $x \in M$ such that $d(x_n, x, y) \to 0$ as $n \to \infty$ for each $y \in M$.*

For simplicity we use the notation $x_n \to x$ as $n \to \infty$ for G-convergence of the sequence $(x_n)_{n=1}^{\infty}$.

Definition 2.11. *We say that a sequence $(x_n)_{n=1}^{\infty}$ in a $(3, 1, \rho)$-metric space (M, d) is G-Cauchy if $d(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.*

Definition 2.12. *We say that a $(3, 1, \rho)$-metric space (M, d) is G-complete if each G-Cauchy sequence is G-convergent (with respect to d).*

Definition 2.13. *Let d be a $(3, 1, \rho)$-metric on M, $x, y \in M$ and $\epsilon > 0$. We define the ϵ-ball with center at (x, y) and radius ϵ to be the set*

$$B(x, y, \epsilon) = \{z | z \in M, d(x, y, z) < \epsilon\}.$$

Definition 2.14. *For a $(3, 1, \rho)$-metric d on M, we define the topology $\tau(G, d)$ on M to be the topology generated by all ϵ-balls $B(x, y, \epsilon)$ for all $x, y \in M$, i.e. the topology whose base is the set of all finite intersections of all ϵ-balls $B(x, y, \epsilon)$ for all $x, y \in M$.*

Definition 2.15. *We say that a topological space (M, τ) is $(3, 1, \rho)$-G-metrizable if there is a $(3, 1, \rho)$-metric d on M such that $\tau = \tau(G, d)$.*

Definition 2.16. *Let M be a set and $\mathcal{M} = \mathcal{P}(M)$ be the power set of M. We say that a sequence $(F_n)_{n=1}^{\infty}$ of subsets of M is decreasing (in $\mathcal{M}$) if $F_{n+1} \subseteq F_n$, for each $n \in \mathbb{N}$.*

Lemma 2.17. *Let (M, d) be a $(3, 1, \rho)$-metric space and $x, y \in M$ are fixed. If $z_n \to z$ as $n \to \infty$, then $d(z_n, x, y) \to d(z, x, y)$ as $n \to \infty$.*

Proof. Let $z_n \to z$ as $n \to \infty$ and $x, y \in M$. Taking $(M1)$ in to account one obtains

$$d(z_n, x, y) \leq d(z, x, y) + d(z_n, z, y) + d(z_n, x, z),$$

i.e.

$$d(z_n, x, y) - d(z, x, y) \leq d(z_n, z, y) + d(z_n, x, z), \tag{1}$$

for arbitrary $n \in \mathbb{N}$. Interchanging z_n with z implies

$$d(z, x, y) - d(z_n, x, y) \leq d(z, z_n, y) + d(z, x, z_n). \tag{2}$$

Combined inequalities (1) and (2) impliy

$$|d(z_n, x, y) - d(z, x, y)| \leq d(z_n, z, y) + d(z_n, x, z).$$

The convergence of the sequence $(z_n)_{n=1}^{\infty}$ implies that the right-hand side tends to 0 when $n \to \infty$. Therefore, the proof is completed. □

Lemma 2.18. [4] Let (M, τ) be a $(3, 1, \nabla)$-metrizable space, via $(3, 1, \nabla)$-metric d. A subset U from M is open iff for any $x \in U$ there are finite number of points $a_1, a_2, ..., a_n \in M$ and $\epsilon_1, \epsilon_2, ..., \epsilon_n > 0$ such that $x \in \cap_{n=1}^{\infty} B(x, a_i, \epsilon_i) \subseteq U$.

Lemma 2.19. *Let (M, τ) be a $(3, 1, \nabla)$-metrizable space, via $(3, 1, \nabla)$-metric d. A sequence $(x_n)_{n=1}^{\infty}$ G-converges to $x \in M$ iff for any $U \in \tau$ such that $x \in U$, there exists an $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$.*

Proof. Follows directly from the previous lemma. □

3 Main results

We impose additional conditions on the set M.
Let (M, τ) be a $(3, 1, \nabla)$-metrizable space, via $(3, 1, \nabla)$-metric d, satisfying the conditions:

(1) For $x, y \in M, x \neq y$ there is a sequence $(z_n)_{n=1}^{\infty}$ in $M \setminus \{x, y\}$ and $z \in M \setminus \{x, y\}$ such that $z_n \to z$ and $d(z_n, x, y) \to 0$ as $n \to \infty$,

(2) If there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ from G-Cauchy sequence $(x_n)_{n=1}^{\infty}$ such that $x_{n_k} \to x$ as $k \to \infty$, then $x_n \to x$ as $n \to \infty$.

Our first concern is the existence of nontrivial example of such space (M, τ). The condition (1) seems really restrictive. Nerveless, such spaces exist. Our example of this type of space is geometrically motivated, so the details are left for an interested reader.

Example 3.1. *Let M be the set consisted of all triples (x, y, z) of not collinear points in $\mathbb{R}^2$ and all the triples (x, y, z) of points in $\mathbb{R}^2$ such that at least two of them are the same. Define $d(x, y, z)$ by the area of the triangle in $\mathbb{R}^2$ with vertices x, y, z. It is not difficult to confirm that $(M0), (M1)$, together with (1) and (2) are satisfied.*

Lemma 3.2. *Let $A \subseteq M$. Then* $\mathrm{diam} A = \mathrm{diam} \overline{A}$.

Proof. It is obvious that $\mathrm{diam} A \leq \mathrm{diam} \overline{A}$.
Let $x, y, z \in \overline{A}$. We consider the following cases.

1^0 If $x, y, z \in A$, then $d(x, y, z) \leq \mathrm{diam} A$.

2^0 If $x \in \overline{A} \setminus A$ and $y, z \in A$, then for each $\epsilon > 0$ there exists $u \in A \cap B(x, y, \epsilon) \cap B(x, z, \epsilon)$. Then

$$\begin{aligned} d(x, y, z) &\leq d(u, y, z) + d(x, u, z) + d(x, y, u) \\ &< 2\epsilon + \mathrm{diam} A. \end{aligned}$$

3^0 If $x, y \in \overline{A} \setminus A$ and $z \in A$, then for each $\epsilon > 0$ there exist u, v such that $u \in A \cap B(x, y, \epsilon) \cap B(x, z, \epsilon)$ and $v \in A \cap B(y, u, \epsilon) \cap B(y, z, \epsilon)$. Then

$$\begin{aligned} d(x, y, z) &\leq d(u, y, z) + d(x, u, z) + d(x, y, u) \\ &< d(u, y, z) + 2\epsilon \\ &\leq d(v, y, z) + d(u, v, z) + d(u, y, v) + 2\epsilon \\ &< 4\epsilon + \mathrm{diam} A. \end{aligned}$$

4^0 If $x, y, z \in \overline{A} \setminus A$, then for each $\epsilon > 0$ there exist u, v, t such that $u \in A \cap B(x, y, \epsilon) \cap B(x, z, \epsilon)$, $v \in A \cap B(y, u, \epsilon) \cap B(y, z, \epsilon)$ and $t \in A \cap B(z, u, \epsilon) \cap B(z, v, \epsilon)$.Then

$$\begin{aligned} d(x, y, z) &\leq d(u, y, z) + d(x, u, z) + d(x, y, u) \\ &< d(u, y, z) + 2\epsilon \\ &\leq d(v, y, z) + d(u, v, z) + d(u, y, v) + 2\epsilon \\ &< 4\epsilon + d(u, v, z) \\ &\leq 4\epsilon + d(t, v, z) + d(u, t, z) + d(u, v, t) \\ &< 6\epsilon + \mathrm{diam} A. \end{aligned}$$

Regardless of the cases, arbitrariness of ϵ implies the claim. □

Next will prove analogous theorem of Cantor's intersection theorem in $(3, 1, \nabla)$-G-metrizable spaces.
Let us consider another condition.

(C) For each decreasing sequence $(F_n)_{n=1}^{\infty}$ of closed subsets of M such that $\lim_{n\to\infty} \mathrm{diam} F_n = 0$ the set $\cap_{n=1}^{\infty} F_n$ consists of a single point.

Theorem 3.3. *If (M, τ) is G-complete, then the condition (C) is satisfied.*

Proof. For each $n \in \mathbb{N}$, let $x_n \in F_n$. Since $(F_n)_{n=1}^{\infty}$ is a decreasing sequence, for $m, l \geq n$ we have $x_m, x_l \in F_n$ and

$$d(x_n, x_m, x_l) \leq \mathrm{diam} F_n \to 0,$$

as $n \to \infty$. Thus, the sequence $(x_n)_{n=1}^{\infty}$ is a G-Cauchy sequence. This means that there is an $x \in M$ such that $x_n \to x$ as $n \to \infty$. We will prove that $x \in \cap_{n=1}^{\infty} F_n$.
Let n be fixed arbitrary positive integer and $U \in \tau$ such that $x \in U$. Then there is a $k_0 \in \mathbb{N}$ such that $x_k \in U$ for all $k \geq k_0$. Thus, $x_k \in U \cap F_n$ for all $k \geq \max\{n, k_0\}$, i.e. $x \in \overline{F_n} = F_n$ (F_n is closed). So, $x \in \cap_{n=1}^{\infty} F_n$. Let us suppose that there is a $y \in M \setminus \{x\}$ such that $y \in \cap_{n=1}^{\infty} F_n$. From the condition (1) it follows that for each $\epsilon > 0$, there are a sequence $(z_n)_{n=1}^{\infty}$ in $M \setminus \{x, y\}$, $z \in M \setminus \{x, y\}$ such that $z_n \to z$, and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$0 \leq d(x, y, z_n) < \epsilon.$$

Letting $n \to \infty$ we obtain that $d(x, y, z) = 0$, which is a contradiction since $x \neq y \neq z \neq x$. □

Theorem 3.4. *If (M, τ) satisfies the condition (C), then (M, τ) is G-complete.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a G-Cauchy sequence in M and for each $n \in \mathbb{N}$ set $F_n = \{x_n, x_{n+1}, ...\}$. Then the sequence $(F_n)_{n=1}^{\infty}$ is decreasing and moreover, $(\overline{F_n})_{n=1}^{\infty}$ is decreasing sequence of closed sets. For $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$d(x_m, x_n, x_l) < \epsilon$$

for each $m, n, l \geq n_0$, and lemma 3.2 infers $\mathrm{diam}\overline{F_{n_0}} = \mathrm{diam} F_{n_0} \leq \epsilon$. But then $\mathrm{diam}\overline{F_n} \leq \epsilon$ for each $n \geq n_0$ meaning that $\lim_{n\to\infty} \mathrm{diam}\overline{F_n} = 0$. So, there is $x \in M$ such that $\cap_{n=1}^{\infty} \overline{F_n} = \{x\}$.
Let $z \in M$ be arbitrary. For each $n \in \mathbb{N}$ there is $y_n \in B(z, x, \frac{1}{n}) \cap F_n$. Thus, $(y_n)_{n=1}^{\infty}$ is a subsequence of $(x_n)_{n=1}^{\infty}$ such that $d(y_n, x, z) \to 0$ as $n \to \infty$, i.e. $y_n \to x$ as $n \to \infty$. From condition (2) it follows that $x_n \to x$ as $n \to \infty$, i.e. (M, τ) is G-complete. □

From the previous two theorems we obtain the analogous Cantor's intersection theorem in these kinds of spaces.

Corollary 3.5. *Let (M, τ) be a $(3, 1, \nabla)$-metrizable space, via $(3, 1, \nabla)$-metric d satisfying the conditions (1) and (2). Then (M, τ) is G-complete iff it satisfies the condition (C).*

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N-TUPLE WEAK ORBITS TENDING TO INFINITY FOR HILBERT SPACE OPERATORS

Sonja Mančevska, Marija Orovčanec

Abstract . In this paper we prove some results on the existence of a dense set of pairs in the direct product of an infinite-dimensional complex Hilbert space with itself such that each pair in this set has an *n*-tuple weak orbit tending to infinity for a specific countable family of mutually commuting bounded linear operators.

1. INTRODUCTION

For bounded linear operators on Banach spaces the concepts of *n*-tuple orbits and *n*-tuple weak orbits are defined as follows. If X is a complex and infinite-dimensional Banach space, $B(X)$ is the algebra of all bounded linear operators on X and $T_1,T_2,\ldots,T_n \in B(X)$ are mutually commuting operators, then the *n-tuple orbit* of the vector $x \in X$ is the set

$$\mathrm{Orb}(\{T_i\}_{i=1}^{n},x)=\left\{T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x : k_i \geq 0; 1\leq i\leq n\right\}. \tag{1.1}$$

The *n*-tuple orbit *tends to infinity* if

$$\lim_{k_i\to\infty}\left\|T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x\right\|=\infty \text{, for all } k_j \geq 0 \text{, } j\neq i \text{, } 1\leq i,j\leq n \text{ .}$$

For $n=1$, the *n*-tuple orbit (1.1) reduces to a simple sequence of form

$$\mathrm{Orb}(T,x)=\left\{T^{n}x : n=0,1,2,\ldots\right\},$$

usually referred as *single orbit* (or simply *orbit*) of the vector $x\in X$ under the operator T . If X^* is the dual space of X , i.e., the space of all bounded linear functionals $x^*: X\to\mathbb{C}$, and for $x\in X$ and $x^*\in X^*$, $\left\langle x,x^*\right\rangle := x^*(x)$, the *n-tuple weak orbit* of the pair $(x,x^*)\in X\times X^*$ is a set of form

$$\mathrm{Orb}(\{T_i\}_{i=1}^{n},x,x^*)=\left\{\left\langle T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x,x^*\right\rangle : k_i\geq 0; 1\leq i\leq n\right\}. \tag{1.2}$$

The *n*-tuple weak orbit *tends to infinity* if

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$$\lim_{k_i\to\infty}\left|\left\langle T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x,x^*\right\rangle\right|=\infty\text{, for all } k_j\geq 0\text{ , } j\neq i\text{ , } 1\leq i,j\leq n\ .$$

For $n=1$, the *n*-tuple weak orbit (1.2) reduces to a simple scalar sequence of form

$$\mathrm{Orb}(T,x,x^*)=\left\{\left\langle T^n x,x^*\right\rangle : n=0,1,2,\dots\right\},$$

usually referred as *weak orbit* of the pair $(x,x^*)\in X\times X^*$ under the operator T .

For the case of Hilbert spaces, by the Riesz Theorem for representation of a bounded linear functional on Hilbert spaces (cf. [7,III.6]), given an infinite-dimensional complex Hilbert space H with an inner product $\langle\cdot|\cdot\rangle$, its dual space H^* can be fully identified with the space itself since

$$H^*=\left\{x\mapsto\left\langle x\middle|y\right\rangle, x\in H : y\in H\right\}.$$

Hence, for a set of mutually commuting operators $T_1,T_2,\dots,T_n\in B(H)$ the *n*-tuple weak orbits will be the sets of form

$$\mathrm{Orb}(\{T_i\}_{i=1}^n,x,y)=\left\{\left\langle T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x\middle|y\right\rangle : k_i\geq 0; 1\leq i\leq n\right\},\ (x,y)\in H\times H\ .$$

In this paper we will consider only the conditions under which the direct product $H\times H$ contains a dense of pairs (x,y) with *n*-tuple weak orbits tending to infinity that do not involve any requirements upon specific subsets of the spectra of the operators. For $H\times H$ we will assume that is endowed with the product topology. Given an operator $T\in B(H)$, $\sigma(T)$ and $r(T)$ will denote the spectrum and the spectral radius of the operator T , respectively.

2. PRELIMINARY RESULTS

Theorem 2.1. ([6, Theorem V.39.8]) *Let H and K be Hilbert spaces, $(T_n)_{n\geq 1}$ be a sequence of operators in $B(H,K)$ and $(a_n)_{n\geq 1}$ be sequence of positive numbers with $\sum_{n\geq 1}a_n<\infty$. Then*

(i) *there are $x\in H$ and $y\in K$ such that and $\left|\left\langle T_n x\middle|y\right\rangle\right|\geq a_n\left\|T_n\right\|$, for all n;*

(ii) *there is a dense subset of pairs $(x,y)\in H\times K$ such that $\left|\left\langle T_n x\middle|y\right\rangle\right|\geq a_n\left\|T_n\right\|$, for all n sufficiently large.*

Corollary 2.2. ([6, Corollary V.39.9]) *Let H be Hilbert space and $T\in B(H)$ is such that $\sum_{k=1}^{\infty}\left\|T^k\right\|^{-1}<\infty$. Then there exist $x,y\in H$ such that $\left|\left\langle T^n x\middle|y\right\rangle\right|\to\infty$. Moreover, the set of such pairs (x,y) is dense in $H\times H$.*

Lemma 2.3. ([6, Lemma V.37.15]) *Let $\varepsilon > 0$ and $(a_n)_{n\geq 1}$ be a sequence of positive numbers satisfying $\sum_{n\geq 1} a_n < \varepsilon$. Then there is a sequence of positive numbers $(b_n)_{n\geq 1}$ such that $b_n \to \infty$ as $n \to \infty$ and $\sum_{n\geq 1} a_n b_n < \varepsilon$.*

3. MAIN RESULTS

Let $F = \{1,2,\ldots,N\}$ for some $N \in \mathbb{N}$, $N \geq 2$, or $F = \mathbb{N}$.

Theorem 3.1. *Let H be a Hilbert space, $\{T_i : i \in F\} \subset B(H)$ and $\{(a_{i,j})_{j\geq 1} : i \in F\}$ be a family of sequences of positive numbers such that $\sum_{i\in F, j\geq 1} a_{i,j} < \infty$. Then for any open balls B_1 and B_2 in H there are vectors $x \in B_1$, $y \in B_2$ and $k_0 \in \mathbb{N}$ such that*

$$\left|\left\langle T_i^k x \middle| y \right\rangle\right| \geq a_{i,k} \left\|T_i^k\right\|, \text{ for all } i \in F \text{ and } k \geq k_0.$$

Proof. Let $T_{i,k} := T_i^k$ ($i \in F$, $k \in \mathbb{N}$), $f : F \times \mathbb{N} \to \mathbb{N}$ be the bijective mapping defined with

$$f(i,j) = \begin{cases} i + N(j-1), & \text{if } F = \{1,2,\ldots,N\} \\ \dfrac{(i+j-2)(i+j-1)}{2} + j, & \text{if } F = \mathbb{N} \end{cases},$$

and let $g : \mathbb{N} \to F \times \mathbb{N}$ denote its inverse mapping. If $(a'_n)_{n\geq 1}$ is a sequence of positive numbers and $(T'_n)_{n\geq 1}$ is a sequence of operators defined with

$$a'_n = a_{g(n)} \text{ and } T'_n = T_{g(n)}, \text{ for all } n \in \mathbb{N},$$

then $\sum_{n\geq 1} a'_n = \sum_{i\in F, j\geq 1} a_{i,j} < \infty$. Hence (by Theorem 2.1. (ii), applied on $(a'_n)_{n\geq 1}$, $(T'_n)_{n\geq 1}$ and $H = K$), if B_1 and B_2 are open balls H, then there are $x \in B_1$, $y \in B_2$ and $n_0 \in \mathbb{N}$ such that

$$\left|\left\langle T'_n x \middle| y \right\rangle\right| \geq a'_n \left\|T'_n\right\|, \text{ for all } n \geq n_0. \tag{3.1}$$

Since $f : F \times \mathbb{N} \to \mathbb{N}$ is bijective, there is a unique pair $(i_0, j_0) \in F \times \mathbb{N}$ such that $n_0 = f(i_0, j_0)$. Let

$$k_0 = \begin{cases} j_0 + 1, & \text{if } F = \{1,2,\ldots,N\} \\ i_0 + j_0, & \text{if } F = \mathbb{N} \end{cases}.$$

If $(i,k) \in F \times \mathbb{N}$ is such that $k \geq k_0$, then by the definition of $f : F \times \mathbb{N} \to \mathbb{N}$ we have:

1. for $F = \{1,2,\ldots,N\}$,

$$f(i,k) = i + N(k-1) \geq N(k_0 - 1) = Nj_0 = N + N(j_0 - 1)$$
$$\geq i_0 + N(j_0 - 1) = n_0 ,$$

2. for $F = \mathbb{N}$,
$$f(i,k) = \frac{(i+k-2)(i+k-1)}{2} + k \geq \frac{(i_0 + j_0 - 2)(i_0 + j_0 - 1)}{2} + j_0 = n_0.$$

Hence, by (3.1) and the definition of $(a'_n)_{n\geq 1}$ and $(T'_n)_{n\geq 1}$ we obtain
$$\left|\left\langle T_i^k x \middle| y \right\rangle\right| = \left|\left\langle T_{i,k} x \middle| y \right\rangle\right| = \left|\left\langle T_{g(n)} x \middle| y \right\rangle\right| \geq a_{g(n)} \left\|T_{g(n)}\right\| = a_{i,k} \left\|T_i^k\right\|,$$
for all $i \in F$ and $k \geq k_0$. ■

Theorem 3.2. *If H is Hilbert space and $\{T_i : i \in F\} \subset B(H)$ is a family of operators such that $\sum_{k=1}^{\infty} \left\|T_i^k\right\|^{-1} < \infty$, for all $i \in F$, then there is a dense set $D \subset H \times H$ such that the weak orbit $\left(\left\langle T_i^k x \middle| y \right\rangle\right)_{k\geq 0}$ tends to infinity for every pair $(x, y) \in D$ and every $i \in F$. If, in addition, $\{T_i : i \in F\}$ is a family of mutually commuting operators such that the sequence $(T_i^k - T_j^k)_{k\geq 1}$ is norm bounded for all $i, j \in F$, then for every $n \in F$ and $1 < m \leq n$, the $m-$tuple weak orbit*
$$\left\{\left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} \ldots T_{i_m}^{k_m} x \middle| y \right\rangle : k_i \geq 0; 1 \leq i \leq m\right\},$$
tends to infinity for all $1 \leq i_1 < i_2 < \ldots < i_m \leq n$.

Proof. Let B_1 and B_2 be open balls H. For $i \in F$, let $\varepsilon_i > 0$ be such that
$$\varepsilon_i \left(\sum_{k=1}^{\infty} \frac{1}{\left\|T_i^k\right\|} \right) < \frac{1}{2^{i+1}},$$
and (by Lemma 2.3) let $(b_{i,k})_{k\geq 1}$ be the sequence of positive numbers such that $b_{i,k} \to \infty$ as $k \to \infty$ and
$$\sum_{k=1}^{\infty} \frac{\varepsilon_i b_{i,k}}{\left\|T_i^k\right\|} < \frac{1}{2^{i+1}}. \tag{3.2}$$
If $a_{i,k} = \varepsilon_i b_{i,k} \left\|T_i^k\right\|^{-1}$, $(i,k) \in F \times \mathbb{N}$, then by (3.2) we have
$$\sum_{i\in F, k\geq 1} a_{i,k} = \sum_{i\in F} \sum_{k=1}^{\infty} \frac{\varepsilon_i b_{i,k}}{\left\|T_i^k\right\|} < \sum_{i\in F} \frac{1}{2^{i+1}} < \frac{1}{2}.$$
Hence, by Theorem 3.1, there are $x \in B_1$, $y \in B_2$ and $k_0 \in \mathbb{N}$ such that

$$\left|\left\langle T_i^k x \middle| y\right\rangle\right| \ge a_{i,k}\left\|T_i^k\right\| = \varepsilon_i b_{i,k}\left\|T_i^k\right\|^{-1}\left\|T_i^k\right\| = \varepsilon_i b_{i,k}\text{, for all } i \in F \text{ and } k \ge k_0 .$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty}\left|\left\langle T_i^k x \middle| y\right\rangle\right| = \infty \text{, for all } i \in F . \tag{3.3}$$

If, in addition, $\{T_i : i \in F\}$ is a family of mutually commuting operators such that the sequence $(T_i^k - T_j^k)_{k\ge 1}$ is norm bounded for all $i, j \in F$, let $M_{i,j} > 0$ is such that $\left\|T_i^k - T_j^k\right\| \le M_{i,j}$, for all $k \ge 0$, and let $(x, y) \in H \times H$ be a pair satisfying (3.3). We continue by induction.

Let $m = 2$ and $1 \le i_1 < i_2 \le n$. By the Cauchy-Bunyakovsky-Schwarz inequality we have

$$\begin{aligned}
\left|\left\langle T_{i_1}^{k_1+k_2} x \middle| y\right\rangle\right| &\le \left|\left\langle T_{i_1}^{k_1+k_2} x - T_{i_1}^{k_1} T_{i_2}^{k_2} x \middle| y\right\rangle\right| + \left|\left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \middle| y\right\rangle\right| \\
&= \left|\left\langle T_{i_1}^{k_1}(T_{i_1}^{k_2} - T_{i_2}^{k_2})x \middle| y\right\rangle\right| + \left|\left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \middle| y\right\rangle\right| \\
&\le \left\|T_{i_1}^{k_1}(T_{i_1}^{k_2} - T_{i_2}^{k_2})x\right\| \cdot \|y\| + \left|\left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \middle| y\right\rangle\right| \\
&\le \left\|T_{i_1}^{k_1}\right\| \cdot \left\|T_{i_1}^{k_2} - T_{i_2}^{k_2}\right\| \cdot \|x\| \cdot \|y\| + \left|\left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \middle| y\right\rangle\right| \\
&\le \left\|T_{i_1}\right\|^{k_1} \cdot M_{i_1,i_2} \cdot \|x\| \cdot \|y\| + \left|\left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \middle| y\right\rangle\right| .
\end{aligned}$$

Since $\left|\left\langle T_{i_1}^{n} x \middle| y\right\rangle\right| \to \infty$ as $n \to \infty$ (hence $\left|\left\langle T_{i_1}^{k_1+k_2} x \middle| y\right\rangle\right| \to \infty$ as $k_2 \to \infty$, for all $k_1 \ge 0$), the above inequalities imply that

$$\left|\left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \middle| y\right\rangle\right| \to \infty \text{, as } k_2 \to \infty \text{, for all } k_1 \ge 0 .$$

Similarly,

$$\begin{aligned}
\left|\left\langle T_{i_2}^{k_1+k_2} x \middle| y\right\rangle\right| &\le \left|\left\langle T_{i_2}^{k_1+k_2} x - T_{i_1}^{k_1} T_{i_2}^{k_2} x \middle| y\right\rangle\right| + \left|\left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \middle| y\right\rangle\right| \\
&= \left|\left\langle T_{i_2}^{k_2}(T_{i_2}^{k_1} - T_{i_1}^{k_1})x \middle| y\right\rangle\right| + \left|\left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \middle| y\right\rangle\right| \\
&\le \left\|T_{i_2}\right\|^{k_2} \cdot M_{i_2,i_1} \cdot \|x\| \cdot \|y\| + \left|\left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \middle| y\right\rangle\right| ,
\end{aligned}$$

which implies that

$$\left|\left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \middle| y\right\rangle\right| \to \infty \text{, as } k_1 \to \infty \text{, for all } k_2 \ge 0 .$$

To complete the proof, it is enough to show the claim is true for $m = n$, under the assumption that the (n-1)-tuple weak orbit

$$\left\{\left\langle T_{i_1}^{k_1}T_{i_2}^{k_2}\ldots T_{i_{n-1}}^{k_{n-1}}x\middle| y\right\rangle : k_j \geq 0; 1\leq j\leq n-1\right\},$$

tends to infinity for all $1\leq i_1<\ldots<i_{n-1}\leq n$. For a fixed $i\in\{1,2,\ldots,n\}$, arbitrary $j\in\{1,2,\ldots,n\}\setminus\{i\}$ and fixed $k_1,k_2,\ldots,k_n\geq 0$ we have

$$\begin{aligned}
&\left|\left\langle T_1^{k_1}\ldots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\ldots T_n^{k_n}x\middle| y\right\rangle\right| \\
&\leq \left|\left\langle T_1^{k_1}\ldots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\ldots T_n^{k_n}x - T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x\middle| y\right\rangle\right| + \left|\left\langle T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x\middle| y\right\rangle\right| \\
&= \left|\left\langle T_1^{k_1}\ldots T_{i-1}^{k_{i-1}}T_{i+1}^{k_{i+1}}\ldots T_n^{k_n}(T_j^{k_i}-T_i^{k_i})x\middle| y\right\rangle\right| + \left|\left\langle T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x\middle| y\right\rangle\right| \\
&\leq \left\|T_1^{k_1}\ldots T_{i-1}^{k_{i-1}}T_{i+1}^{k_{i+1}}\ldots T_n^{k_n}(T_j^{k_i}-T_i^{k_i})x\right\|\cdot\|y\| + \left|\left\langle T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x\middle| y\right\rangle\right| \\
&\leq \left(\prod_{\substack{l=1\\ l\neq i}}^{n}\|T_l\|^{k_l}\right)\cdot M_{i,j}\cdot\|x\|\cdot\|y\| + \left|\left\langle T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x\middle| y\right\rangle\right|.
\end{aligned}$$

Since $j\in\{1,2,\ldots,n\}\setminus\{i\}$, by the inductive assumption, we have

$$\left|\left\langle T_1^{k_1}\ldots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\ldots T_n^{k_n}x\middle| y\right\rangle\right|\to\infty \text{ as } k_i\to\infty, \text{ for all } k_j\geq 0,\ j\neq i.$$

This, together with the above inequalities implies that

$$\left|\left\langle T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x\middle| y\right\rangle\right|\to\infty \text{ as } k_i\to\infty, \text{ for all } k_j\geq 0,\ j\neq i,$$

which completes the proof. ■

Corollary 3.3. *If H is a Hilbert space and $\{T_i : i\in F\}\subset B(H)$ is a family of operators such that $r(T_i)>1$, for all $i\in F$, then there is a dense set $D\subset H\times H$ such that the weak orbit $\left(\left\langle T_i^k x\middle| y\right\rangle\right)_{k\geq 0}$ tends to infinity for every pair $(x,y)\in D$ and every $i\in F$. If, in addition, $\{T_i : i\in F\}$ is a family of mutually commuting operators such that the sequence $(T_i^k - T_j^k)_{k\geq 1}$ is norm bounded for all $i,j\in F$, then for every $n\in F$, every $1<m\leq n$ the $m-$tuple weak orbit*

$$\left\{\left\langle T_{i_1}^{k_1}T_{i_2}^{k_2}\ldots T_{i_m}^{k_m}x\middle| y\right\rangle : k_i\geq 0; 1\leq i\leq m\right\},$$

tends to infinity for all $1\leq i_1<i_2<\ldots<i_m\leq n$.

Proof. If $T\in B(H)$ has a spectral radius $r(T)>1$, then $\sum_{k=1}^{\infty}\|T^k\|^{-1}<\infty$. Namely, if $r(T)>1$, then there is $\lambda\in\sigma(T)$ such that $1<|\lambda|$. By the Spectral Mapping Theorem, $\lambda^n\in\sigma(T^n)$ for every $n\in\mathbb{N}$. Hence $|\lambda|^n\leq r(T^n)\leq\|T^n\|$ and

$$\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} \leq \sum_{n=1}^{\infty} \frac{1}{|\lambda|^n} < \infty .$$

Now the conclusion follows from Theorem 3.2. ■

4. REMARKS ON N-TUPLE ORBITS TENDING TO INFINITY

By the Cauchy-Bunyakovsky-Schwarz inequality we have

$$\left|\left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \middle| y \right\rangle\right| \leq \left\| T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \right\| \cdot \|y\|,$$

for all $(x, y) \in H \times H$, $k_j \geq 0$ and $1 \leq j \leq n$. These inequalities clearly imply that the *n*-tuple orbit $\mathrm{Orb}(\{T_i\}_{i=1}^n, x)$ tends to infinity whenever there is $y \in H$ such that the *n*-tuple weak orbit $\left\{\left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \middle| y \right\rangle : k_i \geq 0; 1 \leq i \leq n\right\}$ tends to infinity. Hence, from the results in the previous section we can derive the following results for *n*-tuple orbits tending to infinity.

Theorem 4.1. *If H is Hilbert space and $\{T_i : i \in F\} \subset B(H)$ is a family of operators such that $\sum_{k=1}^{\infty} \left\|T_i^k\right\|^{-1} < \infty$ for all $i \in F$, then there is a dense set $D \subset H$ such that the orbit $\mathrm{Orb}(T_i, x)$ tends to infinity for every $x \in D$ and every $i \in F$. If, in addition, $\{T_i : i \in F\}$ is a family of mutually commuting operators such that the sequence $(T_i^k - T_j^k)_{k \geq 1}$ is norm bounded for all $i, j \in F$, then for every $n \in F$, every $1 < m \leq n$, the $m-$tuple orbit $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^m, x)$ tends to infinity for all $1 \leq i_1 < i_2 < \dots < i_m \leq n$.*

Corollary 4.2. *If H is Hilbert space and $\{T_i : i \in F\} \subset B(H)$ is a family of operators such that $r(T_i) > 1$ for all $i \in F$, then there is a dense set $D \subset H$ such that the orbit $\mathrm{Orb}(T_i, x)$ tends to infinity or every $x \in D$ and every $i \in F$. If, in addition, $\{T_i : i \in F\}$ is a family of mutually commuting operators such that the sequence $(T_i^k - T_j^k)_{k \geq 1}$ is norm bounded for all $i, j \in F$, then for every $n \in F$, every $1 < m \leq n$, the $m-$tuple orbit $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^m, x)$ tends to infinity for all $1 \leq i_1 < i_2 < \dots < i_m \leq n$.*

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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CHAIN CONNECTED SET IN A SPACE

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Abstract. The paper gives a generalization of connectedness and chain connectedness of a space that is more general than a topological space and it consists of a set and a family of coverings of the set. In these spaces we define the notions of connected and chain connected sets, that are generalization of the notions with the same name in a topological spaces ([1]), and we study their properties. Also, the notions of a chain relation and a chain component in a space that are generalization of a chain relation and a chain component in a topological space ([1]), are defined and their properties are presented. Some new results for topological spaces are also provided.

1. INTRODUCTION

The definitions of connectedness by using the notion of chain as well as chain connectedness in a topological space, are given and theirs properties are studied in [1]-[5].

In this paper we use the notions and properties from article [1] and we generalize the notions to a space that is more general than a topological space and it consists of a set and a family of coverings of the set.

2. Space, subspace and chain

A space is a set X with added structure. By a space in this paper we understand the notion given in the next definition. By a covering of X we understand a covering of X in X.

Definition 2.1. ***The space*** *$X=\left(X,\underline{\mathcal{U}}\right)$ is a set X together with a family of coverings $\underline{\mathcal{U}}=\left\{\mathcal{U}_\alpha \middle| \alpha\in I\right\}$ of X.*

In this paper by a covering $\mathcal{U}$ of X we understand a covering that is an element of the family of coverings of X i.e. $\mathcal{U}\in\underline{\mathcal{U}}$.

Definition 2.2. ***Subspace*** *Y of the space $X=\left(X,\underline{\mathcal{U}}\right)$, is the set Y with the family of coverings $\underline{\mathcal{U}_Y}=\underline{\mathcal{U}}\cap Y=\left\{\mathcal{U}_\alpha\cap Y \middle| \alpha\in I\right\}$.*

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In this paper by subset $Y \subseteq X$ we understand the subspace Y of the space X.

Definition 2.3. *Let $\mathcal{U}$ be a covering of the set X and $x, y \in X$. **A chain in $\mathcal{U}$ that connects** x **and** y (from x to y, from y to x) is a finite sequence of sets $U_1, U_2, ..., U_n$ of $\mathcal{U}$ such that $x \in U_1$, $y \in U_n$ and $U_i \cap U_{i+1} \neq \emptyset$ for every $i = 1, 2, ..., n-1$.*

If X is a topological space, then the topology of X generate a family $\underline{\mathcal{U}}$, of all coverings $\mathcal{U}$ of X that consists of open sets. So, every topological space can be considered as a space that consists of a set X and a family $\underline{\mathcal{U}}$. When a topological space is considered as a space, this space is meant. Topological subspace Y of X is a subspace of the space X.

3. Chain connected set in a space

Using the notion of a chain we define the notion of chain connected set in a space.

Let X be a space and $C \subseteq X$.

Definition 3.1. *The set C is **chain connected** in X, if for every covering $\mathcal{U}$ of X and every $x, y \in C$, there exists a chain in $\mathcal{U}$ that connects x and y.*

Let X be a space and $C \subseteq Y \subseteq X$.

The first property of a chain connected set, shown in the next theorem, is an implication of chain connectedness from a space to each of its super spaces (X is a super space of C, if C is a subspace of X).

Theorem 3.2. *If C is chain connected in Y, then C is chain connected in X.*

Proof. Let C be chain connected in Y and $\mathcal{U}$ be a covering of X. Then:

$$\mathcal{U}_Y = \mathcal{U} \cap Y = \{U \cap Y | U \in \mathcal{U}\}$$

is a covering of Y. Since C is chain connected in Y, it follows that for every two points $x, y \in X$, there exists a chain $U_1 \cap Y, U_2 \cap Y, ..., U_n \cap Y$ of elements of $\mathcal{U}_Y$. Then since $U_i \cap U_{i+1} \neq \emptyset$ for every $i = 1, 2, ..., n-1$ and

$U_i \in \mathcal{U}$ for every $i = 1,2,...,n$, the sequence $U_1, U_2, ..., U_n$ is a chain in $\mathcal{U}$ that connects x and y. It follows that C is chain connected in X.■

Remark 3.3. *The most important case of the previous theorem is when* $Y = C$.■

Example 3.4. Consider the space $X = \{1,2,3\}$ with the family with one covering:

$$\mathcal{U} = \{\{1,2\},\{2,3\}\}.$$

The set $Y = \{1,3\}$ is chain connected in X, but it is not chain connected in Y since there does not exist a chain in $\mathcal{U}_Y = \mathcal{U} \cap Y = \{\{1\},\{3\}\}$ that connects 1 and 3.■

The next claim, which directly follows from the definition, shows that each subset of a chain connected set in a space is chain connected in the same space.

Remark 3.5. *If the set C is chain connected in X, then each subset of C is chain connected in X.*■

We will give criteria for chain connectedness in a space, using the notion of infinite star of a covering [1].

Let X be a space and $C \subseteq X$.

Let $\mathcal{U}$ be a covering of X and $x \in X$. Then the set $st(x,\mathcal{U})$ is a union of all $U \in \mathcal{U}$ which have nonempty intersection with x. The set:

$$st^n(x,\mathcal{U}) = st\left(st^{n-1}(x,\mathcal{U})\right) \text{ and } st^\infty(x,\mathcal{U}) = \bigcup_{n=1}^{\infty} st^n(x,\mathcal{U}).$$

Theorem 3.6. *The set C is chain connected in X, if and only if for every $x \in C$ and every covering $\mathcal{U}$ of X, $C \subseteq st^\infty(x,\mathcal{U})$.*■

Corollary 3.7. *The space X is chain connected in X, if and only if for every $x \in X$ and every covering $\mathcal{U}$ of X, $X = st^\infty(x,\mathcal{U})$.*■

4. Chain relation and chain components

Let $X = (X, \underline{\mathcal{U}})$ be a space and $x \in C \subseteq X$.

Definition 4.1. *The **chain component** of the point x of C in X, denoted by $V_{CX}(x)$, is the maximal chain connected subset of C in X that contains x.*■

Proposition 4.2. *The set $V_{CX}(x)$ consists of all elements $y \in C$, such that for every covering $\mathcal{U} \in \underline{\mathcal{U}}$ there exists a chain in $\mathcal{U}$ that connects x and y.*

If $C = X$, then we use notation $V_X(x)$ or $V(x)$ if we work only with the space X, for $V_{XX}(x)$. Clearly $V_{CX}(x) = C \cap V_X(x)$.

Example 4.3. For the space $X = \{1,2,3\}$ with the family consisting of one covering $\mathcal{U} = \{\{1,2\},\{2,3\}\}$, and the set $C = \{1,3\}$:

$$V_X(1) = \{1,2\} \text{ and } V_{CX}(1) = \{1\}.$$

The set C is chain connected in X if C is subset of $V_X(x)$ for every $x \in C$.

We denote by $U_X(C)$ or $U(C)$, the set that consists of all elements $y \in X$, such that for every covering $\mathcal{U} \in \underline{\mathcal{U}}$ there exists a chain in $\mathcal{U}$ that connects some $x \in C$ and y. This set is a union of chain components.

If C is a chain connected set in X, since for every $x, y \in C$ the chain components $V(x)$ and $V(y)$ coincide i.e. $V(x) = V(y)$, it follows that $U(C)$ is chain component and it is denoted by $V(C)$. Clearly $C \subseteq V_X(C)$ and $V(C) = V(x)$ for every $x \in C$.■

Remark 4.4. *If the set C is chain connected in X, then each subset of $V(C)$ is chain connected in X.*■

When X is a topological space, then it is obvious that the next statement improves the remark 2.2 from [1].

Remark 4.5. *If the set C is chain connected in a topological space X, then each subset of $V(C)$ is chain connected in X.*■

Let X be a space and $x, y \in X$.

Definition 4.6. *Th element x is **chain related** to y in X, and we denote it by $x \sim y$ if for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y.*

The chain relation in a space is an equivalence relation and it depends on the set X and the family of coverings of X.

Remark 4.7. *The set C is chain connected in X if and only if for every $x, y \in C$, $x \sim y$.* ■

Therefore C is not chain connected in X if and only if there exist $x, y \in C$ such that $x \nsim y$.

The chain relation decomposes the space into classes. The classes are chain components.

Let $x, y \in C$. If $y \in V_{CX}(x)$, then $V_{CX}(x) = V_{CX}(y)$. If $V_{CX}(x) \neq V_{CX}(y)$, then $V_{CX}(x) \cap V_{CX}(y) = \emptyset$. As a consequence, the next proposition is valid.

Proposition 4.8. *For every $x \in C$, $V_{CX}(x) = C \cap V_{XX}(x)$. Each chain component of X in X contains at most one chain component of C in X.* ■

Proposition 4.9. *For every $x \in C$,*

$$V_{CC}(x) \subseteq V_{CX}(x) = \bigcup_{y \in V_{CX}(x)} V_{CC}(y) \subseteq V_{XX}(x). ■$$

The proposition shows that every chain component of C in X is a union of chain components of C in C and is a subset of chain component of X in X.

Proposition 4.10. *The set of all chain connected subsets of C in X consist of all chain components and their subsets.* ■

5. Properties of chain connected sets that consist chain components

Next we turn to a union of chain connected sets in a topological space.

Lemma 5.1. *Let $C, D \subseteq X$. If C and D are chain connected in X and $V(C) \cap V(D) \neq \emptyset$, where $V(C)$ and $V(D)$ are chain components of C and D respectively, then the union $V(C) \cup V(D)$ is chain connected in X and*

$$V(C) \cup V(D) = V(C) = V(D).$$

Proof. Let $\mathcal{U}$ be a covering of X and $x, y \in V(C) \cup V(D)$. If $x, y \in V(C)$ or $x, y \in V(D)$, then since $V(C)$ and $V(D)$ are chain connected, there exists a chain in $\mathcal{U}$ that connects x and y. If $x \in V(C)$ and $y \in V(D)$, it follows that firstly there exists $z \in V(C) \cap V(D)$, and secondly that there exist chains in $\mathcal{U}$ that connect x with z, and z with y, from which it follows that there is a chain in $\mathcal{U}$ that connects x and y. So $V(C) \cup V(D)$ is chain connected in X.

Since $V(C) = V(x)$ is chain component of some $x \in C$, and $V(C) \cup V(D)$ is a chain connected set that contain x it follows that $V(C) \cup V(D) = V(C)$. Similarly $V(C) \cup V(D) = V(D)$.■

Corollary 5.2. *Let $C, D \subseteq X$. If C and D are chain connected in X and $V(C) \cap V(D) \neq \emptyset$, where $V(C)$ and $V(D)$ are chain components of C and D respectively, then the union $C \cup D$ is chain connected in X.*

Theorem 5.3. *Let $C_i, i \in I$ be a family of chain connected subspaces of X. If there exists $i_0 \in I$ such that for every $i \in I$, $V(C_{i_0}) \cap V(C_i) \neq \emptyset$, then the union $\bigcup_{i \in I} V(C_i)$ is chain connected in X* and *$\bigcup_{i \in I} V(C_i) = V(C_i)$ for every $i \in I$,*

Proof. Let $\mathcal{U}$ be a covering of X and $C_i, i \in I$ be a family of chain connected subspaces of X. Let $x, y \in \bigcup_{i \in I} V(C_i)$, i.e. $x \in V(C_x)$ and $y \in V(C_y)$ for some $x, y \in I$.

Since $V(C_{i_0}) \cap V(C_i) \neq \emptyset$ for every $i \in I$, from the previous lemma, it follows that $V(C_{i_0}) \cup V(C_x)$ is chain connected in X. Similarly $V(C_{i_0}) \cup V(C_y)$ is chain connected in X. Then because $C_{i_0} \neq \emptyset$, from the previous lemma it follows that $V(C_{i_0}) \cup V(C_x) \cup V(C_y)$ is chain connected in X, i.e. for every covering $\mathcal{U}$ of X, there exists a chain in $\mathcal{U}$ that connects x and y. So $\bigcup_{i \in I} V(C_i)$ is chain connected in X.

Since $V(C_i) = V(x_i)$ is chain component of some $x_i \in C_i$ for every $i \in I$, and $\bigcup_{i \in I} V(C_i)$ is a chain connected set that contain x it follows that:

$$\bigcup_{i \in I} V(C_i) = V(C_i) \text{ for every } i \in I. \blacksquare$$

Corollary 5.4. *Let $C_i, i \in I$ be a family of chain connected subspaces of X. If there exists $i_0 \in I$ such that for every $i \in I$, $C_{i_0} \cap C_i \neq \varnothing$, then the union $\bigcup_{i \in I} C_i$ is chain connected in X.*

If X is a topological space then the next two statements which directly follow from the lemma 5.1 and the theorem 5.3, respectively, improves the lemma 3.1 and theorem 3.6 from [1].

Corollary 5.5. *If C and D are chain connected sets in a topological space X and $V(C) \cap V(D) \neq \varnothing$, where $V(C)$ and $V(D)$ are chain components of C and D respectively, then the union $V(C) \cup V(D)$ is chain connected in X.*

Theorem 5.6. *Let $C_i, i \in I$ be a family of chain connected subspaces of a topological space X. If there exists $i_0 \in I$ such that for every $i \in I$, $V(C_{i_0}) \cap V(C_i) \neq \varnothing$, then the union $\bigcup_{i \in I} V(C_i)$ is chain connected in X.* $\blacksquare$

Corollary 5.7. *If every two points x and y of $C \subseteq X$ are in a chain connected set C_{xy} in X, then C is chain connected in X.* $\blacksquare$

6. CONCLUSIONS

The notion of connectedness by using the standard definition cannot be generalized from a topological space to a more general space, without generalizing the topology of the space, since it is related to it. However connectedness defined by chain ([1]), as well as its generalization chain connectedness to a pair of a topological space and its subspace, can, such that instead of families of all coverings of open sets, subfamilies of coverings of arbitrary sets will be considered. The generalizations can also be defined on even more general structures, such as a set of subsets of a given set, i.e. to define connectedness in this set.

The paper gives a generalization of connectedness and chain connectedness of a space that is more general than a topological space and it consists of a set and a family of coverings of the set. In these spaces, the notion of chain connected set, as well as the notion of chain relation are defined and their properties are presented. A number of statements from [1] cannot be generalized to the space level. Two examples for spaces that are not topological, are given, to be shown that one statement must not be true in the converse direction and in one statement two sets are not equal. Also, the special cases of claims 5.1 and 5.3 of the paper provide a new results at a level of topological spaces. They reduce to stronger claims than the corresponding claims lemma 3.1 and theorem 3.6 from [1], since they are expressed by using chain components instead of closed sets.

The generalizations to a space can be done to other topological notions and spaces.

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DEFORMED SPHERICAL CURVES

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Abstract. This paper is devoted to the study of spherical curves in Euclidean three-dimensional space using the theory of infinitesimal bending. Some interesting infinitesimal bending fields are obtained and discussed. In particular, the infinitesimal bending of a spherical curve such that all bent curves are approximately on the initial sphere (with a given precision) is studied. Some examples are analyzed and graphically presented.

1. Introduction

A spherical curve is a curve traced on a sphere. The curvature-to-torsion ratio completely describes a spherical curve in the sense that certain relations between curvature and torsion are necessary and sufficient for the space curve to lie on the sphere.

Deformations of spherical curves represent an interesting field of research. Some of the recent results in this regard are in the papers [3], [4], [6], [14]. A special type of small deformation is the so-called infinitesimal bending. A concept of infinitesimal bending first appeared in the description of the deformation of surfaces in three-dimensional Euclidean space, and then further extended to the curves and the manifolds. The theory of infinitesimal bending deals with vector fields and quantities associated with them, defined at the points of observed geometric objects and satisfying deformation equations. Under infinitesimal bending, the length of the arc is invariant with appropriate precision. In other words, in the initial moment of a deformation, the arc length is stationary, i.e. the initial velocity of its change is zero. Some papers related to infinitesimal bending of curves, knots and surfaces are [1], [2], [5], [7-14]. In this paper, we pay attention to spherical curves and their behavior during this type of deformation.

The paper is organized as follows: In Sec. 2 preliminary results and notation regarding infinitesimal bending of curves are presented. In Sec. 3 spherical curves and corresponding infinitesimal bending fields are studied. We suppose that the curves suffer a small deformation such that they remain on the same sphere with a given precision. In Sec. 4 and Sec. 5, respectively, Viviani's curve

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and spherical curve with constant slope (spherical helix) are discussed both analytically and graphically.

2. INFINITESIMAL BENDING

Let us consider a regular curve

$$C\colon \mathbf{r} = \mathbf{r}(t), \quad t \in J \subseteq \mathbb{R}^3 \tag{1}$$

of a class C^k, $k \geq 3$, included in a family of the curves

$$C_\epsilon\colon \mathbf{r}_\epsilon(t) = \mathbf{r}(t) + \epsilon\mathbf{z}(t) \tag{2}$$

where $\epsilon \geq 0, \epsilon \to 0$, and we get C for $\epsilon = 0$ ($C = C_0$).

Definition 1. *A continuous one-parameter family of curves C_ϵ, given by Eq.* (2), *is called an* **infinitesimal deformation** *of the curve C, given by Eq.* (1). *A field $\boldsymbol{z}(t) \in C^k, k \geq 3,$ is a vector function defined in the points of C called a* **deformation field**.

Definition 2. [2] *An infinitesimal deformation C_ϵ is an* **infinitesimal bending** of the curve C *if*

$$ds_\epsilon^2 - ds^2 = o(\epsilon). \tag{3}$$

The field $\mathbf{z}(t)$ *is the* **infinitesimal bending field** *of the curve C.*

According to Def. 2, the next theorem states.

Theorem 1. [2] *Necessary and sufficient condition for the curves C_ϵ to be infinitesimal bending of the curve C is to be valid*

$$d\mathbf{r} \cdot d\mathbf{z} = 0. \tag{4}$$

If infinitesimal bending is reduced to rigid motion of the curve, without internal deformations, we say it is **trivial** infinitesimal bending. The corresponding bending field is also called trivial.

Based on [11] we have the following theorem.

Theorem 2. *Under infinitesimal bending of curves each line element gets non-negative addition, which is the infinitesimal value of the order higher than the first with respect to ϵ, i. e.*

$$ds_\epsilon - ds \geq o(\epsilon).$$

The following theorem is related to determination of the infinitesimal bending field of a curve C.

Theorem 3. [12] *The infinitesimal bending field for the curve C is*

$$\mathbf{z}(t) = \int [p(t)\mathbf{n}_1(t) + q(t)\boldsymbol{n}_2(t)]\, dt, \tag{5}$$

where $p(t)$ and $q(t)$ are arbitrary integrable functions, and vectors $\mathbf{n}_1(t)$ and $\mathbf{n}_2(t)$ are respectively unit principal normal and binormal vector fields of the curve C.

3. SPHERICAL CURVES UNDER INFINITESIMAL BENDING

Let

$$C: \mathbf{r}(t) = (a\cos u(t)\cos v(t), a\sin u(t)\cos v(t), a\sin v(t)) \tag{6}$$

be a spherical curve on the sphere

$$S: \mathbf{r}(u,v) = (a\cos u\cos v, a\sin u\cos v, a\sin v),$$

with radius a and

$$C_\epsilon: \mathbf{r}_\epsilon(t) = \mathbf{r}(t) + \epsilon\mathbf{z}(t)$$

be an infinitesimal deformation of C, where $\mathbf{z}(t)$ is a deformation field. The following theorem gives the explicit expression for the field $\mathbf{z}(t)$ to be infinitesimal bending field for the spherical curve C.

Theorem 4. Let $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$ be the vector field such that

$$\begin{aligned} z_1(t) &= \int \cos u(t)\cos v(t)\,dt + c_1, \\ z_2(t) &= \int \sin u(t)\cos v(t)dt + c_2, \\ z_3(t) &= \int \sin v(t)dt + c_3, \end{aligned} \tag{7}$$

c_1, c_2, c_3 are constants. Then $\mathbf{z}(t)$ is the infinitesimal bending field for the spherical curve C given by Eq. (6).

Proof. We have

$$\dot{\mathbf{r}}(t) = \begin{cases} -a\dot{u}(t)\sin u(t)\cos v(t) - a\dot{v}(t)\cos u(t)\sin v(t) \\ a\dot{u}(t)\cos u(t)\cos v(t) - a\dot{v}(t)\sin u(t)\sin v(t) \\ a\dot{v}(t)\cos v(t) \end{cases}$$

and

$$\dot{\mathbf{z}}(t) = (\cos u(t)\cos v(t), \sin u(t)\cos v(t), \sin v(t)),$$

where 'dot' denotes derivative with respect to t. From the previous two equations we obtain $\dot{\mathbf{r}} \cdot \dot{\mathbf{z}} = 0$, which means that $\mathbf{z}$ is infinitesimal bending field.

■

Note that the previous equations do not determine all infinitesimal bending fields of the spherical curve C. If $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$, where $z_1(t), z_2(t), z_3(t)$ are arbitrary real continuous differentiable functions, then $\mathbf{z}$ can be determined from the following equation

$$\begin{aligned} &(-a\dot{u}(t)\sin u(t)\cos v(t) \\ &- a\dot{v}(t)\cos u(t)\sin v(t))\dot{z_1} + (a\dot{u}(t)\cos u(t)\cos v(t) \\ &- a\dot{v}(t)\sin u(t)\sin v(t))\dot{z_2} + a\dot{v}(t)\cos v(t)\,\dot{z_3} = 0, \end{aligned}$$

which has many solutions. Our field is obtained having in mind that $\left\|\mathbf{r}\right\|^2 = a^2 \to 2\mathbf{r}\cdot\dot{\mathbf{r}} = 0$ and we put $\dot{\mathbf{z}} = \frac{1}{a}\mathbf{r}$.

We posed the question whether it is possible to infinitesimally bend the spherical curve so that all bent curves are on the initial sphere. Regarding that we have the following theorem.

Theorem 5. [14] Let $C\colon \mathbf{r}\colon (t_1, t_2) \to \mathbb{R}^3$ be a curve on the sphere S. It does not exist nontrivial vector field $\mathbf{z}(t)$ so that the family of curves

$$C_\epsilon\colon\ \mathbf{r}_\epsilon(t) = \mathbf{r}(t) + \epsilon\mathbf{z}(t)$$

belongs to the sphere S.

Further, we are weakening the previous condition by requiring that the bent curves lie on a given sphere with a predetermined precision. Precisely, let us determine an infinitesimal bending field so that all bent spherical curves are on the initial sphere with a given precision, i.e. let be valid

$$F\big(x(t), y(t), z(t)\big) = 0,$$
$$F(x_\epsilon(t), y_\epsilon(t), z_\epsilon(t)) = o(\epsilon),$$

where $F(x, y, z) = 0$ is the implicit sphere equation in Cartesian coordinates x, y, z and $o(\epsilon)$ is an infinitesimal of at least second order with respect to ϵ. In connection with that we have the following theorem.

Theorem 6. Necessary and sufficient condition for the infinitesimal deformation of the spherical curve C to be on the sphere S with a given precision is that the field $\mathbf{z}$ satisfies the condition

$$\mathbf{r}\cdot\mathbf{z} = 0. \tag{8}$$

Proof. A vector equation of a sphere S of radius a is

$$||\mathbf{r}||^2 = a^2,$$

$\mathbf{r}$ is the position vector of an arbitrary point on S. Let $\mathbf{z}$ be deformation field which given spherical curve leaves on the initial sphere with a given precision, i.e. let the following condition be valid

$$||\mathbf{r}_\epsilon||^2 = a^2 + o(\epsilon),$$

where $o(\epsilon)$ is an infinitesimal of at least second order with respect to ϵ. Since $\mathbf{r}_\epsilon = \mathbf{r} + \epsilon\mathbf{z}$, the previous equation reduces to

$$||\mathbf{r} + \epsilon\mathbf{z}||^2 = a^2 + o(\epsilon),$$

wherefrom we obtain

$$||\mathbf{r}||^2 + 2\epsilon\mathbf{r}\cdot\mathbf{z} + \epsilon^2||\mathbf{z}||^2 = a^2 + o(\epsilon)$$

which leads to Eq. (8). ■

In the case of the sphere it is easy to see that the unit normal $\boldsymbol{\nu}(u, v)$ satisfies $\boldsymbol{\nu}(u, v) = \frac{1}{a}\mathbf{r}(u, v)$. So, based on the condition (8), we conclude that $\mathbf{z} \perp \boldsymbol{\nu}$, i.e. $\mathbf{z}$ lies in the tangent plane of the sphere along the curve C. It means that we can present the vector $\mathbf{z}$ as the linear combination of the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ that determine tangent plane along C, i.e.

$$\mathbf{z}(t) = f(t)\mathbf{r}_u + g(t)\mathbf{r}_v,$$

where $f(t)$ and $g(t)$ are arbitrary real continuous differentiable functions. Since

$$\dot{\mathbf{z}}(t) = \dot{f}(t)\mathbf{r}_u + f(t)(\mathbf{r}_{uu}\dot{u} + \mathbf{r}_{uv}\dot{v}) + \dot{g}(t)\mathbf{r}_v + g(t)(\mathbf{r}_{vu}\dot{u} + \mathbf{r}_{vv}\dot{v})$$

and

$$\dot{\mathbf{r}}(t) = \mathbf{r}_u\dot{u} + \mathbf{r}_v\dot{v}$$

using the condition for infinitesimal bending $\dot{\mathbf{r}} \cdot \dot{\mathbf{z}} = 0$, we obtain the following equation

$$\dot{f}(t)\mathbf{r}_u \cdot \mathbf{r}_u\dot{u} + \mathbf{r}_v\dot{v} + f(t)(\mathbf{r}_{uu}\dot{u} + \mathbf{r}_{uv}\dot{v}) \cdot \mathbf{r}_u\dot{u} + \mathbf{r}_v\dot{v} + \dot{g}(t)\mathbf{r}_v \cdot \mathbf{r}_u\dot{u} + \mathbf{r}_v\dot{v}$$
$$+g(t)(\mathbf{r}_{vu}\dot{u} + \mathbf{r}_{vv}\dot{v}) \cdot \mathbf{r}_u\dot{u} + \mathbf{r}_v\dot{v} = 0.$$

Using one of the functions $f(t)$ and $g(t)$ arbitrarily, we obtain the other from the previous linear differential equation. In this way we find infinitesimal bending of a spherical curve such that all bent curves are on the initial sphere with a given precision.

Example 1. A circle $\mathbf{r}(t) = (a\cos t\,, a\sin t, 0)$ has an infinitesimal bending field $\mathbf{z}(t) = (-\sin t\,, \cos t\,, f(t))$, where $f(t)$ is an arbitrary real continuous differential function. Corresponding infinitesimal bending is on the sphere with a given precision, i.e. it is valid $||\mathbf{r}_\epsilon||^2 = a^2 + \epsilon^2\big(1 + f^2(t)\big) = a^2 + o(\epsilon)$. In Fig.1 we can see deformed circle for $f(t) = \sin t \cos t$. The red color denotes original curve, and blue deformed ones for ϵ= 0.5 and ϵ= 1.

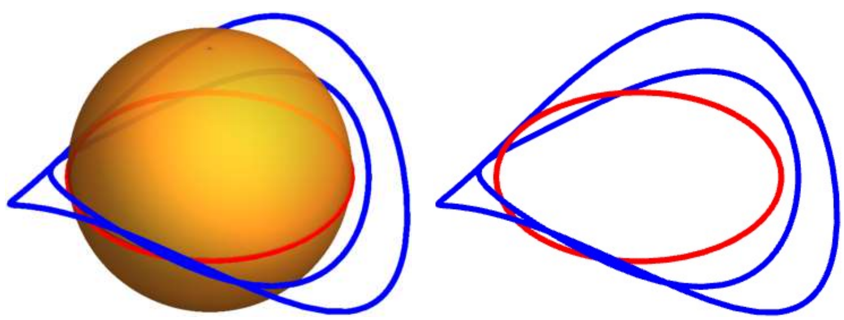

Figure 1: Circle and its infinitesimal bending for $\epsilon = 0.5$ and $\epsilon = 1$.

Based on Theorems 4 and 6 we obtain the following corollary.

Corollary 1. Necessary and sufficient condition for the infinitesimal bending of the spherical curve C, determined by the field $\mathbf{z}$ given by Eqs. (7), to be on the sphere S with a given precision is that the field $\mathbf{z}$ is of constant intensity.

Proof. Since for the field $\mathbf{z}$ holds $\dot{\mathbf{z}} = \frac{1}{a}\mathbf{r}$, the condition (8) reduces to $\dot{\mathbf{z}} \cdot \mathbf{z} = 0 \leftrightarrow (\mathbf{z} \cdot \mathbf{z})^{\cdot} = 0 \leftrightarrow \big||\mathbf{z}|\big| = const.$ ■

4. VIVIANI'S CURVE

Viviani's curve is obtained as an intersection of a sphere with a cylinder that is tangent to the sphere and passes through two poles of the sphere. The parametric representation is

$$\mathbf{r}(t) = (a(1+\cos t), a\sin t, 2a\sin\frac{t}{2}), \quad t \in [-2\pi, 2\pi], \quad a > 0.$$

It is easy to see that the vector field

$$\mathbf{z}(t) = (\sin t, -\cos t, 0)$$

is an infinitesimal bending field for which all bent curves are closed, i.e. $\mathbf{z}(-2\pi) = \mathbf{z}(2\pi)$. In Fig.2 we can see deformed Viviani's curve for $\epsilon = 0.5$ and $\epsilon = 1$. The red color denotes original curve, and blue deformed ones.

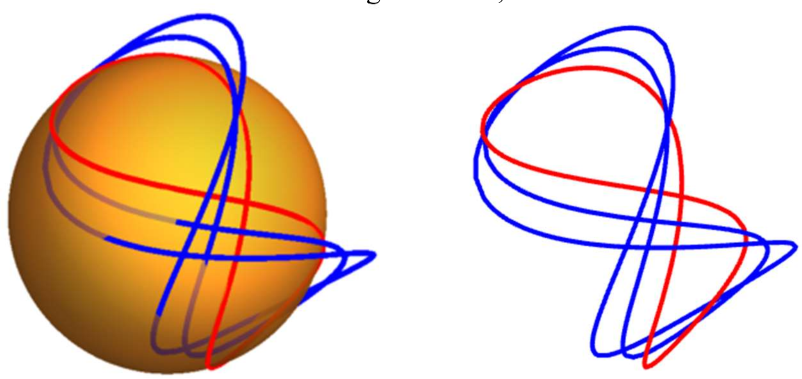

Figure 2: Viviani's curve and its infinitesimal bending for $\epsilon = 0.5$ and $\epsilon = 1$.

Let us examine whether this infinitesimal bending lies with a given precision on the sphere

$$S: ||\mathbf{r}||^2 = 4a^2$$

containing the initial Viviani's curve. Since

$$\mathbf{r}_\epsilon = (a(1+\cos t) + \epsilon\sin t, a\sin t - \epsilon\cos t, 2a\sin\frac{t}{2})$$

we easy obtain

$$||\mathbf{r}_\epsilon||^2 = 4a^2 + 2a\epsilon\sin t + \epsilon^2.$$

We conclude that bent curves are not approximately on the sphere S. However, the points for $t = k\pi \leftrightarrow \sin t = 0$ are on the sphere S with a given precision.

5. SPHERICAL CURVES WITH CONSTANT SLOPE

Spherical curve with constant slope (spherical helix) has the following parametric equation

$$\boldsymbol{r}(t) = \begin{cases} (a+b)\cos t - b\cos\dfrac{a+b}{b}t \\ (a+b)\sin t - b\sin\dfrac{a+b}{b}t \\ 2\sqrt{ab+b^2}\cos\dfrac{a}{2b}t \end{cases}$$

$a \geq b > 0$. It lies on the sphere of the radius $a + 2b$. The vector field

$$\mathbf{z}(t) = \left(\frac{2b}{a+2b}\cos\frac{a+2b}{2b}t, \frac{2b}{a+2b}\sin\frac{a+2b}{2b}t, 0\right)$$

is an infinitesimal bending field. In Fig.3 we have deformed spherical helix $(a = 1, b = 1)$ for $\epsilon = 0.5$ and $\epsilon = 1$ together with original curve (red colour).

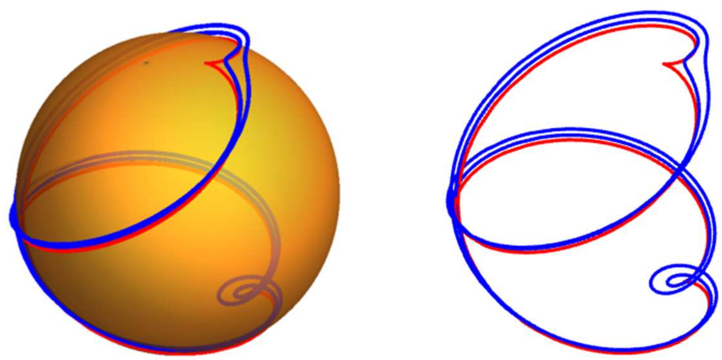

Figure 3: Spherical helix (1,1) (spherical cardioid) and its infinitesimal bending for $\epsilon = 0.5$ and $\epsilon = 1$.

By checking the condition

$$||\mathbf{r}_\epsilon||^2 = (a+2b)^2 + o(\epsilon)$$

we conclude that this infinitesimal bending is not on the initial sphere. More precisely, it holds

$$||\mathbf{r}_\epsilon||^2 = (a+2b)^2 + \epsilon\frac{4ab}{a+2b}\cos\frac{a}{2b}t + \epsilon^2\frac{4b^2}{(a+2b)^2}.$$

The points for $\cos\frac{a}{2b}t = 0$ are on the initial sphere with a given precision.

6. Conclusions

In this paper we point out to the possibility of infinitesimal bending of spherical curves. We give explicit formulas for bending fields with appropriate graphic illustration using program packet Mathematica.

Many geometric magnitudes of curves are changed during the process of infinitesimal bending. For instance, in Fig.4 we can see how the curvature and the torsion of the Viviani's curve is changed for different values of the infinitesimal ϵ. Our next step is investigation of these changes and finding the appropriate variations.

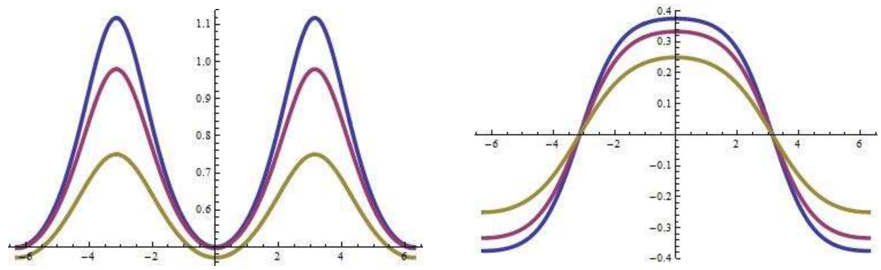

Figure 4: Curvature and torsion of Viviani's curve under infinitesimal bending for $\epsilon = 0,\ \epsilon = 0.5$ and $\epsilon = 1$.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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INVERTIBILITY OF LINEAR COMBINATIONS OF K-POTENT MATRICES

Marina Tošić, Jelena Vujaković

Abstract. We study the problem of the invertibility of linear combinations of two or three k-potent matrices under various conditions. In these cases, we give explicit formulae of their inverses.

1. Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of all $m\times n$ complex matrices. Specially, let $\mathbb{C}^{n\times n}$ denote the set of all $n\times n$ square complex matrices. The symbols $R(A)$ and $N(A)$ denote the range (column space) and the null space of a matrix A, respectively, while $r(A)$ is rank of A. $\mathbb{C}_r^{n\times n}$ is symbol of the set of all $n\times n$ matrices with rank r. Also, I_n denotes the identity matrix of order n. We say that integers k and l are congruent modulo the positive integer m, and we use notation $k\equiv l\ (mod\ m)$, if m divide $k-l$.
In this paper, we deal with *k*-potent matrices, where *k* is a positive integer greater than one. This type of matrices is defined as follows.

Definition 1. ([4]) *A matrix* $A\epsilon\mathbb{C}^{n\times n}$ *is k-potent if* $A^k=A$, *where* k *is a positive integer greater than one.*

Any *k*-potent matrix is group invertible. For $A\epsilon\mathbb{C}^{n\times n}$, the group inverse of A is the unique, if it exists (see [2]), matrix $A^{\#}\epsilon\mathbb{C}^{n\times n}$ such that:

$$A=AA^{\#}A, A^{\#}=A^{\#}AA^{\#}, AA^{\#}=A^{\#}A.$$

Thus, if $A\epsilon\mathbb{C}^{n\times n}$ is a *k*-potent matrix, then $A^{\#}=A^{k-2}$.

The research dealing with *k*-potent matrices is quite extensive (see [1], [4]-[6], [8], [9]) because they have a wide application, for example in statistics (see [7], [10]). A particularly interesting research topic related to *k*-potent matrices is the invertibility of a linear combination of *k*-potent matrices. In this paper, we study the invertibility of linear combinations c_1A+c_2B and $c_1A+c_2B+c_3C$, where A,B,C are k-potent matrices and c_1,c_2,c_3 are nonzero complex numbers. Also, we give some formulae for $(c_1A+c_2B)^{-1}$ and $(c_1A+c_2B+c_3C)^{-1}$ under various conditions.

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Key words and phrases. *k*-potent matrix, linear combination, invertibility

2. INVERTIBILITY OF A LINEAR COMBINATION OF TWO K-POTENT MATRICES

Recently, there has been interest in investigating the invertibility of a linear combination of two *k*-potent matrices. In [3], J. Benitez, X. Liu and T. Zhu proved the following results.

Theorem 1. [3] *Let $A, B \in \mathbb{C}^{n\times n}$ be two k-potent matrices such that $A^{k-1}B = B^{k-1}A$ or $BA^{k-1} = AB^{k-1}$. If a linear combination $d_1A + d_2B$ is nonsingular for some $d_1, d_2 \in \mathbb{C} \setminus \{0\}$ satisfying $d_1 + d_2 \neq 0$, then $c_1A + c_2B$ is nonsingular for all $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ satisfying $c_1 + c_2 \neq 0$.*

Theorem 2. **[3]** *Let $A, B \in \mathbb{C}^{n\times n}$ be two k-potent matrices such that $I_n - A^{k-1}B^{k-2}$ is nonsingular. If there exist $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such $c_1A + c_2B$ is nonsingular, then $A - B$ is also nonsingular.*

Theorem 3. [3] *Let $A, B \in \mathbb{C}^{n\times n}$ be two commuting k-potent matrices. If there exists $a \in \mathbb{C} \setminus \{0\}$ such that $A + aB$ is nonsingular, then $c_1A + c_2B$ and $c_1I_n + c_2AB$ are nonsingular for all $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ with $\frac{-c_1}{c_2} \notin \sqrt[k-1]{1}$.*

Theorem 4. [3] *Let $A, B \in \mathbb{C}^{n\times n}$ be two k-potent matrices, and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. Then the following statements are equivalent:*

(i) *$c_1AB^{k-1} + c_2BA^{k-1}$ is nonsingular.*

(ii) *$c_1B^{k-1}A + c_2A^{k-1}B$ is nonsingular.*

(iii) *$c_1A + c_2B$ and $I_n - A^{k-1} - B^{k-1}$ are nonsingular.*

Theorem 5. [3] *Let $A, B \in \mathbb{C}^{n\times n}$ be two k-potent matrices such that $A^{k-1}B = B^{k-1}A$, and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. If A or B are nonsingular, then $c_1A + c_2B$ are nonsingular if and only if $c_1 + c_2 \neq 0$. In this case,*

(i) *If A is nonsingular, then*

$$(c_1 + c_2)(c_1A + c_2B)^{-1} = A^{-1} + c_2c_1^{-1}A^{-1}\left(I_n - B^{k-1}\right).$$

(ii) *If B is nonsingular, then*

$$(c_1 + c_2)(c_1A + c_2B)^{-1} = B^{-1} + c_1c_2^{-1}B^{-1}\left(I_n - A^{k-1}\right).$$

Theorem 6. [3] *Let $A, B \in \mathbb{C}^{n\times n}$ be two k-potent matrices such that $AB = 0$, and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. Then $N(c_1A + c_2B) = N(A + B)$ and $R(c_1A + c_2B) =$*

$R(A+B)$. *In particular,* c_1A+c_2B *is nonsingular if and only if* $A+B$ *is nonsingular, and in this case, we have*

$$(c_1A+c_2B)^{-1}=c_1^{-1}(A+B)^{-1}+(c_2^{-1}-c_1^{-1})B^{k-2}\left(I_n-A^{k-1}\right).$$

However, the invertibility of a linear combination can be studied under other conditions.
Let $A,B\epsilon\mathbb{C}^{n\times n}$ be two k-potent matrices for some natural $k>1$. Since $A\epsilon\mathbb{C}^{n\times n}, r(A)=r$ is k-potent, this matrix can be written as:

$$A=U\begin{bmatrix}K & 0\\ 0 & 0\end{bmatrix}U^{-1}, \tag{1}$$

where $U\epsilon\mathbb{C}^{n\times n}$ is nonsingular, $K=diag(\lambda_1,\dots,\lambda_r)$, $\lambda_i^{K-1}=1$ for $i=1,\dots,r$. Obviously, $K\epsilon\mathbb{C}^{r\times r}$ is nonsingular and $K^{k-1}=I_r$. Furthermore, we can write $B\epsilon\mathbb{C}^{n\times n}$ as follows:

$$B=U\begin{bmatrix}B_1 & B_2\\ B_3 & B_4\end{bmatrix}U^{-1}, \tag{2}$$

where $B_1=\mathbb{C}^{r\times r}$ and $B_4\epsilon\mathbb{C}^{(n-r)\times(n-r)}$.

Theorem 7. *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices such that* $AB=0=BA$, *and let* $c_1,c_2\epsilon\mathbb{C}\setminus\{0\}$. *Then* c_1A+c_2B *is nonsingular if and only if* $A^{k-1}+\left(I_n-A^{k-1}\right)B$ *is nonsingular. Furthermore,*

$$(c_1A+c_2B)^{-1}=c_1^{-1}A^{k-2}+c_2^{-1}\left(I_n-A^{k-1}\right)B^{k-2}.$$

Proof. Let $A\epsilon\mathbb{C}_r^{n\times n}$ be of the form (1). Since $AB=0=BA$, then B has the form:

$$B=U\begin{bmatrix}0 & 0\\ 0 & B_4\end{bmatrix}U^{-1}, \tag{3}$$

where $B_4\epsilon\mathbb{C}^{(n-r)\times(n-r)}$. From $B^k=B$, it follows that $B_4^k=B_4$, i.e. B_4 is k-potent. In addition,

$$c_1A+c_2B=U\begin{bmatrix}c_1K & 0\\ 0 & c_2B_4\end{bmatrix}U^{-1}.$$

Based on the invertibility of K, we conclude that c_1A+c_2B is nonsingular if and only if B_4 is nonsingular. Since

$$A^{k-1}+\left(I_n-A^{k-1}\right)B=U\begin{bmatrix}I_r & 0\\ 0 & B_4\end{bmatrix}U^{-1},$$

we have that c_1A+c_2B is nonsingular if and only if $A^{k-1}+\left(I_n-A^{k-1}\right)B$ is nonsingular. Furthermore,

$$(c_1A+c_2B)^{-1} = U\begin{bmatrix} c_1^{-1}K^{-1} & 0 \\ 0 & c_2^{-1}B_4^{-1}\end{bmatrix}U^{-1} = U\begin{bmatrix} c_1^{-1}K^{k-2} & 0 \\ 0 & c_2^{-1}B_4^{k-2}\end{bmatrix}U^{-1}$$
$$= c_1^{-1}A^{k-2} + c_2^{-1}\left(I_n - A^{k-1}\right)B^{k-2}. \quad \square$$

Corollary 1. *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices such that* $AB=0=BA$, *and let* $c_1,c_2\epsilon\mathbb{C}\setminus\{0\}$. *Then,* c_1A+c_2B *is nonsingular if and only if* $A+B$ *is nonsingular.*

Beside forms (1) and (2), the following fact: *If* $E,F\epsilon\mathbb{C}^{n\times n}$ *and* $EF=FE$, *then*

$$E^k+(-1)^{k+1}F^k = (E+F)\sum_{i=0}^{k-1}(-1)^i E^{k-1-i}F^i,\ k\epsilon N,\ k>1,$$

is very useful for next results. First, note that $A^l=A^s$, where A is a *k*-potent matrix and $l\equiv s\big(mod(k-1)\big)$.

Theorem 8. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices, and let* $c_1,c_2\epsilon\mathbb{C}\setminus\{0\}$.

(i) *If* $AB=A^s$, $s\epsilon\{0,1,3,\dots,k-2\}$, *and* $c_1^{k-1}+(-1)^k c_2^{k-1}\neq 0$, *then* c_1A+c_2B *is nonsingular if and only if* $A^{k-1}+B\left(I_n-A^{k-1}\right)$ *is nonsingular. Furthermore,*

$$(c_1A+c_2B)^{-1} = A_1 - B^{k-2}\left(I_n-A^{k-1}\right)^2 BA^{k-1}A_1 + \tfrac{1}{c_2}B^{k-2}\left(I_n-A^{k-1}\right),$$

where: $A_1=\frac{1}{c_1^{k-1}+(-1)c_2^{k-1}}\sum_{i=0}^{k-2}(-1)^i c_1^{k-2-i}c_2^i A^{k-2-i+(s-1)i}$.

(ii) *If* $AB=A^2$ *and* $c_1+c_2\neq 0$, *then* c_1A+c_2B *is nonsingular if and only if* $A^{k-1}+B\left(I_n-A^{k-1}\right)$ *is nonsingular. Furthermore,*

$$(c_1A+c_2B)^{-1} = \tfrac{1}{c_1+c_2}A^{k-2} - \tfrac{1}{c_1+c_2}B^{k-2}\left(I_n-A^{k-1}\right)^2 BA^{k-1}A^{k-2} + \tfrac{1}{c_2}B^{k-2}\left(I_n-A^{k-1}\right).$$

Corollary 2. *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices, and let* $c_1,c_2\epsilon\mathbb{C}\setminus\{0\}$.

(i) *If* $AB=A^s$, $s\epsilon\{0,1,3,\dots,k-2\}$, *and* $c_1^{k-1}+(-1)^k c_2^{k-1}\neq 0$, *then* c_1A+c_2B *is nonsingular if and only if* $A+B$ *is nonsingular.*

(ii) *If* $AB=A^2$ *and* $c_1+c_2\neq 0$, *then* c_1A+c_2B *is nonsingular if and only if* $A+B$ *is nonsingular.*

Theorem 9. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices, and let* $c_1, c_2\epsilon\mathbb{C}\setminus\{0\}$.

(i) *If* $BA = A^s$, $s\epsilon\{0,1,3,\dots,k-2\}$, *and* $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, *then* $c_1A + c_2B$ *is nonsingular if and only if* $A^{k-1} + \left(I_n - A^{k-1}\right)B$ *is nonsingular. Furthermore,*

$$(c_1A + c_2B)^{-1} = A_1 - A_1A^{k-1}B\left(I_n - A^{k-1}\right)^2B^{k-2} + \frac{1}{c_2}\left(I_n - A^{k-1}\right)B^{k-2}, \tag{6}$$

where: $A_1 = \frac{1}{c_1^{k-1}+(-1)^k c_2^{k-1}}\sum_{i=0}^{k-2}(-1)^i c_1^{k-2-i}c_2^iA^{k-2-i+(s-1)i}$.

(ii) *If* $BA = A^2$ *and* $c_1 + c_2 \neq 0$, *then* $c_1A + c_2B$ *is nonsingular if and only if* $A^{k-1} + \left(I_n - A^{k-1}\right)B$ *is nonsingular. Furthermore,*

$$(c_1A + c_2B)^{-1} = \frac{1}{c_1+c_2}A^{k-2} - \frac{1}{c_1+c_2}A^{k-2}A^{k-1}B\left(I_n - A^{k-1}\right)^2B^{k-2} + \frac{1}{c_2}\left(I_n - A^{k-1}\right)B^{k-2}.$$

Corollary 3. *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices, and let* $c_1, c_2\epsilon\mathbb{C}\setminus\{0\}$.

(i) *If* $BA = A^s$, $s\epsilon\{0,1,3,\dots,k-2\}$, *and* $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, *then* $c_1A + c_2B$ *is nonsingular if and only if* $A + B$ *is nonsingular.*

(ii) *If* $BA = A^2$ *and* $c_1 + c_2 \neq 0$, *then* $c_1A + c_2B$ *is nonsingular if and only if* $A + B$ *is nonsingular.*

Theorem 10. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be commuting k-potent matrices, and let* $c_1, c_2\epsilon\mathbb{C}\setminus\{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. *Then* $c_1A + c_2B$ *is nonsingular if and only if* $A^{k-1} + \left(I_n - A^{k-1}\right)B$ *is nonsingular. Furthermore,*

$$(c_1A + c_2B)^{-1} = \frac{1}{c_1^{k-1}+(-1)^k c_2^{k-1}}\sum_{i=0}^{k-2}(-1)^i c_1^{k-2-i}c_2^iA^{k-2-i}\left(A^{k-1}B\right)^i + \frac{1}{c_2}\left(I_n - A^{k-1}\right)B^{k-2}.$$

Corollary 4. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be commuting k-potent matrices, and let* $c_1, c_2\epsilon\mathbb{C}\setminus\{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. *Then,* $c_1A + c_2B$ *is nonsingular if and only if* $A + B$ *is nonsingular.*

Lemma 1. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *be a k-potent matrix, and let* $c_1, c_2\epsilon\mathbb{C}, c_1 \neq 0, c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. *Then* $c_1 I_n + c_2 A$ *is nonsingular and:*

$$(c_1 I_n + c_2 A\,)^{-1} = \frac{1}{c_1^{k-1}+(-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i A^i + \frac{1}{c_1}\left(I_n - A^{k-1}\right).$$

Theorem 11. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices, and let* $c_1, c_2\epsilon\mathbb{C}\setminus\{0\}, c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. *If* $AB = B$ *or* $BA = B$, *then* $c_1 A + c_2 B$ *is nonsingular if and only if* A *is nonsingular. Furthermore,*

$$(c_1 A + c_2 B)^{-1} = A^{-1}\left(\frac{1}{c_1^{k-1}+(-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i B^i + \frac{1}{c_1}\left(I_n - B^{k-1}\right)\right).$$

Corollary 5. *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices, and let* $c_1, c_2\epsilon\mathbb{C}\setminus\{0\}, c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. *If* $AB = B$ *or* $BA = B$, *then* $c_1 A + c_2 B$ *is nonsingular if and only if* $A + B$ *is nonsingular.*

3. INVERTIBILITY OF A LINEAR COMBINATION OF THREE K-POTENT MATRICES

Now, we study the invertibility of a linear combination of three k-potent matrices.

Theorem 12. *Let* $A\epsilon\mathbb{C}^{n\times n}$ *and* $B, C\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices such that* $AB = 0 = BA, AC = 0 = CA$ *and* $BC = CB$, *and let* $c_1, c_2, c_3\epsilon\mathbb{C}\setminus\{0\}$ *such that* $c_2^{k-1} + (-1)^k c_3^{k-1} \neq 0$. *Then* $c_1 A + c_2 B + c_3 C$ *is nonsingular if and only if* $A^{k-1} + (B + C)\left(I_n - A^{k-1}\right)$ *is nonsingular. Furthermore,*

$$(c_1 A + c_2 B + c_3 C)^{-1} = c_1^{-1} A^{k-2} + \left[(c_2 B + c_3 C)\left(I_n - A^{k-1}\right)\right]^{\#}. \tag{4}$$

Proof. Let $A\epsilon\mathbb{C}_r^{n\times n}$ be of the form (1). Since $AB = 0 = BA$, then B has the form (3). Suppose that $C\epsilon\mathbb{C}^{n\times n}$ has the next representation:

$$C = U\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} U^{-1},$$

where $C_1\epsilon\mathbb{C}^{r\times r}$ and $C_4\epsilon\mathbb{C}^{(n-r)\times(n-r)}$. From $AC = 0 = CA$, it follows that $C_1 = C_2 = C_3 = 0$, i.e.

$$C = U\begin{bmatrix} 0 & 0 \\ 0 & C_4 \end{bmatrix} U^{-1},$$

where $C_4\epsilon\mathbb{C}^{(n-r)\times(n-r)}$ is the *k*-potent matrix because C is the *k*-potent matrix. Now, $c_1 A + c_2 B + c_3 C$ can be represented as:

$$c_1A + c_2B + c_3C = U\begin{bmatrix} c_1K & 0 \\ 0 & c_2B_4 + c_3C_4 \end{bmatrix}U^{-1},$$

where $B_4, C_4 \epsilon \mathbb{C}^{(n-r)\times(n-r)}$ are k-potent matrices. Since $BC = CB$, then $B_4C_4 = C_4B_4$. Based on the invertibility of K, we conclude that $c_1A + c_2B + c_3C$ is nonsingular if and only if $c_2B_4 + c_3C_4$ is nonsingular for all constants $c_1, c_2, c_3 \epsilon \mathbb{C} \setminus \{0\}$. By Corollary 4, we deduce that $c_2B_4 + c_3C_4$ is nonsingular if and only if $B_4 + C_4$ is nonsingular for all constants $c_2, c_3 \epsilon \mathbb{C} \setminus \{0\}$ such that $c_2^{k-1} + (-1)^k c_3^{k-1} \neq 0$. Furthermore,

$$A^{k-1} + (B + C)(I_n - A^{k-1}) = U\begin{bmatrix} I_r & 0 \\ 0 & B_4 + C_4 \end{bmatrix}.$$

Thus, the necessary and sufficient condition of the invertibility of $c_1A + c_2B + c_3C$ is invertibility of $A^{k-1} + (B + C)(I_n - A^{k-1})$.
In addition, by direct computation, we get

$$\begin{aligned}(c_1A + c_2B + c_3C)^{-1} &= U\begin{bmatrix} c_1^{-1}K^{-1} & 0 \\ 0 & (c_2B_4 + c_3C_4)^{-1} \end{bmatrix}U^{-1} \\ &= U\begin{bmatrix} c_1^{-1}K^{k-2} & 0 \\ 0 & (c_2B_4 + c_3C_4)^{-1} \end{bmatrix}U^{-1}.\end{aligned}$$

Note that $(c_2B + c_3C)(I_n - A^{k-1}) = U\begin{bmatrix} 0 & 0 \\ 0 & (c_2B_4 + c_3C_4)^{-1} \end{bmatrix}U^{-1}$.

Hence, the formula (4) holds. □

Corollary 6. *Let $A \epsilon \mathbb{C}^{n\times n}$, and $B, C \epsilon \mathbb{C}^{n\times n}$ be k-potent matrices such that $AB = 0 = BA$, $AC = 0 = CA$ and $BC = CB$, and let $c_1, c_2, c_3 \epsilon \mathbb{C} \setminus \{0\}$ such that $c_2^{k-1} + (-1)^k c_3^{k-1} \neq 0$. Then, $c_1A + c_2B + c_3C$ is nonsingular if and only if $A + B + C$ is nonsingular.*

The next results are proved in [11].

Theorem 13. [11] *Let $A \epsilon \mathbb{C}_r^{n\times n}$ and $B, C \epsilon \mathbb{C}^{n\times n}$ be k-potent matrices such that $AC = 0 = CA$ and $BC = CB$, and let $c_1, c_2, c_3 \epsilon \mathbb{C} \setminus \{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$.*

(i) If $AB = A^s$, $s \epsilon \{0, 1, 3, \dots, k-2\}$, and $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, then $c_1A + c_2B + c_3C$ is nonsingular if and only if $A^{k-1} + (B + C)(I_n - A^{k-1})$ is nonsingular. Furthermore,

$$(c_1A + c_2B + c_3C\,)^{-1} = A_1 - \tfrac{1}{c_2}\left[(c_2B + c_3C)\left(I_n - A^{k-1}\right)\right]^{\#},$$

where:

$$A_1 = \frac{1}{c_1^{k-1} + (-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i A^{k-2-i+(s-1)i}.$$

(ii) If $AB = A^2$, *then* $c_1A + c_2B + c_3C$ *is nonsingular if and only if* $A^{k-1} + (B + C)\left(I_n - A^{k-1}\right)$ *is nonsingular. Furthermore,*

$$(c_1A + c_2B + C_3C)^{-1} = \tfrac{1}{c_1+c_2}A^{k-2} - \tfrac{1}{c_2(c_1+c_2)}\left[(c_2B + c_3C)\left(I_n - A^{k-1}\right)\right]^{\#}\left(I_n - A^{k-1}\right)BA^{k-1}A^{k-2} + \left[(C_2B + C_3C)\left(I_n - A^{k-1}\right)\right]^{\#}.$$

Corollary 7. *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B, C\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices such that* $AC = 0 = CA$ *and* $BC = CB$, *and let* $c_1, c_2, c_3\epsilon\mathbb{C} \setminus \{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$.

(i) If $AB = A^s$, $s\epsilon\{0, 1, 3, \ldots, k-2\}$, *and* $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, *then* $c_1A + c_2B + c_3C$ *is nonsingular if and only if* $A + B + C$ *is nonsingular.*

(ii) If $AB = A^2$, *then* $c_1A + c_2B + c_3C$ *is nonsingular if and only if* $A + B + C$ *is nonsingular.*

Theorem 14. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B, C\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices such that* $AC = 0 = CA$, *and let* $c_1, c_2, c_3\epsilon\mathbb{C} \setminus \{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. *If* $AB = B$ *or* $BA = B$, *then* $c_1A + c_2B + c_3C$ *is nonsingular if and only if* $A^{k-1} + C\left(I_n - A^{k-1}\right)$ *is nonsingular. Furthermore,*

$$(c_1A + c_2B + c_3C)^{-1} = A^{k-2}\left(\tfrac{1}{c_1^{k-1}+(-1)^k c_2^{k-1}} \textstyle\sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i B^i + \tfrac{1}{c_1}\left(I_n - B^{k-1}\right)\right) + \tfrac{1}{c_3}C^{k-2}.$$

Corollary 8. *Let Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B, C\epsilon\mathbb{C}^{n\times n}$ *be k-potent matrices such that* $AC = 0 = CA$, *and let* $c_1, c_2, c_3\epsilon\mathbb{C} \setminus \{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$. *If* $AB = B$ *or* $BA = B$, *then* $c_1A + c_2B + c_3C$ *is nonsingular if and only if* $A + B + C$ *is nonsingular.*

Theorem 15. [11] *Let* $A\epsilon\mathbb{C}_r^{n\times n}$ *and* $B, C\epsilon\mathbb{C}^{n\times n}$ *be commuting k-potent matrices such that* $AC = 0 = CA$, *and let* $c_1, c_2, c_3\epsilon\mathbb{C} \setminus \{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, *and* $c_2^{k-1} + (-1)^k c_3^{k-1} \neq 0$. *Then,* $c_1A + c_2B + c_3C$ *is nonsingular if and only if* $A^{k-1} + \left(I_n - A^{k-1}\right)(B + C)$ *is nonsingular. Furthermore,*

$$(c_1A + c_2B + c_3C)^{-1} = \frac{1}{c_1^{k-1}+(-1)^k c_2^{k-1}} \sum_{i=0}^{k-2} (-1)^i c_1^{k-2-i} c_2^i A^{k-2-i} \left(A^{k-1}B\right)^i + \frac{1}{c_2}\left(I_n - A^{k-1}\right) B^{k-2} + \left[(c_2B + c_3C)\left(I_n - A^{k-1}\right)\right]^{\#}.$$

Corollary 9. *Let $A \epsilon \mathbb{C}_r^{n\times n}$ and $B, C \epsilon \mathbb{C}^{n\times n}$ be commuting k-potent matrices such that $AC = 0 = CA$, and let $c_1, c_2, c_3 \epsilon \mathbb{C} \setminus \{0\}$, $c_1^{k-1} + (-1)^k c_2^{k-1} \neq 0$, and $c_2^{k-1} + (-1)^k c_3^{k-1} \neq 0$. Then, $c_1A + c_2B + c_3C$ is nonsingular if and only if $A + B + C$ is nonsingular.*

4. Conclusions

The paper provided investigations of the invertibility of linear combinations of k-potent complex matrices. Several new properties of the invertibility of k-potent matrices are identtified. Furthermore, some results in the literature are restablished. The most important conclusion is that the invertibility of the linear combination of k-potent matrices is equivalent to the invertibility of the sum of given matrices. Thus, the invertibility of the linear combination of two or three k-potent matrices is independent of the choice of the nonzero complex constants.

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STATISTICAL ANALYSIS OF REPAIR RATE FOR MAINTENANCE DECISION-MAKING

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Abstract The most important part of any maintenance process is to achieve the optimal level of system availability. Having reviewed the available literature, we noticed that the availability mostly depends on the number of spare parts in stock, reliability, and repair time. Due to that fact, in this paper, we are analyzing the stochastic nature of the repair process. The aim is to determine the repair time for the required level of availability. First, we analyze the repair time of a single component. The final equation of the mathematical model that we present herein is the probability density function of the repair rate which allows us to determine the repair time for the related level of availability and mean time to failure. Further, based on this equation, we present the approach for the determination of the repair time of a series system comprised of two components. The model's output can be used in making important decisions such as the planning of maintenance activities, capacity, labor planning, etc.

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1. INTRODUCTION

Maintenance comprises a set of procedures and methods for keeping the system in an operational state or returning the system to that state after failure [1]. So, when the component or system fails, it needs to be repaired or replaced. Each of these activities requires a certain time for their implementation. This period is usually called downtime. Since many factors influence the delay duration, they can be divided into waiting and active downtimes. Waiting downtimes are delays that occur due to waiting for spare parts, administrative procedures, deliveries, staff, etc. Active downtimes are used to repair or replace a component or system, so the repair time can be observed as a random variable. In this paper, we will observe only repairable systems i.e. systems that can be returned to their functional state with certain activities, after the occurrence of failure.

The key performance measures of both repairable and non-repairable systems are availability and reliability. Availability is defined as the probability that a system will perform its function in a time [2]. Conversely, reliability is defined as the probability that the equipment or the system will complete a specific task under specified conditions for a stated time [3]. The main goal of each maintenance system (MRO) is to achieve the desired level of availability and for that purpose, maintenance contracts have been used. Their characteristic is that no specific maintenance activities such as servicing, repairs, and required materials are paid for, but only performances of the system resulting from the undertaken maintenance activities. This concept originates from the military industry i.e. it is related to the maintenance of military aircraft and weapon systems. Maintenance contracts have also found their use in civilian companies, under the name Performance Based Contracts (PBC) [4]. In practice, it comes down to this, when the component under a PBC contract is serviced, maintenance is not charged by the number of working hours used for engine repair or by the number of used spare parts, but by the time in which the airplane is available after repairs i.e. number of hours the engine is in an operational state. [5]

Kang et al. [6] have observed systems whose maintenance was regulated with maintenance contracts. They concluded that the mean time to failure (MTTF), mean time to repair (MTTR), and the number of spare parts impact availability. Evaluating the availability of a certain component or system is a common topic in related literature. Inherited availability and methods for its evaluation in repairable systems have been researched in papers [7 - 9]. Papers [10-13] contributed significantly to determining the availability of repairable systems and components under maintenance contracts. A similar issue was researched in the paper [14] and it was concluded that repair time and reliability have a significantly greater effect on the system's availability than the number of spare parts in inventory.

Based on the previous, this paper presents the model for the repair rate of one component and the series system comprised of two components.

2. MODELING OF REPAIR RATE

Stochastic modeling of the component or system repair time is not new and has been already presented in the paper [15]. The method presented in [16] observes the repair rate as a stochastic process and aims to determine this parameter for the preferred level of unit availability. Only repairable components and systems were considered, actually the systems that alternate between successive up and down intervals. We assumed that at the start system is operative. It remains in that state for a certain time T (failure time), then it stops operating for time R (repair time) and after being repaired system is back in the operative state. This process of failure and repair will repeat. In the literature, this approach is known as the alternating renewal process [17] and it is defined as a sequence of independent and non-negative random variables. In this case, the random variables are the times-to-failure and the times-to-repair/restore. Each time a component fails and is restored to working order, a renewal occurred.

We also assumed that perfect repair has been carried out at a constant rate after which the system behaves the same as the new one. As already stated, the main purpose of maintenance contracts is to optimize system availability and in the case when we have a maintenance contract, inherited or steady state availability is often used availability measure. The steady-state availability is inherited availability when considering only the corrective downtime of the system and is equal to:

$$A(\infty) = \lim_{t\to\infty} A(t) \tag{1}$$

According to the key renewal theorem the limited probability that the system is available can be expressed as the ratio of the mean of the period when the system is operative and the mean of the period which represents one renewal cycle [18]:

$$A = \lim_{t\to\infty} A(t) = \frac{E[T]}{E[T]+E[R]} = \frac{MTTF}{MTTF+MTTR}, \tag{2}$$

where E is the expected value operator. Further, $MTTF$ is a random variable and if there exists probability density function $p(t)$, then the $MTTF$ can be defined as:

$$MTTF = \int_0^\infty tp(t)\,dt. \tag{3}$$

We assumed that failure time has Rayleigh distribution so the probability density function is:

$$p(t) = {2t}/{x} \exp\left(-{t^2}/{x}\right) \tag{4}$$

where the distribution parameter x is determined by relation $E(t^2)= x$.
By replacing (4) with (3) the equation for $MTTF$ is:

$$MTTF = \int_0^\infty \frac{2t^2}{x} \exp\left(-\frac{t^2}{x}\right) dt. \tag{5}$$

From the maintenance theory, we know that the rate of repair μ can be observed as a reciprocal value of $MTTR$ as in Eq. (6)

$$\mu = 1/MTTR. \tag{6}$$

Based on the all presented facts in the paper [16] and according to the previous, the probability density function (PDF) of the repair rate can be stated as:

$$p(\mu) = \frac{8A^2}{(1-A)^2 \mu^3 \pi x_0} \exp\left(\frac{-4A^2}{(1-\mathrm{A})^2 \mu^2 \pi x_0} \right), \quad (7)$$

while the CDF can be calculated as:

$$F(\mu) = \int_0^{\mu_r} p(\mu)\,d\mu = 1 - \exp\left(\frac{-4A^2}{\left(1-\mathrm{A}^2\right) \mu^2 \pi x_0} \right). \quad (8)$$

Further, we will observe the repair rate of the series system of two components. If one component fails, the system will fail too. This system can be illustrated in Fig. 1

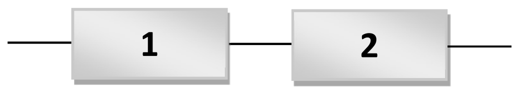

Fig1. The series system of two components

The repair rate of such a system can be expressed as $\mu_s = \mu_1 \cdot \mu_2$, so the PDF function of the system repair rate is:

$$\mathrm{p}(\mu_s) = \int_0^{+\infty} \frac{1}{\mu_2} \mathrm{p}_1\left(\frac{\mu_s}{\mu_2} \right) \mathrm{p}_2(\mu_2)\, d\mu_2 . \quad (9)$$

Since the repair rate of part 1 is $\mu_1 = \dfrac{\mu_s}{\mu_2}$ then the PDF function of the first part is:

$$p\left(\frac{\mu_s}{\mu_2} \right) = \frac{8A_1^2}{(1-A_1)^2 \left(\frac{\mu_s}{\mu_2} \right)^3 \pi x_{01}} \exp\left(\frac{-4A_1^2}{(1-A_1)^2 \left(\frac{\mu_s}{\mu_2} \right)^2 \pi x_{01}} \right). \quad (10)$$

Similarly, the PDF of the second part is:

$$p(\mu_2) = \frac{8A_2{}^2}{(1-A_2)^2 \mu_2^3 \pi x_{02}} \exp\left(\frac{-4A_2^2}{(1-A_2)^2 \mu_2^2 \pi x_{02}} \right). \quad (11)$$

Now, we can determine the PDF of the series system repair rate as:

$$p(\mu_s) = \frac{64A_1A_2}{\left(1-A_1^2\right)\left(1-A_2^2\right) \pi x_{01} x_{02} \mu_s^3} \int_0^{+\infty} \frac{1}{\mu_2} \exp\left(\frac{-4A_1^2 \mu_2}{\left(1-A_1^2\right) \mu_s^2 \pi x_{01}} + \frac{-4A_2^2}{\left(1-A_2^2\right) \mu_2^2 \pi x_{02}} \right) d\mu_2 . \quad (12)$$

By solving the integral in the previous in Eq., we get the following:

$$p(\mu_s) = \frac{64A_1A_2}{\left(1-A_1^2\right)\left(1-A_2^2\right)\pi x_{01}x_{02}\mu_s^3} K_0\left(\frac{8A_1A_2}{\mu_s\pi\sqrt{\left(1-A_1^2\right)\left(1-A_2^2\right)x_{01}x_{02}}}\right), \tag{13}$$

where K_0 is BesselK function [19].

NUMERICAL RESULTS

In this section, the model presented in the previous section will be tested for two components. We assumed the failure rates λ of both components are known ($\lambda_1 = 1.5$ and $\lambda_2 = 2$).

Numerical analysis was conducted to calculate the annual expected time for the repair to acquire availability of A=0.85, A=0.9, A=0.95 by emphasizing the stochastic nature of this process. A similar analysis can also be conducted for other values of parameter A.

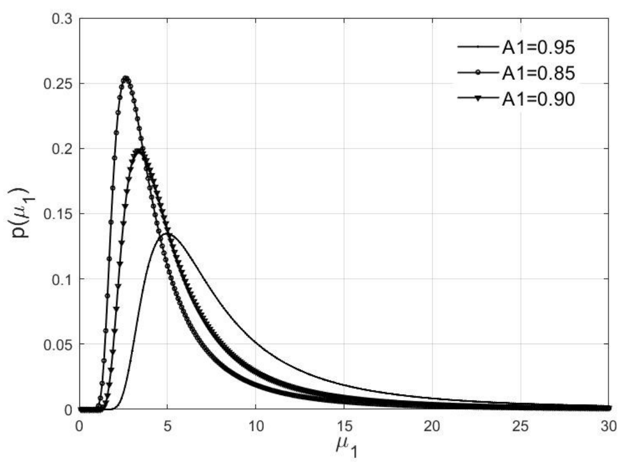

Fig 2. PDF function of the first component (λ=1.5)

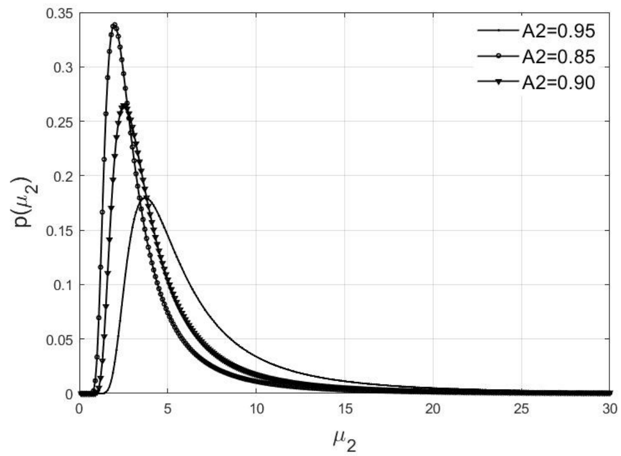

Fig 3. PDF function of the second component (λ=2)

Fig.2 represents the probability of repair rate of the first part depending on time for cases when it is expected that availability of this component is 85%, 90%, and 95%. Likewise, Fig.3 provides data related to the second part.

Finally, we have determined the repair rate of the series system comprised of these two parts based on Eq. (13). The results have been illustrated in Fig4.

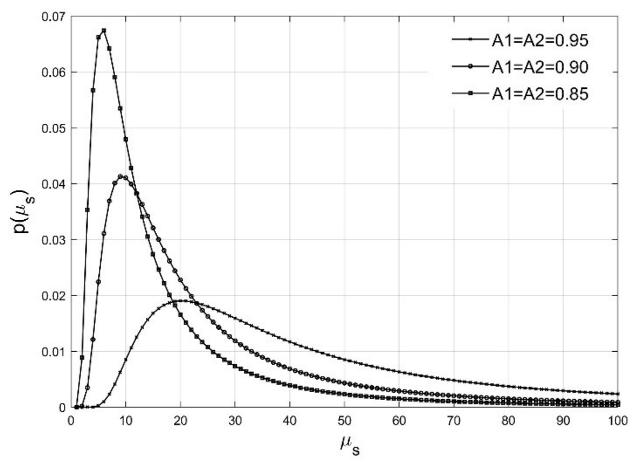

Fig 4. The repair rate of the series system comprised two components

The time necessary to repair such a system, as can be seen in Fig. 4, is known to be longer than the time necessary to repair individual parts.

5. CONCLUSION

Related research on repairable systems' maintenance processes showed that reliability and repair rate have a significant impact on the availability of such systems. In this paper, we examined a repairable system that can be modeled with an alternating renewal process. First, we observed separate units/components of such a system. We assumed that the failure rate is Rayleigh distributed and that the *MTTF* is a predetermined value. Also, after repairs, the unit returned to its original state and performed as new. By observing repair time as a stochastic process, we presented the exact expressions for the repair rate's PDF. After determining the repair rate characteristics of a single unit or subsystem, we calculated the PDF of a series system of two components. In the Numerical section, the proposed model was applied to the system comprised of two components. The PDF of the repair rate for each component was graphically presented as well as the PDF of the series system. Based on this information we can conclude in which time interval maintenance action should be completed to achieve the desired level of availability. Even though we set availability on certain levels, the numerical analysis can be repeated with different values of availability. This model can be applied in the same manner to other repairable systems with the alternating renewal process. The obtained results can be used in the planning of maintenance activities, inventory, service systems, and the number of required employees, in the process of system maintenance.

COMPETING INTERESTS

The authors have declared that no competing interests exist.

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A PARTICULAR SOLUTION TO THE SPECIAL CASE OF A FOURTH-ORDER SHORTENED LORENZ SYSTEM

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Abstract. In this paper, from the expanded class of the second-order linear differential equations, a subclass of the second-order linear differential equations will be obtained. For this subclass, a new condition for reductability according to Frobenius, as well as explicit formulas of its particular solution will be received. This subclass of the second-order linear differential equations and its particular solution, for obtaining a particular solution of the special case of the fourth-order shortened Lorenz system which was obtained from the Modified Lorenz system will be applied.

1. INTRODUCTION

For the class of second-order linear differential equations of B.S. Popov in [1] necessary and sufficient condition for reductable according to Frobenius is obtained. In mathematical literature [2-7] the following theorem is known:

Theorem 1. Let the differential equation

$$(Ax^2+Bx+C)y''+(Dx+E)y'+Fy=0, \quad A,B,C,D,E,F\in\mathbb{R} \qquad (1)$$

is given. The differential equation (1) is integrable if there exists an integer $n\in\mathbb{Z}$ (the smallest number after absolute value if there are such numbers) that satisfies the condition

$$n(n-1)A+nD+F=0 \qquad (2)$$

In doing so, the differential equation (1) has a particular solution which is given by the formula

$$y_p(x)=P_n(x)=(Ax^2+Bx+C)e^{-\int\frac{Dx+E}{Ax^2+Bx+C}dx}[(Ax^2+Bx+C)^{n-1}e^{\int\frac{Dx+E}{Ax^2+Bx+C}dx}]^{(n)} \qquad (3)$$

if $n\in\mathbb{N}$ (a polynomial solution).

But, if $n\in\mathbb{Z}^-, k=-(n+1)\in\mathbb{N}$ then a particular solution will be given by the formula

$$y_p(x)=[(Ax^2+Bx+C)^{k+1}e^{-\int\frac{Dx+E}{Ax^2+Bx+C}dx}]^{(k)} \qquad (4)$$

The Lorenz system in mathematical literature (e.g. [8-20]) is already known. Its explicit solutions are unknown and its behavior is analyzed through graphical visualization (e.g. [8-14]). It has the following form

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$$\begin{gathered}\dot{x} = \sigma(y - x) \\ \dot{y} = x(r - z) - y \\ \dot{z} = xy - bz \\ \sigma, r, b > 0\end{gathered}$$

and initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0)$. The Modified Lorenz system in [21] with initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0)$, $z_p = \overset{(p)}{z}(0)$, $p \in \{1,2,3,4\}$

$$\begin{gathered}\dot{x} = \sigma(y - x) \\ \dot{y} = x(r - z) - y \qquad (5) \\ \overset{(5)}{z} = -(A + b)\overset{(4)}{z} + (B - Ab)\dddot{z} - (C - Bb)\ddot{z} + (D - Cb)\dot{z} + Dbz \\ \sigma, r, b > 0,\ A = 1 + \sigma + b, B = \sigma(r - z_0) - x_0{}^2, C = \sigma\, x_0 y_0, D = -\sigma^2 y_0{}^2\end{gathered}$$

is presented.

The third equation of the Modified Lorenz system is a five-order linear homogeneous differential equation with the constant coefficients. Its characteristic equation is

$$m^5 + (A + b)m^4 - (B - Ab)m^3 + (C - Bb)m^2 - (D - Cb)m - Db = 0 \qquad (6)$$

which solutions are m_1= $-b$, $m_{2/3/4/5}$=$k(A,B,C,D,b)$. The explicit solutions of the Modified Lorenz system in [21] for any value of the parameters $\sigma, r, b > 0$ σ>0 and initial values x_0, y_0, z_0 are obtained.

By using of two solutions from the solutions $m_{1/2/3/4/5}$ of the equation (5), the 7th order Modified Lorenz system (5) in [22] is transformed in a fourth-order subsystem Modified Lorenz system

$$\begin{gathered}\dot{x} = \sigma(y - x) \\ \dot{y} = x(r^* - z) - y \qquad (6) \\ \dot{z} = u \\ \dot{u} = (m_1 + m_2)u - m_1 m_2 z \\ \sigma, r^* > 0, m_1, m_2 \in \mathbb{R}\end{gathered}$$

with the initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0),\ u_0 = u(0) = \dot{z}(0) = z_1$ .The fourth-order subsystem from the Lorenz system will be called fourth-order shortened Lorenz system. For this system (6) in [23] is done dynamical analysis. When the system condition

$$m_1, m_2 = 2m_1 \in \mathbb{R} \qquad (*)$$

is true, we will speak for a special case of a fourth-order shortened Lorenz system (6).

Remark 1. By the notation r^* in the fourth-order shortened Lorenz system (6), the parameter *r* from Modified Lorenz system (5) has been replaced. Because the notation *r* will be used with another meaning in this paper.

The explicit solutions of the Modified Lorenz system (5) from [21] can be used for solving of the fourth-order shortened Lorenz system (6). But, these solutions are complex for use. In this paper under specific conditions with proving integrability of a subclass of differential equations from the extended class linear differential equations [24] which are presented in [1], we will be offered simpler obtaining of a particular solution for a special case of the fourth-order shortened Lorenz system (6).

Remark 2. In the paper [25] in the same way, a particular solution of the third-order shortened Lorenz system via integrability of a class of differential equations is already offered. Therefore, this paper will follow the already published paper [25].

The integrability of this extended class of differential equations gives us explicit formulas for one particular solution. A subclass from this extended class of differential equations will be obtained, which will be used for solving the fourth-order shortened Lorenz system (6).

This paper gives only theoretical access without examples, which is small, but an essential contribution to solving differential equations.

2. MAIN RESULTS

In this part, the subclass from the extended class of second-order linear differential equations of B.S. Popov is obtained, which can be used for solving of the fourth-order shortened Lorenz system (6). For this goal, the following Theorem 2 will be proved.

Theorem 2. Let the differential equation

$$\ddot{z} + \beta\dot{z} + (A_1 e^{2t} + B_1 e^t + C_1)z = 0, \quad \beta, A_1, B_1, C_1 \in \mathbb{R} \qquad (7)$$

is given. The differential equation (7) is integrable if there exists an integer $n \in \mathbb{Z}$ that satisfies the condition

$$B_1 + (\mp\sqrt{-A_1})[2n + 1 + (\mp\sqrt{\beta^2 - 4C_1})] = 0, A_1 < 0 \quad (8)$$

In doing so, the differential equation (7) has a particular solution which is given by the formula

$$z_p(t) = e^{-\int (re^t + s)dt} y_p(e^t) \qquad (9)$$

where

$$y_p(x) = P_n(x) = x^{1-E} e^{-Dx} [x^{n-1+E} e^{Dx}]^{(n)} \qquad (10)$$

if $n \in \mathbb{N}$
or

$$y_p(x) = [x^{k+1-E} e^{-Dx}]^{(k)}, \qquad (11)$$

if $n \in \mathbb{Z}^-$, $k = -(n+1) \in \mathbb{N}$.
By the relations

$$\begin{aligned} &D = 2(\mp\sqrt{-A_1}), \qquad E = 1 + (\mp\sqrt{\beta^2 - 4C_1}), \qquad r = \pm\sqrt{-A_1}, \\ &s = \frac{1}{2}(\beta \pm \sqrt{\beta^2 - 4C_1}), \qquad F = B_1 + (\mp\sqrt{-A_1})[1 + (\mp\sqrt{\beta^2 - 4C_1})] \end{aligned} \qquad (12)$$

the coefficients in the formulas (9), (10) and (11) are obtained.

Proof. Let us consider the differential equation

$$xy'' + (Dx + E)y' + Fy = 0, \quad D, E, F \in \mathbb{R} \qquad (13)$$

where

$$y = y(x), y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}.$$

By the substitution

$$x = e^t \qquad (14)$$

the differential equation (13) can be written as

$$\ddot{y} + [De^t + E - 1]\dot{y} + Fe^t y = 0 \quad (15)$$

where

$$y = y(x), \dot{y} = \frac{dy}{dt}, \ddot{y} = \frac{d^2y}{dt^2}.$$

By the substitution

$$y(t) = e^{\int (re^t + s)\,dt} z(t), \quad r, s \in \mathbb{R} \qquad (16)$$

the equation (15) is transformed in the differential equation

$$\begin{aligned} &\ddot{z} + [(2r + D)e^t + 2s + E - 1]\dot{z} + [(r^2 + rD)e^{2t} + (2rs + rE + sD + F)e^t \\ &\quad + s^2 + sE - s]z = 0 \end{aligned} \qquad (17)$$

where

$$z = z(t), \dot{z} = \frac{dz}{dt}, \ddot{z} = \frac{d^2z}{dt^2}.$$

The equation (7) is equal of the equation (17) if the following relations

$$\begin{aligned}
&2r + D = 0 \\
&2s + E - 1 = \beta \\
&r^2 + rD = A_1 \qquad\qquad (18) \\
&2rs + rE + Ds + F = B_1 \\
&s^2 + sE - s = C_1
\end{aligned}$$

are satisfied. From (18), the relations (12) are obtained.
By using the Theorem 1, the equation (13) is integrable if there exists an integer $n \in \mathbb{Z}$ that satisfies the condition

$$nD + F = 0 \qquad (19)$$

In accordance with the relations (12), the condition (19) is equal to the condition (8). By using the formulas (3) and (4) from Theorem 1 applied to the equation (13), the formulas (10) and (11) are obtained. Finally, in accordance with the substitutions (14) and (16), the formula (9) is obtained. □

Remark 3. In connections (12) the sign before the roots is equal to the sign before the roots the condition (8).

By Theorem 3, the last two equations of the fourth-order shortened Lorenz system (6) for given initial values offered a particular solution.

Theorem 3. The last two equations of the fourth-order shortened Lorenz system (6) are transformed in a second-order linear homogeneous differential equation with the constant coefficients

$$\ddot{z} - (m_1 + m_2)\dot{z} + m_1 m_2 z = 0 \qquad (20).$$

The differential equation (20) with the initial values $z_0 = z(0),\ z_1 = \dot{z}(0)$, and the condition (*) for $m_1 = m$ has the particular solution

$$z_p(t) = We^{mt} + Le^{2mt}, W = \frac{2mz_0 - z_1}{m},\ L = \frac{z_1 - mz_0}{m}. \qquad (21)$$

Proof. By help of the fourth equation and differentiation of the third equation of the fourth-order shortened Lorenz system (6), a second-order linear homogeneous differential equation with the constant coefficients (20) is obtained. The characteristic equation of the differential equation (20) is

$$m^2 - (m_1 + m_2)m + m_1 m_2 = 0$$

by solutions m_1, m_2. The general solution of the differential equation (20) is

$$z(t) = We^{m_1 t} + Le^{m_2 t}, \quad W, L = \mathrm{c}onst\,.$$

Nothing is lost from generality, if we assume that $m_1 = m$. For the initial values $z_0 = z(0),\ z_1 = \dot{z}(0)$ and the condition $m_1 = m,\ m_2 = 2m_1 = 2m$, the particular solution

$$z_p(t) = We^{mt} + Le^{2mt}, \quad W = \frac{2mz_0 - z_1}{m}, L = \frac{z_1 - mz_0}{m}$$

is obtained. □

By Theorem 4, the first two equations of the fourth-order shortened Lorenz system (6) in a second-order linear differential equation are transformed.

Theorem 4. The first two equations of the fourth-order shortened Lorenz system (6) are transformed in a second-order linear differential equation

$$\ddot{x} + (\sigma+1)\dot{x} + \sigma(1 - r^* + We^{mt} + Le^{2mt})x = 0, \quad \sigma, r^* > 0, \quad m \in \mathbb{R} \quad (22)$$

where

$$x = x(t), \dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}, W = \frac{2mz_0 - z_1}{m}, L = \frac{z_1 - mz_0}{m}.$$

Proof. By help of the second equation and differentiation of the first equation of the fourth-order shortened Lorenz system (6), a second-order linear differential equation

$$\ddot{x} + (\sigma+1)\dot{x} + \sigma(1 - r^* + z_p(t))x = 0, \quad \sigma, r^* > 0$$

is obtained, where

$$x = x(t), \dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}$$

By using of the particular solution (21), the second-order linear differential equation (22) is obtained. □

The condition for integrability of the differential equation (22) in Theorem 4 is given by Theorem 5. In accordance with the formulas of Theorem 2, one particular solution is obtained by Theorem 5.

Theorem 5. Let the differential equation (22) is given. The differential equation (22) is integrable if there exists an integer $n \in \mathbb{Z}$ that satisfies the condition

$$\sigma W + (\mp\sqrt{-\sigma L})[(2n+1)m + (\mp\sqrt{(\sigma-1)^2 + 4\sigma r^*}) = 0, L < 0, m \in \mathbb{R}, \quad (23)$$

where $W = \dfrac{2mz_0 - z_1}{m}$, $L = \dfrac{z_1 - mz_0}{m}$, $z_0, z_1 \in \mathbb{R}$.

In doing so, the differential equation (20) has a particular solution which is given by the formula

$$x_p(t^*) = e^{-\int (re^{t^*}+s)dt^*} y_p(e^{t^*}) \qquad (24)$$

where

$$y_p(x) = P_n(x) = x^{1-E} e^{-Dx} [x^{n-1+E} e^{Dx}]^{(n)}$$

if $n \in \mathbb{N}$

or

$$y_p(x) = [x^{k+1-E} e^{-Dx}]^{(k)}$$

if $n \in \mathbb{Z}^-, k = -(n+1) \in \mathbb{N}$ for $t^* = mt$.

By the relations

$$D = \frac{2}{m}\left(\mp\sqrt{-\sigma L}\right), \quad E = 1 \mp \frac{1}{m}\sqrt{(\sigma-1)^2 + 4\sigma r^*}, \quad r = \frac{1}{m}\left(\pm\sqrt{-\sigma L}\right),$$

$$s = \frac{1}{2m}\left[(\sigma+1) \pm \sqrt{(\sigma-1)^2 + 4\sigma r^*}\right],$$

$$F = \frac{1}{m}\left[\frac{1}{m}\sigma W + (\mp\sqrt{-\sigma L})(1 \mp \frac{1}{m}\sqrt{(\sigma-1)^2 + 4\sigma r^*}\right]$$

the coefficients D, r, s, E, F are obtained.

Proof. By the substitution

$$mt = t^* \qquad (25)$$

the equation (20) are transformed in the differential equation

$$\ddot{x}(t^*) + \frac{1}{m}(\sigma+1)\dot{x}(t^*) + \frac{1}{m^2}\sigma(Le^{2t^*} + We^{t^*} + 1 - r^*)x(t^*) = 0 \qquad (26)$$

where

$$x = x(t), \dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}.$$

The differential equation (26) is equal by the equation (7), if the relations

$$\beta = \frac{1}{m}(\sigma+1), A_1 = \frac{1}{m^2}\sigma L, B_1 = \frac{1}{m^2}\sigma W, C_1 = \frac{1}{m^2}\sigma(1-r^*)$$

are valid.

The condition (8) of Theorem 2 applied to the equation (26) is the condition (23). By using the formula (9) of Theorem 2, formula (24) is obtained. In accordance with the substitution (25), the formula of one particular solution is obtained. □

A particular solution $(x_p(t), y_p(t), z_p(t))$ of a special case of the fourth-order shortened Lorenz system (6) is obtained by the following Theorem 6.

Theorem 6. A particular solution $(x_p(t), y_p(t), z_p(t))$ of a special case of the fourth-order shortened Lorenz system (6) when the condition (23) is satisfied is obtained as follows:

- for $x_p(t)$ with the formula (24);
- $y_p(t)=\frac{1}{\sigma}\dot{x}_p(t)+x_p(t)$ where $\dot{x}_p(t)=\frac{dx_p}{dt}$;
- for $z_p(t)$ with the formula (21).

Proof. It is clear that a particular solution $(x_p(t),y_p(t),z_p(t))$ of the fourth-order shortened Lorenz system (6) can be found in the condition (23) of Theorem 5 is satisfied. The particular solution is obtained by using the formulas for one particular solution (24) for one particular solution from Theorem 5 for $x_p(t)$, the formula (21) from Theorem 3 for $z_p(t)$ and by using the first equation of the fourth-order shortened Lorenz system (6) with $y_p(t)=\frac{1}{\sigma}\dot{x}_p(t)+x_p(t)$, where $\dot{x}_p(t)=\frac{dx_p}{dt}$. □

3. Conclusions

In this paper for the special case of the fourth-order shortened Lorenz system (6), a way for theoretically obtaining one particular solution was presented. We speak for a finding of a particular solution for a small class of systems of differential equations, but solving such a nonlinear system is complex even with a computer.

It would be good to be given an appropriate example with concrete initial values and its geometrical visualization. But, the choice of such an example is a difficult and complex process even with a computer.

Therefore, this paper gives only theoretical access, which is an essential contribution to solving differential equations.

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APPLICATION OF MARKOV CHAINS IN BIOLOGY

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Abstract. The Markov chain is a random process with Markov characteristics, which exists in the discrete index set and state space in probability theory and mathematical statistics. Markov chains are powerful tools for stochastic modelling that can be useful in any science discipline. In this paper, we give an overview of some basic applications of the Markov chains in biology. We will describe the application in crossbreeding of the animals in close relation and carcinogenesis**.**

1. INTRODUCTION

Markov Chain is a powerful and effective technique to model stochastic processes with discrete time space and states space. Markov chains can be used to model many real-life processes. They have very wide applications in various fields such as: physics, chemistry, biology, medicine, music, game theory, sports, economics etc. They can be used for animal life populations mapping, to search engine algorithms, for music composition, speech recognition etc., [1].

In chemistry, Markov chains are used when physical systems closely approximate the Markov property. More chemical reactions can be considered as Markov chains. The reaction networks can be modelled with Markov chains, also the model of enzyme activity, Michaelis–Menten kinetics, can be viewed as a Markov chain. There are many advantages of using the Markov-chain model in chemistry. Some advantages are: physical models can be presented by state vector and a one-step transition probability matrix, it is easy to obtain all distributions of the state vector from the Markov-chain solution. Also, with Markov chain can be modelled various processes in chemical engineering by combination of flows, recycle streams, plug-flow and perfectly mixed reactors [2,3].

Claude Shannon, the father of Information theory, used Markov chains to model the English language. Through this model, he introduced the concept of entropy. In this language model, he assumed that letters from some text have a certain degree of randomness and are dependent on each of others. Also, this Markov model allow to produced text similar to text written to English language. Hence Markov models are widely used in Natural Language Processing and Computational Linguistics. Markov model can be used for effective data compression through entropy encoding techniques. Even without describing the full structure of the system perfectly, such signal models can make possible very effective data compression through entropy encoding techniques such as arithmetic coding [4].

Also, Markov chains are a base for hidden Markov models (HMM). These models are used in telephone networks, speech recognition and bioinformatics [5].

Process of birth and death that are basis of queueing theory are homogeneous Markov process. More of the queue systems (M/M/n, M/m/,∞) can be modelled

by using Markov process or Markov chain. For example, for the queue M/M/m, the time spent by a client in the queue is a Markov process and the number of clients in the queue is a Markov chain [6,7].
The Markov chain has applications in Internet applications. For example, Google's PageRank algorithm of a webpage is defined by a Markov chain [8].
In economics and finance, Markov chains are used to predict macroeconomic situations like market crashes and cycles between recession and expansion. Other areas of application include predicting asset and option prices and calculating credit risks [9].
In this paper, we will consider the application of Markov chains in biology. Concretely, we will regard two applications of Markov chains: the genetic problem of interbreeding animals in close relatives and application in carcinogenesis.

2. RANDOM (STOCHASTIC) PROCESS

The neediness for introducing the concept of stochastic (random) process follows from the work of different systems and variables that are random by their nature. Also, those variables depend on one or more parameters such as time, length, elevation, and others. Stochastic processes are widely used as mathematical models of systems and phenomena that appear to vary in a random manner. The variables that are considered by meteorological research like: temperature, humidity, pressure, concentration of smoke, sulfur dioxide, winds speed on the certain place are functions of the time, latitude, altitude of the place, are random.

Brownian motion of particles, voltage and power of electric current, number of car accidents, number of earthquakes, the speed of the vehicles etc., are the real processes that are random functions of the time [10,11].

Let (Ω, F, P) is a probability space and nonempty parameter set T. The stochastic or random process $\{X_t, t \in T\}$ is usually defined as a family of random variables.
Depending on the parameter set T, random process can be:

1. Random process with discrete parameter set (discrete random sequence).
2. Random process with continuous parameter set.

If the distribution of random variables $\{X_t, t \in T\}$ is considered, then random process can be:

1. Discrete random process.
2. Continuous random process.

Usually, T is one-dimensional set and the parameter can be interpreted as **time**.

For each fixed $t \in T, X_t$ is a random variable that is called intersection of process at the moment t. For fixed $\omega \in \Omega, X_t(\omega)$ is a function of t, that is called realization (trajectory) of the random process $\{X_t, t \in T\}$.

Definition 2.1 A random process $\{X_t, t \in T\}$ is said to be Markov, if for each $n \in \mathbb{N}, t_1 < t_2 < \ldots t_{n,} t_i \in T, i = 1, 2, \ldots n$ and for each $x_1, x_2, \ldots x_n$:

$$P\{X_{t_n} < x_n \mid X_{t_{n-1}} = x_{n-1}, X_{t_{n-2}} = x_{n-2}, \ldots, X_{t_1} = x_1\} =$$
$$= P\{X_{t_n} < x_n \mid X_{t_{n-1}} = x_{n-1}\}$$

Because of that, it is said that a random process $\{X_t, t \in T\}$ is a Markov if the "future" state X_{t_n} is independent of the "past state" $X_{t_{n-2}}$, if the "present state" $X_{t_{n-1}}$ is given.

For discrete Markov process the following holds:

$$P\{X_{t_n} = x_n \mid X_{t_{n-1}} = x_{n-1}, X_{t_{n-2}} = x_{n-2}, \ldots, X_{t_1} = x_1\} =$$
$$= P\{X_{t_n} = x_n \mid X_{t_{n-1}} = x_{n-1}\}.$$

A discrete Markov process with a discrete set of states and a discrete parameter set is called a Markov chain, i.e.

Defeniton 2.2 A discrete random process $\{X_i\}_{i=0}^{\infty}$ is said to be Markov chain, if for each $n \in \mathbb{N}$, and for each $x_1, x_2, \ldots x_n$, the following holds:

$$P\{X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_1 = x_1\} =$$
$$= P\{X_{n+1} < x_{n+1} \mid X_n = x_n\}.$$

The random process is a chain when the state space is discrete. The name Markov refers to Andrei. A. Markov, a Russian mathematician, who works described the Markov chains.

Definition 2.3 The one step transition probability, denoted as $p_{ij}^{(n)}$ is defined by the condition probability:

$$p_{ij}^{(n)} = P\{X_n = j \mid X_{n-1} = i\}.$$

This is the probability that process is in state j at the moment n, given that the process was in state i at the moment $n-1$. The probability $p_{ij}^{(n)}$ are called transition probability from state i to state j at the moment n. These probabilities form matrix of transition probabilities $\mathbf{P}^{(n)} = \left[p_{ij}^{(n)} \right]$.

3. Application of discrete Markov Chains in biology

The discrete Markov chains have wide application in biology.
With Markov chain the general process of birth and death in discrete time can be modelled. In this model is assumed that the size of the population is maximal. The theory developed from the random walk can be used for analysing the birth and death process. This theory is used to analyse, the probability of population extinction, the expected time of population extinction, and the distribution conditioned on nonextinction, known as the quasistationary distribution [12].
The other application of Markov chains in discrete time are the epidemiology models. Discrete Markov chains can be used for modelling of some chain based as SI, SIS and SIR models in epidemiology. The Susceptible-Infected-Susceptible (SIS) model describes the transmission of disease when recovered individual from the population do not have permanent immunity. Recovered individual can immediately become infectious again. The results for SIS model in [13] show normal distribution nature of the quasi-stationary distribution in the case when the population is large, and the reproduction number is greater than 1.
With the SIR model population is divided into three subgroups: susceptible, infected, and recovered individuals. In this model a susceptible individual gets infected with disease and recovers from it and have a permanent immunity. The main aim of this model is to predict the trajectory of epidemic transmission. The transitions are made from one to another population [14-16].
The other application of discrete Markov chains in epidemiology is known as the binomial chain model.
Epidemiological models of binomials chains were first developed in 1920 and 1930 by Reed, Frost and Greenwood, so according to them is the model named. For these models, the duration and extent of the epidemic are calculated [17].
Also, discrete Markov chain is used to proliferating epithelial cells.
In this paper, two classical biological applications of Markov chains in discrete time will be considered [17].

3.1 The genetic problem of interbreeding animals in close relatives

Genetics is an important area in the biology which studies genes, genetic variation, and heredity. It studies how living organisms receive common characteristic from previous generation. Heredity depends on the information that are contained in chromosomes. Every cell of a human being contains 46 chromosomes (23 from the mother and another 23 from the father), or 23 pairs named as ‘diploid’: 22 pairs of autosomes, and one pair of sex chromosomes, called X and Y.

Genes are arranged, one after another, at specific locations on chromosomes in the nucleus of cells. The genes are made of sequence of DNA. A location on the chromosome where the gene is found are called locus. The gene is located within a determined region on the chromosome and is composed of the different base pairs (GATC). Alleles are variants of the same gene that occur on the same place on a chromosome [18,19].

Assume that there are only two types of alleles for a given gene, denoted as a and A. A diploid individual can have one of three different allele combinations: AA, Aa or aa, known as locus genotypes. As aa and AA have the same homogenous composition, they are called homozygotes, while Aa is called heterozygotes. [20]:

In [21] the problem of genetic pairing of animals in close relatives is regarded. This process form Markov chain. Suppose that two individuals are randomly paired. Process of pairing between siblings and close relatives continues every year. This process can be formulated as a finite Markov chain in discrete time, whose states consist of 6 types of mating:

1. $AA\times AA$
2. $AA\times Aa$
3. $Aa\times Aa$
4. $Aa\times aa$
5. $AA\times aa$
6. $aa\times aa$

It is assumed that the parents are type 1, $AA\times AA$. Then the next generation descendants of these parents will be AA individual, and then the pairing of siblings will be only type 1, $p_{11}=1$. Analogously for parents type 6, $aa\times aa$, where $p_{66}=1$.

Now, it is assumed that the parents are type 2, $AA\times Aa$. Let X represents their randomly chosen descendant. Let $Y_1\in\{A,A\}$ is allele that will be transmitted on to descendant from parents with genotype AA, and $Y_2\in\{A,a\}$ represents the allele that will be transmitted to descendants from parents with genotype Aa. Then the following holds:

$$P\{X=AA\}=P\{Y_1=A,Y_2=A\}=P\{Y_1=A\}P\{Y_2=A\}=1\frac{1}{2}=\frac{1}{2},$$

$$P\{X=Aa\}=P\{Y_1=A,Y_2=a\}=P\{Y_1=a\}P\{Y_2=A\}=$$

$$=P\{Y_1=1\}P\{Y_2=a\}=P\{Y_1=a\}P\{Y_2=A\}=1\frac{1}{2}+0=\frac{1}{2},$$

$$P\{X=aa\}=P\{Y_1=a,Y_2=a\}=P\{Y_1=a\}P\{Y_2=a\}=0$$

From this generations there are two genotypes. If X_1 and X_2 are two randomly chosen generations. Then:

$$P\{X_1 \times X_2 = AA \times AA\} = P\{X_1 = AA\} P\{X_2 = AA\} = \frac{1}{4}$$

$$P\{X_1 \times X_2 = AA \times Aa\} = P\{X_1 = AA\} P\{X_2 = Aa\} + P\{X_1 = aA\} P\{X_2 = AA\} = \frac{1}{2},$$

$$P\{X_1 \times X_2 = Aa \times Aa\} = P\{X_1 = Aa\} P\{X_2 = Aa\} = \frac{1}{4}.$$

The probabilities are:

$$p_{12} = \frac{1}{4}, p_{22} = \frac{1}{2}, p_{32} = \frac{1}{4}.$$

If the parents are type 3, $Aa \times Aa$ the descendant is in proportions $\frac{1}{4}AA, \frac{1}{2}Aa$ and $\frac{1}{4}aa$, then the pairing between siblings give $\frac{1}{16}$ type 1, $\frac{1}{4}$ type 2, $\frac{1}{4}$ type 3, $\frac{1}{4}$ type 4, $\frac{1}{8}$ type 5 and $\frac{1}{16}$ type 6.

The transition matrix P is:

$$P = \begin{pmatrix} 1 & 1/4 & 1/16 & 0 & 0 & 0 \\ 0 & 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1 & 0 \\ 0 & 0 & 1/4 & 1/2 & 0 & 0 \\ 0 & 0 & 1/8 & 0 & 0 & 0 \\ 0 & 0 & 1/16 & 1/4 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & | & 1/4 & 1/16 & 0 & 0 & | & 0 \\ - & | & - & - & - & - & - & - \\ 0 & | & 1/2 & 1/4 & 0 & 0 & | & 0 \\ 0 & | & 1/4 & 1/4 & 1/4 & 1 & | & 0 \\ 0 & | & 0 & 1/4 & 1/2 & 0 & | & 0 \\ 0 & | & 0 & 1/8 & 0 & 0 & | & 0 \\ - & - & - & - & - & - & - & - \\ 0 & | & 0 & 1/16 & 1/4 & 0 & | & 1 \end{pmatrix} = \begin{pmatrix} 1 & A & 0 \\ 0 & T & 0 \\ 0 & B & 1 \end{pmatrix}.$$

The Markov chain is reducible and there are three classes of communication, $\{1\},\{6\}$ and $\{2,3,4,5\}$. The first two classes are positive recurrent, and the third class is transient. The states 1 and 6 are absorbing states, $p_{ii}=1, i=1$ and $i=6$.
Let notice

$$p^n=\begin{pmatrix} 1 & A_n & 0 \\ 0 & T^n & 0 \\ 0 & B_n & 1 \end{pmatrix},$$

Where A_n and B_n are functions of T, A and B, $A_n=A\sum_{i=0}^{n-1}T^i$ and $B_n=B\sum_{i=0}^{n-1}T^i$. First T^n need to be found. Because of T belongs to the transient class, it holds that $\lim\limits_{n\to\infty}T^n=0$.
Also,

$$\lim_{n\to\infty}B_n=B(I-T)^{-1},\lim_{n\to\infty}A_n=A(I-T)^{-1}.$$

3.2 Restricted random walk and its application in carcinogenesis

The random walk is one of the most fundamental models in probability theory, demonstrating power of mathematical properties.
A one-dimensional random walk is a Markov chain with finite or infinite state space. In the simple one-dimension random walk, two movement are allowed. The movement to the right for one position, i.e. from the position x to the position $x+1$ and movement from the position x to the position $x-1$ i.e. movement to left for one position. Let p is a probability of moving to the right and q is a probability of moving to the left, [22].
A restricted random walk corresponds to a random walk in the presence of a boundary. If the state space is finite, $\{0,1,2,\ldots,N\}$ then 0 and N are boundary. If the state space is, $\{0,1,2,\ldots\}$ then boundary is in 0. There are three types of boundary behaviour: absorbed, reflected and elastic. Absorbed boundary in the $x=0$ assumes that transition probabilities for one step $p_{00}=1$. Reflected boundary in the $x=0$ assumes that transition probabilities for one step are: $p_{00}=1-p,\ p_{10}=p,\ 0<p<1$. And the elastic boundary in the $x=0$ assumes that one-step transition probabilities are: $p_{21}=p,\ p_{11}=sq,\ p_{01}=(1-s)q,\ p+q=1, p_{00}=1$ for $0<p,s<1$.
Cancer is a human genetic disease. It is caused by mutations that occurred in a more number of genes that controlling growth. Cancer is multi-stage process. The genes that caused cancer can exist from birth, increasing a chance of getting cancer. The transition of a normal cell into a cancerous cell are happened in more

steps (stages). The number of stages is a number of mutations that are required to creating a cancerous cell.
The number of stages can be regarded as the number the state in the random walk. For this reason, we will study a simple random walk model. Let $S=\{0,1,2,\ldots,N\}$ is a number of states. One movement in the random walk (when the process transits from one to other state) corresponds on the transition from one to other stage of the one cancerous cell.
The state 0 represents the stage (state) of total recovery. This model requires several successive mutations, each of which produces a clone of mutated cells. State N indicates completion of the mutation process in which malignant cells are created. [17]
Let $\{X_n\}, n \geq 0$ represents random walk, that corresponds to the mutation process. A step forward implies transition in the next stage(state). This transition is occurred with following probability: $P\{x \to x+1\} = p_x = \frac{x}{N}$. A step back implies transition in the previous stage (this is a move toward recovery) and this transition is occurred with probability: $P\{x \to x-1\} = q_x = \frac{N-x}{N}$.
The state 0 and state N are absorbing states. If the process come in this state stay here:

$$P\{N \to N\} = P\{0 \to 0\} = 1.$$

The other states different from state 0 and state N are reflecting states.
Probability the process to stay in these states is equal to zero: $P\{x \to x\} = 0$, for $x = 0,1,2,\ldots,N-1$.

Let π_0 is a stationary probability of complete recovery (the process is in state 0), π_N is a stationary probability in cancerous state and π_x is a stationary probability in state x, $1 \leq x \leq N-1$. For these probabilities following differential equation is obtained:

$$\pi_x = \frac{x}{N}\pi_{x+1} + \left(1-\frac{x}{N}\right)\pi_{x-1}, 1 \leq x \leq N-1$$

with initial conditions

$$\pi_0 = 0,\ \pi_N = 1.$$

Let $A(t) = \sum_x \pi_x t^x$ is a probability generating function for $\{\pi_x\}$. By using of simple mathematical operations, we write the differential equation in the following way:

$$\pi_x = \frac{x+1}{N}\pi_{x+1} - \frac{1}{N}\pi_{x+1} + \left(1 - \frac{x-1}{N}\right)\pi_{x-1} - \frac{1}{N}\pi_{x-1}, 1 \le x \le N-1$$

If is taken $\pi_{N+k} = 1,\ k \ge 0$,

$$A(t) = \sum_{x=0}^{\infty}\left\{\frac{x}{N}\pi_x t^{x-1} - \frac{1}{N}\pi_x t^{x-1} + \left(1-\frac{1}{N}\right)\pi_x t^{x+1} - \frac{x}{N}\pi_x t^{x+1}\right\}$$

are obtained that

$$\left(A(t)\right)^{-1}\frac{dA}{dt} = t^{-1} + (1-t)^{-1} + (N-1)(1-t)^{-1}.$$

With solving of the differential equation, the following equation is obtained:

$$\left(A(t)\right)^{-1} = Ct(1+t)^{N+1}(1-t)^{-1} = Ct\left[\sum_{x=0}^{N-1} C\binom{N-1}{x} t^x\right]\sum_{y>0} t^y,$$

where C is a constant.

With using the limit conditions,

$$1 = \sum_{x=0}^{N-1} C\binom{N-1}{y} C2^{N-1},$$

is obtained that

$$\pi_x = \sum_{x=0}^{x-1} C\binom{N-1}{y} 2^{1-N}, 0 \le x \le N.$$

After the initial process of the carcinogenesis, it is assumed the state is 1. From there:

$$\pi_N = \pi_1\binom{N-1}{0}2^{N-1} = 2^{N-1}.$$

Because of the random walk is simple, we obtain that:

$$\pi_0 = 1 - \pi_N = 1 - 2^{N-1}.$$

4.CONCLUSION

Markov chains are useful tools in statistics modeling in all fields of applied mathematics. They have great application in the modeling of natural phenomena and sciences. In this paper, we consider application of Markov chains in biology. By help, of two applications of Markov chain in biology, we can conclude that they are powerful tool for modeling of many problems in real life. In this paper, we have considered application of Markov chain in the genetic problem of interbreeding animals in close relatives and application in carcinogenesis which is very important for their analysis.

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COMMON FIXED POINTS OF TWO T_f REICH-TYPE CONTRACTIONS IN COMPLETE METRIC SPACE

Samoil Malcheski

Abstract. In this work, it considered theorems about common fixed points of two T_f Reich-type contractions in complete metric space (X,d). In doing so, it is used that the mapping T is continuous, injection and sequentially convergent, and function f is from the class Θ continuous monotonically nondecreasing functions $f:[0,+\infty)\to[0,+\infty)$ such that $f^{-1}(0)=\{0\}$, where it is additionally taken the function to be subadditive, i.e. $f(p+q)\le f(p)+f(q)$, for each $p,q\in[0,+\infty)$.

1. INTRODUCTION

In literature there are well known Banach's fixed point principle and its generalizations given by R. Kannan ([4]), S. K. Chatterjea ([7]), P.V. Koparde, B. B. Waghmode ([3]) and Reich ([10]).

In [9] S. Moradi and D. Alimohammadi generalize the result of R. Kannan, using the sequentially convergent mappings.

Then, in [1] several generalizations of the theorems of R. Kannan, S. K. Chatterjea and P. V. Koparde, B. B. Waghmode were proved, using the sequentially convergent mappings, and in 2016 in [5] with the help of sequentially convergent mappings are proven more common fixed point results for two type mappings by R. Kannan, S. K. Chatterjea and P.V. Koparde, B.B. Waghmode.

In 2010 in [8] S. Moradi and A. Beiranvand introduce the concept of T_f contractive mapping, using the Θ class of continuous monotonically nondecreasing functions $f:[0,+\infty)\to[0,+\infty)$ such that $f^{-1}(0)=\{0\}$. Note here that, if $f\in\Theta$, then from $f^{-1}(0)=\{0\}$ follows that $f(t)>0$, for each $t>0$.

S. Moradi and A. Beiranvand prove that if S is T_f contractive mapping, and then S has a unique fixed point.

Then, in 2014 M. Kir and H. Kiziltunc generalize the result of S. Moradi and A. Beiranvand to mappings of type R. Kannan and S. K. Chatterjea.

In 2021 in [11], [12] and [13] are proven more generalizations for common fixed points of two T_f contractions of the type of R. Kannan, S. K. Chatterjea and P. V. Koparde, B. B. Waghmode on a complete metric space, while for the

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function f from the class Θ it is further assumed that it is subadditive, i.e. that $f(p+q) \le f(p) + f(q)$, for each $p, q \in [0, +\infty)$. In further considerations under the same assumptions we will give some results for common fixed points of two contractions of the Reich type.

2. MAIN RESULT

Definition 1 ([8]). Let (X,d) be a metric space. A mapping $T: X \to X$ is sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ also is convergence.

Definition 2 ([8]). Let (X,d) be a metric space, $S,T: X \to X$ and $f \in \Theta$. A mapping S is T_f – contraction if there exist $\lambda \in (0,1)$ such that

$$f(d(TSx, TSy)) \le \lambda f(d(Tx, Ty)),$$

for all $x, y \in X$.

Theorem 1. Let (X,d) is a complete metric space, $S_1, S_2 : X \to X$, $f \in \Theta$ is such that $f(p+q) \le f(p) + f(q)$, for each $p, q \in [0, +\infty)$ and mapping $T: X \to X$ is continuous, injection and sequentially convergent. If there are any $a, b > 0$ and $c \ge 0$ such that $a+b+c \in (0,1)$ and

$$f(d(TS_1x, TS_2y)) \le af(d(Tx, TS_1x)) + bf(d(Ty, TS_2y)) + cf(d(Tx, Ty)) \tag{1}$$

for each $x, y \in X$, then S_1 and S_2 have a single common fixed point.

Proof. Let x_0 is an arbitrary point from X and let the sequence $\{x_n\}$ is defined by

$$x_{2n+1} = S_1 x_{2n},\ x_{2n+2} = S_2 x_{2n+1},\ n = 0,1,2,3,\ldots .$$

If it exists $n \ge 0$, such that $x_n = x_{n+1} = x_{n+2}$, then it is easily proved that $u = x_n$ is a common fixed point of S_1 and S_2. Let us therefore assume that there are no three consecutive equal members of the sequence $\{x_n\}$. Then, using inequality (1), it is easy to prove that the following inequalities are true:

$$f(d(Tx_{2n+1}, Tx_{2n})) \le \tfrac{b+c}{1-a} f(d(Tx_{2n}, Tx_{2n-1}))$$

and

$$f(d(Tx_{2n}, Tx_{2n-1})) \le \tfrac{a+c}{1-b} f(d(Tx_{2n-1}, Tx_{2n-2})).$$

From the last two inequalities it follows that for each $n = 0,1,2,\ldots$ and for

$$\lambda = \min\{\tfrac{a+c}{1-b}, \tfrac{b+c}{1-a}\} \in (0,1)$$

is true:

$$f(d(Tx_{n+1},Tx_n)) \le \lambda f(d(Tx_n,Tx_{n-1})) . \tag{2}$$

Furthermore, from inequality (2) it follows

$$f(d(Tx_{n+1},Tx_n)) \le \lambda^n f(d(Tx_1,Tx_0)) , \tag{3}$$

for each $n = 0,1,2,\ldots$. Now from the metric properties, the function properties f and the inequality (3) follows that for each $m,n \in R$, $n > m$ is true

$$\begin{aligned} f(d(Tx_n,Tx_m)) &\le f(\sum_{k=m}^{n-1} d(Tx_{k+1},Tx_k)) \le \sum_{k=m}^{n-1} f(d(Tx_{k+1},Tx_k)) \\ &\le \sum_{k=m}^{n-1} \lambda^k f(d(Tx_1,Tx_0)) < \frac{\lambda^m}{1-\lambda} f(d(Tx_1,Tx_0)). \end{aligned}$$

It follows from the last inequality that

$$\lim_{m,n\to\infty} f(d(Tx_n,Tx_m)) = 0 ,$$

and because $f \in \Theta$ we have $\lim\limits_{m,n\to\infty} d(Tx_n,Tx_m) = 0$. Therefore, the sequence $\{Tx_n\}$ is Cauchy and because (X,d) is a complete metric space it is convergent. Further, the mapping $T: X \to X$ e sequentially convergent, so therefore the sequence $\{x_n\}$ is convergent, i.e. exists $u \in X$ such that $\lim\limits_{n\to\infty} x_n = u$. Now, from the continuity of T it follows $\lim\limits_{k\to\infty} Tx_n = Tu$.

We will prove that $u \in X$ is a fixed point for the mapping S_1 . We have:

$$\begin{aligned} f(d(Tu,TS_1u)) &\le f(d(Tu,Tx_{2n+2})) + f(d(Tx_{2n+2},TS_1u)) \\ &= f(d(Tu,Tx_{2n+2})) + f(d(TS_2x_{2n+1},TS_1u)) \\ &\le f(d(Tu,Tx_{2n+2})) + af(d(Tu,TS_1u)) + bf(d(Tx_{2n+1},TS_2x_{2n+1})) + cf(d(Tu,Tx_{2n+1})) \\ &= f(d(Tu,Tx_{2n+2})) + af(d(Tu,TS_1u)) + bf(d(Tx_{2n+1},Tx_{2n+2})) + cf(d(Tu,Tx_{2n+1})) \end{aligned}$$

The mappings f and T are continuous, therefore, from the properties of the metric, works follows if in the last inequality we take $n \to \infty$, we will have

$$(1-a)f(d(Tu,TS_1u)) \le (1+b+c)f(0)$$

But, $1-a>0$ and $f^{-1}(0)=\{0\}$, so from so from the last inequality we have $d(Tu,TS_1u)=0$, i.e. $TS_1u = Tu$. Finally, T is an injection, therefore $S_1u = u$, which means that u is a fixed point for the mapping S_1 . Analogously it is proved that u is a fixed point for the mapping S_2, i.e. u is a common fixed point for the mappings S_1 and S_2 .

We will prove that S_1 and S_2 have a single common fixed point. Let $v \in X$ is a fixed point for S_2 , i.e. $S_2v = v$. Then

$$f(d(Tu,Tv)) = f(d(TS_1u,TS_2v)|) \le af(d(Tu,TS_1u)) + bf(d(Tv,TS_2v))) + cf(d(Tu,Tv)$$
$$= af(d(Tu,Tu)) + bf(d(Tv,Tv))) + cf(d(Tu,Tv)$$
$$= (a+b)f(0) + cf(d(Tu,Tv)).$$

Now, $1-c>0$ and $f^{-1}(0)=\{0\}$, so from the last inequality we have $d(Tu,Tv)=0$, i.e. holds that $Tu=Tv$. But, T is an injection, so $u=v$, which means that S_1 and S_2 have a single common fixed point. ■

Corollary 1. Let (X,d) is a complete metric space, $S_1,S_2 : X \to X$, $f \in \Theta$ is such that $f(p+q) \le f(p)+f(q)$, for each $p,q \in [0,+\infty)$ and mapping $T: X \to X$ is continuous, injection and sequentially convergent. If $\lambda \in (0,1)$ exists such that

$$f(d(TS_1x,TS_2y)) \le \lambda \sqrt[3]{f(d(Tx,TS_1x)) \cdot f(d(Ty,TS_2y)) \cdot f(d(Tx,Ty)}$$

for each $x,y \in X$, then S_1 and S_2 have a single common fixed point.

Proof. It follows from the inequality between the arithmetic mean and the geometric mean and Theorem 1 for $a=b=c=\frac{\lambda}{3}$.■

Corollary 2. Let (X,d) is a complete metric space, $S_1,S_2 : X \to X$, $f \in \Theta$ is such that $f(p+q) \le f(p)+f(q)$, for each $p,q \in [0,+\infty)$ and mapping $T: X \to X$ is continuous, injection and sequentially convergent. If $a,b>0$ and $c \ge 0$ exits such that $a+b+c \in (0,1)$ and

$$f(d(TS_1x,TS_2y)) \le \frac{af^2(d(Tx,TS_1x))+bf^2(d(Ty,TS_2y))}{f(d(Tx,TS_1x))+f(d(Ty,TS_2y))} + cf(d(Tx,Ty),$$

for each $x,y \in X$, then S_1 and S_2 have a single common fixed point.

Proof. It folloes from the inequality given in the condition, the inequality (1). ■

Corollary 3. Let (X,d) is a complete metric space, $S_1,S_2 : X \to X$, $f \in \Theta$ is such that $f(p+q) \le f(p)+f(q)$, for each $p,q \in [0,+\infty)$ and mapping $T: X \to X$ is continuous, injection and sequentially convergent. If $a,b>0$ exits such that $a+b \in (0,1)$ and

$$f(d(TS_1x,TS_2y)) \le af(d(Tx,TS_1x)) + bf(d(Ty,TS_2y)),$$

for each $x,y \in X$, then S_1 and S_2 have a single common fixed point.

Proof. The corollary follows from Theorem 1, for $c=0$. ■

Corollary 4. Let (X,d) is a complete metric space, $S_1,S_2 : X \to X$ and $f \in \Theta$ is such that $f(p+q) \le f(p)+f(q)$, for each $p,q \in [0,+\infty)$. If $a,b>0$ exits and $c \ge 0$ such that $a+b+c \in (0,1)$ and

$$f(d(S_1x,S_2y)) \le af(d(x,S_1x)) + bf(d(y,S_2y)) + cf(d(x,y))$$

for each $x, y \in X$, then S_1 and S_2 have a single common fixed point.

Proof. Mapping $T: X \to X$ defined with $Tx = x$ is an uninterrupted injection and is sequentially convergent. So, the corollary follows from Theorem 1 for $Tx = x$. ■

Corollary 5. Let (X,d) is a complete metric space, $S_1, S_2 : X \to X$ and $f \in \Theta$ is such that $f(p+q) \le f(p) + f(q)$, for each $p, q \in [0, +\infty)$. If $a, b > 0$ exits such that $a + b \in (0,1)$ and

$$f(d(S_1 x, S_2 y)) \le af(d(x, S_1 x)) + bf(d(y, S_2 y)),$$

for each $x, y \in X$, then S_1 and S_2 have a single common fixed point.

Proof. The corollary follows from corollary 3 for $Tx = x$ or from corollary 4 for $c = 0$. ■

Corollary 6. Let (X,d) is a complete metric space, $S_1, S_2 : X \to X$, $f \in \Theta$ is such that $f(p+q) \le f(p) + f(q)$, for each $p, q \in [0, +\infty)$ and mapping $T: X \to X$ is continuous, injection and sequentially convergent. If $p, q \in N$ exits and $a, b > 0$ and $c \ge 0$ such that $a + b + c \in (0,1)$ and

$$f(d(TS_1^p x, TS_2^q y)) \le af(d(Tx, TS_1^p x)) + bf(d(Ty, TS_2^q y)) + cf(d(Tx, Ty))$$

for each $x, y \in X$, then S_1 and S_2 have a single common fixed point. ■

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GENERALIZATION OF A REICH-TYPE CONTRACTIVE MAPPING IN A COMPLETE METRIC SPACE

Risto Malcheski[1], Samoil Malcheski[2]

Abstract. In this work, is given a generalization of the fixed point theorem of the Reich-type mapping on a complete metric space (X,d). Continuous, injective and sequentially convergent mapping T was used, as well as function f is from the class Θ continuous monotonically nondecreasing functions $f:[0,+\infty)\to[0,+\infty)$ such that $f^{-1}(0)=\{0\}$, where it is additionally taken the function to be subadditive, i.e. $f(x+y)\le f(x)+f(y)$, for each $x,y\in[0,+\infty)$.

1. INTRODUCTION

Banach's principle for a fixed point is well known in the literature, namely:

Let (X,d) is a metric space. Mapping $S:X\to X$ we will call it a contraction if there exits $\lambda\in(0,1)$ such that for each $x,y\in X$ is true

$$d(Sx,Sy)\le\lambda d(x,y). \tag{1}$$

If metric space (X,d) is complete, then the mapping T for which condition (1) is satisfied has a unique fixed point. In 1968, R. Kannan ([4]) generalized Banach's fixed point principle as follows:

Theorem 1. If mapping $S:X\to X$ where (X,d) is complete metric space, satisfies the inequality

$$d(Sx,Sy)\le\lambda(d(x,Sx)+d(y,Sy)), \tag{2}$$

where $\lambda\in(0,\frac{1}{2})$ and $x,y\in X$, then S has a single fixed point. ■

If S satisfies the condition (2), then for S we say it is a Kannan-type mapping.

In 1972, similar contraction conditions were introduced by S. K. Chatterjea ([7]), as follows:

Theorem 2. If mapping $S:X\to X$ where (X,d) is a complete metric space satisfies the inequality

$$d(Sx,Sy)\le\lambda(d(x,Sy)+d(y,Sx)), \tag{2}$$

where $\lambda\in(0,\frac{1}{2})$ and $x,y\in X$, then S has a single fixed point. ■

If S satisfies condition (2), then we say that is a Chatterjea-type mapping.

In 1971, S. Reich ([3]), gave a new generalization of Banach's fixed point principle as follows:

Theorem 3. If mapping $S: X \to X$ where (X, d) is a complete metric space satisfies the inequality

$$d(Sx, Sy) \leq ad(x, Sx) + bd(y, Sy) + cd(x, y), \tag{3}$$

where $a > 0, b > 0$ and $c > 0$ are such that $a + b + c < 1$ and $x, y \in X$, then S има has a single fixed point. ■

If it satisfies condition (3), then we say that is a Reich-type mapping.

In [9] S. Moradi and D. Alimohammadi generalize R. Kannan's result, using the sequentially convergent mappings, and in [1] several generalizations of Kannan and Chatterjea's theorems are proved, using the sequentially convergent mappings and , which are defined as follows:

Definition 1 ([8]). Let (X, d) be a metric space. A mapping $T: X \to X$ is said sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ also is convergence.

In [8] S. Moradi and A. Beiranvand introduce the concept of T_f contractive mapping, whereby they use the class Θ of continuous monotonically nondecreasing functions $f: [0, +\infty) \to [0, +\infty)$ such that $f^{-1}(0) = \{0\}$, which is defined as follows.

Definition 2 ([8]). Let (X, d) be a metric space, $S, T: X \to X$ and $f \in \Theta$. A mapping S is said T_f −contraction if there exist $\lambda \in (0,1)$ such that

$$f(d(TSx, TSy)) \leq \lambda f(d(Tx, Ty)),$$

for all $x, y \in X$.

Let us note here that, if $f \in \Theta$, then from $f^{-1}(0) = \{0\}$ follows that $f(t) > 0$, for each $t > 0$. S. Moradi and A. Beiranvand prove that if S is T_f contractive mapping, then S has a single fixed point. Then, in [2] M. Kir and H. Kiziltunc generalize the result of S. Moradi and A. Beiranvand for the mappings of the type of Kannan and Chatterjea. In [10] are generalized the results of Kir and Kiziltunc and is given their application.

In the following considerations we will give an analogous generalization for the Reich-type mapping.

2. MAINS RESULTS

Theorem 4. Let (X, d) is a complete metric space $S: X \to X$, $f \in \Theta$ is such that $f(p+q) \leq f(p) + f(q)$, for each $p, q \in [0, +\infty)$ and mapping $T: X \to X$ is continuous, injection and sequentially convergent. If exits $a > 0, b > 0$ and $c > 0$ such that $a + b + c < 1$ and

$$f(d(TSx,TSy)) \le af(d(Tx,TSx)) + bf(d(Ty,TSy)) + cf(d(Tx,Ty)) \tag{4}$$

for each $x, y \in X$, then S has a single fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to fixed point.

Proof. Let x_0 is an arbitrary point from X and let the array $\{x_n\}$ is determined by $x_{n+1} = Sx_n$, $n = 0,1,2,3,\ldots$. It follows from the inequality (4).

$$\begin{aligned} f(d(Tx_{n+1},Tx_n)) &= f(d(TSx_n,TSx_{n-1})) \\ &\le af(d(Tx_n,TSx_n)) + bf(d(Tx_{n-1},TSx_{n-1})) + cf(d(Tx_n,Tx_{n-1})) \\ &= af(d(Tx_n,Tx_{n+1})) + (b+c)f(d(Tx_{n-1},Tx_n)), \end{aligned}$$

i.e.

$$f(d(Tx_{n+1},Tx_n)) \le \tfrac{b+c}{1-a} f(d(Tx_n,Tx_{n-1})) .$$

Therefore, for $\lambda = \frac{b+c}{1-a} < 1$ the following holds true

$$f(d(Tx_{n+1},Tx_n)) \le \lambda f(d(Tx_n,Tx_{n-1})) , \tag{5}$$

for each $n = 1,2,3,\ldots$. From the inequality (5) it follows that

$$f(d(Tx_{n+1},Tx_n)) \le \lambda^n f(d(Tx_1,Tx_0)) , \tag{6}$$

for each $n = 1,2,3,\ldots$. Now from inequality (6) the properties of the metric and the monotonicity and subadditivity of the function f it follows that for each $m,n \in R$ $n > m$ the following holds true

$$\begin{aligned} f(d(Tx_n,Tx_m)) &\le f(\sum_{k=m}^{n-1} d(Tx_{k+1},Tx_k)) \le \sum_{k=m}^{n-1} f(d(Tx_{k+1},Tx_k)) \\ &\le \sum_{k=m}^{n-1} \lambda^k f(d(Tx_1,Tx_0)) < \tfrac{\lambda^m}{1-\lambda} f(d(Tx_1,Tx_0)). \end{aligned}$$

It follows from the last inequality

$$\lim_{m,n\to\infty} f(d(Tx_n,Tx_m)) = 0 ,$$

and because $f \in \Theta$ we have $\lim\limits_{m,n\to\infty} d(Tx_n,Tx_m) = 0$. According to that, $\{Tx_n\}$ is Cauchy sequence. But, X is a complete metric space, so the sequence $\{Tx_n\}$ is convergent. Further, the mapping $T : X \to X$ e sequentially convergent, so therefore the sequence $\{x_n\}$ is convergent i.e. exists $u \in X$ such that $\lim\limits_{n\to\infty} x_n = u$. From the continuity of T it follows $\lim\limits_{n\to\infty} Tx_n = Tu$.

We will prove that $u \in X$ is a fixed point for the mapping S. We have

$$\begin{aligned} f(d(TSu,Tx_{n+1})) &= f(d(TSu,TSx_n)) \\ &\le af(d(TSu,Tu)) + bf(d(TSx_n,Tx_n)) + cf(d(Tu,Tx_n)) \\ &= af(d(TSu,Tu)) + bf(d(Tx_{n+1},Tx_n)) + cf(d(Tu,Tx_n)). \end{aligned}$$

If in the last inequality we take $n \to \infty$, then form $\lim_{n\to\infty} Tx_n = Tu$ and the continuity of metric and function f follows the inequality

$$f(d(TSu,Tu)) \le \tfrac{b+c}{1-a} f(0).$$

But, $0 < \tfrac{b+c}{1-a} < 1$ and $f^{-1}(0) = \{0\}$, so it follows from inequality $d(TSu,Tu) = 0$, i.e. $TSu = Tu$. Finally, T is injection and therefore $Su = u$, i.e. mapping S has a fixed point.

Let $u, v \in X$ are two fixed points for S, i.e. $Su = u$ and $Sv = v$. From the inequality, (4) it follows that

$$f(d(Tu,Tv)) = f(d(TSu,TSv)) \le af(d(Tu,TSu)) + bf(d(Tv,TSv)) + cf(d(Tu,Tv))$$

i.e.

$$f(d(Tu,Tv)) \le \tfrac{a+b}{1-c} f(0),$$

so similarly as above we conclude that $d(Tu,Tv) = 0$. Therefore, $Tu = Tv$. But, T is an injection, and therefore $u = v$, i.e. S has a single fixed point.

Finally, from the arbitrariness of the point x_0 it follows that for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to the fixed point. ■

Corollary 1. Let (X,d) is a complete metric space, $S: X \to X$ and $f \in \Theta$ is such that $f(p+q) \le f(p) + f(q)$, for each $p,q \in [0,+\infty)$. If $a > 0, b > 0, c \ge 0$ exits such that $a+b+c < 1$ and

$$f(d(Sx,Sy)) \le af(d(x,Sx)) + bf(d(y,Sy)) + cf(d(x,y)),$$

for each $x, y \in X$, then S has a single fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to the fixed point.

Proof. The mapping $Tx = x$, for each $x \in X$ is continuous, injection and sequentially convergent. Therefore the corollary follows directly from Theorem 4 for $Tx = x$. ■

Corollary 2. Let (X,d) is a complete metric space, $S: X \to X$ and mapping $T: X \to X$ is continuous and sequentially convergent. If $a > 0, b > 0, c \ge 0$ exits such that $a+b+c < 1$ and

$$d(TSx,TSy) \le ad(Tx,TSx) + bd(Ty,TSy) + cd(Tx,Ty)$$

for each $x, y \in X$, then S has a single fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to the fixed point.

Proof. The function $f(t) = t$, $t \ge 0$ is monotonically nondecreasing, $f^{-1}(0) = \{0\}$ and is such that $f(p+q) \le f(p) + f(q)$, for each $p,q \in [0,+\infty)$. Therefore the corollary follows directly from Theorem 4 for $f(t) = t$. ■

Comment. If we consider that the mapping $Tx = x$, for each $x \in X$ is continuous, injection and sequentially convergent, from corollary 2 follows theorem 3, [3], i.e. follows that if for the mapping $S:X\to X$ exits $a>0, b>0, c\geq 0$ such that $a+b+c<1$ and

$$d(Sx,Sy)\leq ad(x,Sx)+bd(y,Sy)+cd(x,y)$$

for each $x,y\in X$, then S has a single fixed point.

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SEVERAL LOCI GENERATED BY A MOVING TRIANGLE BETWEEN TWO FIXED CIRCLES

Prof. Sava Grozdev, Prof. Veselin Nenkov, Assoc. Prof. Tatiana Madjarova

Abstract. According to a famous theorem belonging to the French mathematician Poncelet, if two conics are located in the plane in such a way that a polygon exists which is inscribed in one of the curves and circumscribed with respect to the other one, then each point on any of the two conics generates an inscribed-circumscribed polygon with respect to them. Particularly, two circles could be located in the plane in such a way that one of them is circumscribed with respect to a triangle and the other one is inscribed in it. In connection with this configuration when the triangle moves between the two circles, several loci are considered which are determined by some notable points in the plane of the triangle. It turns out that the loci are circles and ellipses with centers on the central line of the two fixed circles.

Keywords: inscribed circle, circumscribed circle, center of gravity, orthocenter, Euler line, Euler circle, Nagel point, GSP (Geometer's sketchpad)

1. Introduction

It is well-known a remarkable theorem of Poncelet in the Euclidean geometry, a particular case of which is the following assertion:

Theorem. *If the circles Γ and ω are positioned in the plane in such a way that a triangle exists which is inscribed and circumscribed with respect to Γ and ω, respectively, then:*

1) each point on Γ is a vertex of a unique triangle, which is inscribed in Γ and circumscribed with respect to ω;

2) each point on ω is a tangent point on a side of a unique triangle, which is inscribed in Γ and circumscribed with respect to ω.

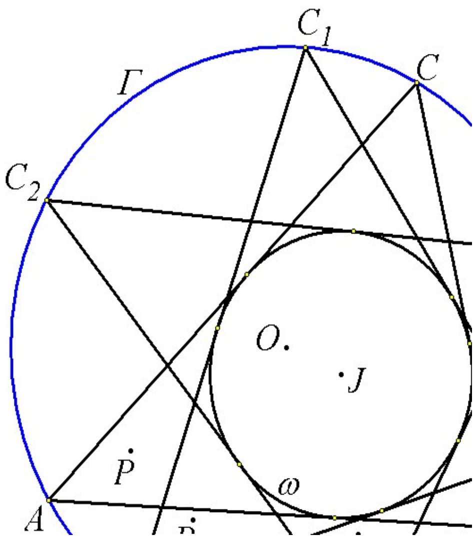


Figure 1

Let Γ and ω be two non-concentric circles satisfying the theorem. It follows from the first item that if A is an arbitrary point on the circle Γ, then there exist such points B and C on the circle in question, that the triangle ABC is circumscribed with respect to ω (Fig. 1). If the point A moves on Γ (the point A occupies consecutive positions A_1, A_2, $\dots$ on Γ), the triangle ABC will move (ABC occupies consecutive positions $A_1B_1C_1$, $A_2B_2C_2$, $\dots$) between the circles Γ and ω in such a way that it is inscribed in Γ always and also circumscribed with respect to ω (Fig. 1). Along this motion an arbitrary point P connected with ΔABC in some way will move together with the triangle (P occupies consecutive positions P_1, P_2, $\dots$ together with the corresponding triangles $A_1B_1C_1$, $A_2B_2C_2$, $\dots$) (Fig. 1) and at the same time it will describe a determined trajectory in the plane of the circles. Thus, a problem appears to determine the trajectories which some notable points of the triangle describe when the triangle moves between the circles Γ and ω in the mentioned way.

Let O and R be the center and the radius of the circle Γ, respectively, while J and r be the center and the radius of the circle ω, respectively. Loci will be presented that are described by some classic notable points of the triangle. They are noticed by means of the program software Geometer's sketchpad (GSP).

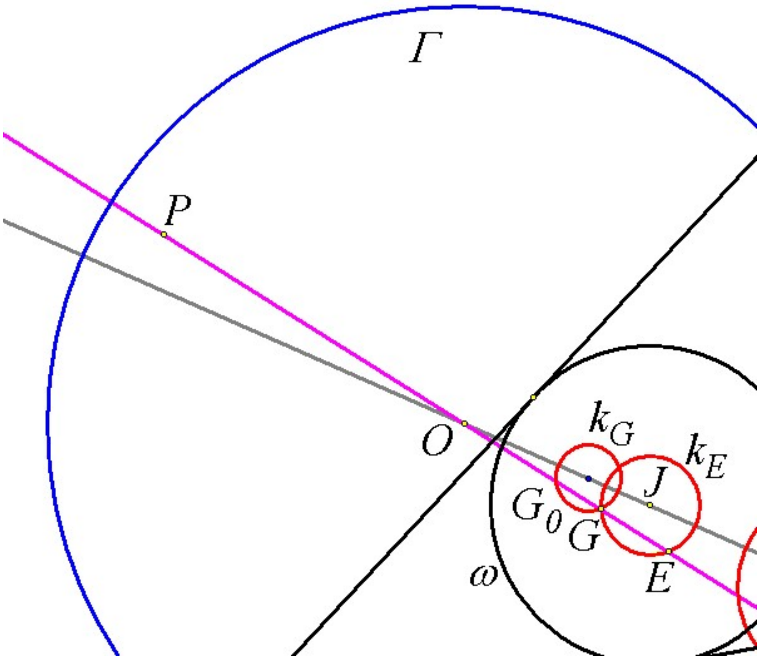


Figure 2

2. Some notable circles in the plane of the triangle

Some notable points of the triangle ABC describe circles along the motion of the triangle between the circles Γ and ω. The following assertions are satisfied for three points which are characteristic for the Euler line:

Theorem 1. *The orthocenter* H *of* ΔABC *describes a circle* k_H *with center* H_0 *on the line* OJ *and radius* $R-2r$ (Fig. 2).

Theorem 2. *The center of gravity* G *of* ΔABC *describes a circle* k_G *with center* G_0 *on the line* OJ *and radius* $\dfrac{R-2r}{3}$ (Fig. 2).

Theorem 3. *The center* E *of the Euler circle describes a circle* k_E *with center* J *and radius* $\dfrac{R-2r}{2}$ (Fig. 2).

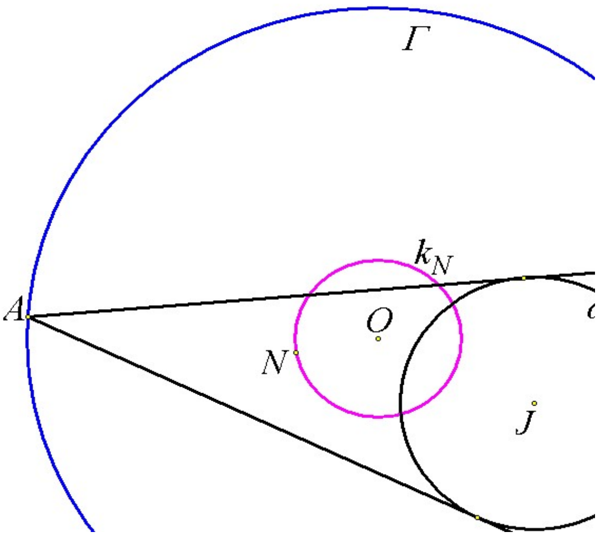
Proofs and generalizations of these assertions are included in [2].

Other notable points of ΔABC are the points of Nagel and Gergonne. If r_a, r_b and r_c are the radii of the ex-circles of ΔABC, which are tangent to the sides BC, CA and AB, respectively, then the Nagel point N and the Gergonne one G could be determined with the following vector equalities, respectively:

$$\overrightarrow{ON}=\frac{r_br_c\overrightarrow{OA}+r_cr_a\overrightarrow{OB}+r_ar_b\overrightarrow{OC}}{r_br_c+r_cr_a+r_ar_b},\ \overrightarrow{OG}=\frac{r_a\overrightarrow{OA}+r_b\overrightarrow{OB}+r_c\overrightarrow{OC}}{r_a+r_b+r_c}.$$

The points in question satisfy the following two assertions:

Theorem 4. *The Nagel point* N *describes a circle* k_N *with center* O *and radius* $R-2r$ (Fig. 3).



Figure

A proof and a generalization of this assertion are included in [2].

Theorem 5. *If* G *is the Gergonne point of a moving triangle* ABC *between the circles* Γ *and* ω, *then it describes a circle* $k(G)$ *with center* P *on the line* OJ, *where* $OP = \dfrac{4(R+r).OJ}{4R+r}$, *and a radius* $\rho = \dfrac{(R-2r)r}{4R+r}$ (Fig. 4).

Two different proofs of this assertion are included in [3] and [4]. The circle $k(G)$ will be called Poncelet-Gergonne circle.

The points on the Euler circle of ΔABC describe a special set of circles. More precisely, it is satisfied the following:

Theorem 6. *If* M *is a point on the Euler circle of the triangle ABC, moving between the circles* Γ *and* ω, *then it describes a circle* $k(M)$ *with radius* $\rho = \dfrac{1}{2}(R-2r)$, *which is tangent to* ω *exteriorly* (Fig. 5).

The proof of this theorem will be published in Mathematics Plus journal later. The circles $k(M)$ will be called Poncelet-Euler circles.

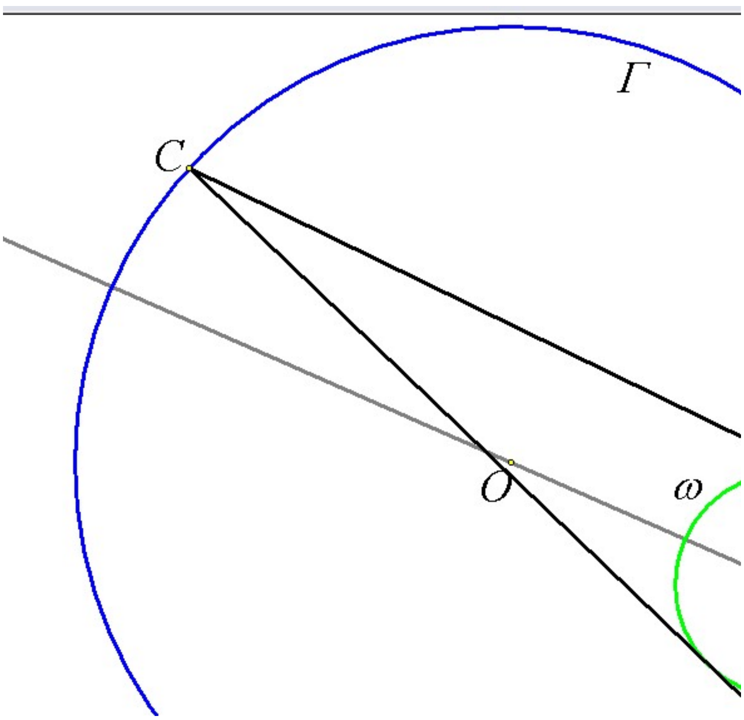


Figure 4

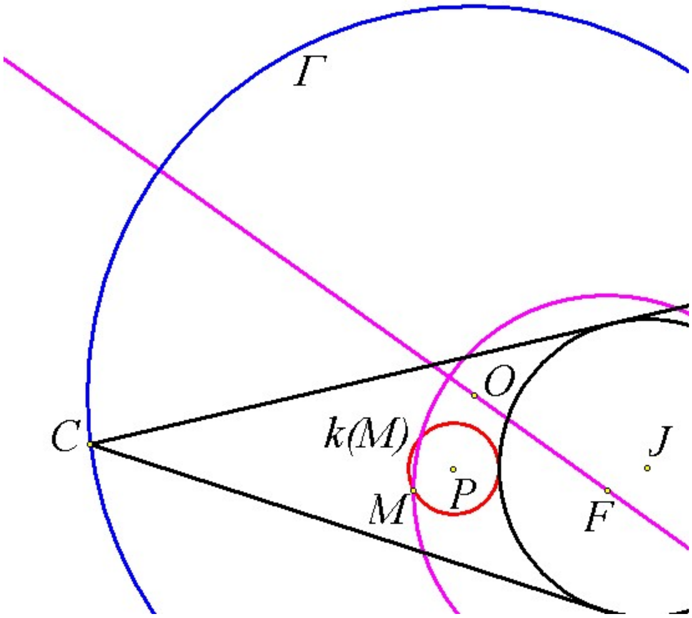


Figure 5

3. Two notable ellipses in the plane of the triangle

The points L, L_a, L_b and L_c, determined with the vector equalities

$$\overrightarrow{OL} = \frac{a_0^2\overrightarrow{OA} + b_0^2\overrightarrow{OB} + c_0^2\overrightarrow{OC}}{a_0^2 + b_0^2 + c_0^2},\ \overrightarrow{OL_a} = \frac{-a_0^2\overrightarrow{OA} + b_0^2\overrightarrow{OB} + c_0^2\overrightarrow{OC}}{-a_0^2 + b_0^2 + c_0^2},$$

$$\overrightarrow{OL_b} = \frac{a_0^2\overrightarrow{OA} - b_0^2\overrightarrow{OB} + c_0^2\overrightarrow{OC}}{a_0^2 - b_0^2 + c_0^2},\ \overrightarrow{OL_c} = \frac{a_0^2\overrightarrow{OA} + b_0^2\overrightarrow{OB} - c_0^2\overrightarrow{OC}}{a_0^2 + b_0^2 - c_0^2},$$

where $BC = a_0$, $CA = b_0$ and $AB = c_0$, are called Lemoine points of ΔABC.

It turns out that the Lemoine points of ΔABC describe ellipses with interesting properties. The characteristic properties of these ellipses are described in the following assertions.

Theorem 7. *If L is the Lemoine point of the triangle ABC, moving between the circles Γ and ω, then it describes an ellipse $k(L)$ with center T on the line OJ, where $OT = \dfrac{3R^2}{3R^2 - 2Rr + r^2} \cdot OJ$. The value of the small semiaxis α of*

$k(L)$ *on* OJ *is equal to* $\alpha = \frac{Rr(R-2r)}{3R^2 - 2Rr + r^2}$, *while the value of the big one is equal to* $\beta = R(R-2r)\sqrt{\frac{r}{(4R+r)(3R^2-2Rr+r^2)}}$ (Fig. 6).

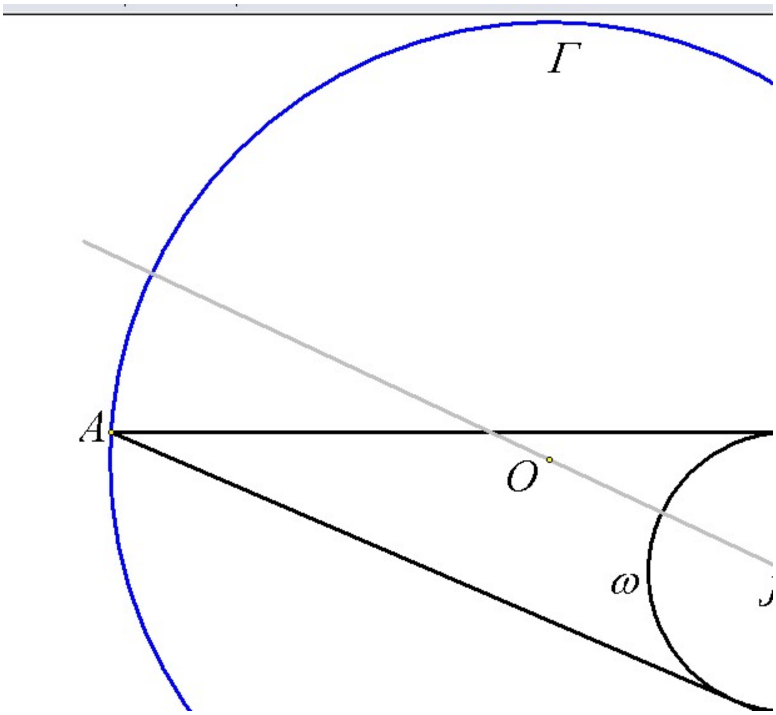


Figure 6

The ellipse will be called Poncelet-Lemoine ellipse.

Theorem 8. *If* L_a, L_b *and* L_c *are the outer Lemoine points of the triangle ABC moving between the circles* $\Gamma(O,R)$ *and* $\omega(J,r)$, *then they describe a curve* $\bar{k}(L)$ *of second degree with a focus* O *and a focal axis* OJ. *The curve* $\bar{k}(L)$ *has the following properties:*

1) If $2r \le R < (\sqrt{2}+1)r$, *then the curve* $\bar{k}(L)$ *is an ellipse with a small semiaxis* $\alpha = \frac{R^2 r}{r^2 + 2Rr - R^2}$, *big semiaxis* $\beta = \frac{R^2}{\sqrt{r^2 + 2Rr - R^2}}$ *and a center* T, *satisfying the equality* $OT = -\frac{R^2.OJ}{r^2 + 2Rr - R^2}$;

2) If $R > \left(\sqrt{2}+1\right)r$*, then the curve* $\bar{k}\left(L\right)$ *is a hyperbola with a small semiaxis* $\alpha = \dfrac{R^2 r}{R^2 - 2Rr - r^2}$*, big semiaxis* $\beta = \dfrac{R^2}{\sqrt{R^2 - 2Rr - r^2}}$ *and a center* T*, satisfying the equality* $OT = \dfrac{R^2 .OJ}{R^2 - 2Rr - r^2}$*;*

3) If $R = \left(\sqrt{2}+1\right)r$*, then the curve* $\bar{k}\left(L\right)$ *is a parabola with a focal parameter* $p = \left(\sqrt{2}+1\right)^2 r$ *and a vertex* V*, satisfying the equality* $OV = \dfrac{\left(\sqrt{2}+1\right)^2 r}{2}$.

The ellipse $\bar{k}\left(L\right)$ will be called outer Poncelet-Lemoine ellipse.

Detailed proofs of these theorems will be published in the Mathematics Plus journal.

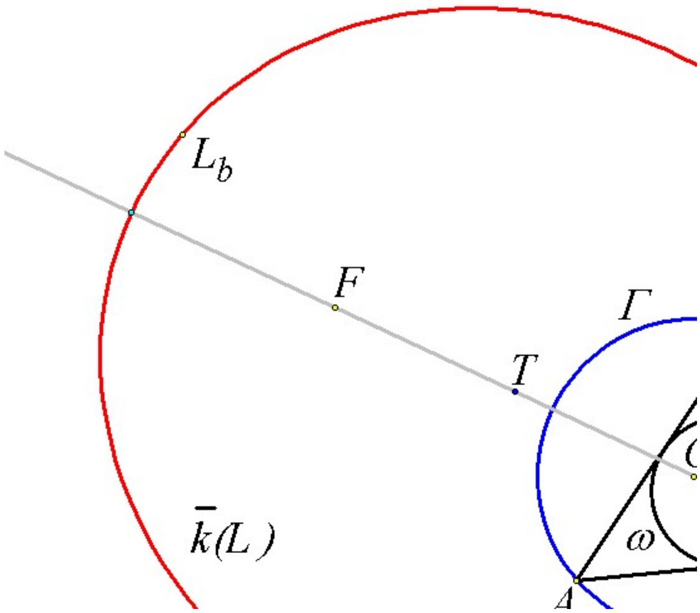


Figure 7

Concluding we could state that the described circles and ellipses in the listed theorems are noticed by means of the Poncelet theorem and the geometric capabilities of the program software GSP. They are located in the plane of a scalene triangle moving between two fixed circles. The curves themselves exist for any triangle, no matter it is regarded as moving or stationary one. For this reason they could be called notable circles and ellipses of the triangle.

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SOLVING TASKS FROM LINEAR PROGRAMMING USING GEOGEBRA

Elena Karamazova Gelova [1], Mirjana Kocaleva Vitanova [2], Sonja Mančevska [3]

Abstract. Each student from every new generation, soon or latter, encounters difficulties when solving mathematical problems, even the best ones. Some students have problems with mastering the material, others with solving homework assignments, others have problems with quickly forgetting what they have learned, others think that they would lose a lot of time for solving the task and, in the end, they might get an incorrect result, which reduces their motivation for finding a solution and so on. Mathematical content that is presented graphically and mathematical problems whose solution may be obtained graphically remains best in the student's memory. Moreover, if an appropriate software is used during the solving of a given mathematical problem, if its graphical representation is precise enough and the final solution can be clearly seen from it, then most of the stumbling blocks for students will be surmounted. In this paper we will present the graphical solution method for linear programming problems using GeoGebra. The software allows graphic editing to be done in a quick and simple way which is very important for students.

1. INTRODUCTION

Whenever possible, a graphical representation that can be done in the fastest and most accurate way and its use to obtain a solution of a given mathematical problem enables a permanent memorization of what has been learned. Using an appropriate software for graphical solving will enable obtaining a solution in a much shorter time and perceiving the solution from the drawing itself. The graphic of the solved problem usually gives a complete picture of the solution which, for most of the students, is crucial for permanent memorization of what was perceived and learned. The best way to perform the graphical solution is with the use of an appropriate educational software. The software which we are going to use in this paper is GeoGebra. It is a free, open source, simple to use mathematical program that connects geometry, algebra, calculus, and statistics. The possibilities of GeoGebra as an educational software for mathematics are enormous. It can be used at all levels of education, from primary schools to universities, for drawing basic geometrical shapes to three dimensional objects, for performing basic analysis of functions with one variable to determining a conditional extremity of a function with two variables and visually presenting the conditional extremum (see for example [13]).

2010 *Mathematics Subject Classification:* 90C05.
Key words and phrases. GeoGebra, Linear Programming, LP problems, graphical solution method, questionnaire

One of the main features of GeoGebra is that it is a dynamic software. Unlike a sketch on paper, which is a static model, in GeoGebra it is possible to change certain parameters in the Graphics window simply by making changes in the Algebra window. GeoGebra's user interface is flexible and customizable as needed.

All about the main features of GeoGebra, along with its usage for solving mathematical problems from different mathematical topics, can be found in [2] and [6].

Several studies show that the use of educational software as well as other information technologies increases students' motivation, desire, and interest in solving problems. The main goal of the research in paper [12] is the question "Does the technical equipment of the classrooms bring better results in mastering the teaching program by the students?". The authors determine the quality of knowledge that the students get when learning the topic "Construction of triangle and quadrangle" with use of GeoGebra and informatics/mathematics approach, by comparing the achieved results on the diagnostic and the final test, of the experimental and the control group. The experimental group of students is learning the topic with use of software and constructions are made on computer, while the control group is learning by the traditional method of constructions made in notebook with ruler and compass.

In [1] authors analyze the perceptions and attitudes about the use of ICT tools for visualization as a "modern" approach for solving geometry problems in primary schools in Macedonia. In [15] the concept of a discrete random variable is introduced following the standard definitions, but by use of information technology, with emphasis on modeling probability situations with only two outcomes. Also, examples of discrete random variable with a geometric distribution are given, which are visually represented with GeoGebra.

The number of countries worldwide that have the development of the Information Society as one of their highest priorities has increased in the last decades. One of the key segments for the promotion and development of the Information Society is the education. The quality of the educational process is closely related to the application of the information and its communication technologies. In [16], the authors have presented the results of their research that has been conducted to investigate the factors that affect the motivation of teachers to use ICT in their classroom. There is more research in which the main goal is to see the importance of ICT in the teaching process in mathematical subjects. In [14] are given the results of the research which was conducted with students from two Universities: Mother Teresa, Skopje and Goce Delcev, Stip. Students were split into two groups. With one group, the mathematical content (algebra, geometry, analysis) was processed by using GeoGebra and on a computer, while with the other group the same material was processed without any kind of visualization. After that the testing was done. The comparison of the results led to the conclusion that the visualization of problems, the inclusion of visualization software in the curriculum, introducing students to the importance of mathematics and its extensive application is very important to do during the educational process.

For increasing the motivation for learning mathematics and increasing the level of knowledge, a web application http://mathlabyrinth.azurewebsites.net is presented in [11]. This application is for students in the secondary education and contains mathematical problems that are related to real-life problems the students may encounter.

Many high school teachers face questions from their students about the applicability of the mathematical contents. In [3], the authors address students' questions related to linear programming problems and solved them using GeoGebra. The article [9] describes the observations of the experimental teaching conducted in the high school in Košice, where GeoGebra was used. GeoGebra was used for the first time in students' lives for solving a linear optimization word problem. The findings in [5] show that the use of GeoGebra enhanced the students' performance in learning linear programming, hence it was recommended that teachers employ GeoGebra software in teaching and learning Linear Programming and any other mathematics topics. The study [10] aims to determine the increase in students' critical thinking skills in linear programming learning through the Problem Based Learning (PBL) model assisted by GeoGebra Software. This research is a semi-experimental study with one group pre-test post-test design. The group in this study involved 24 students. The instrument used was the pre-test and post-test questions on critical thinking skills. The data were analyzed using SPSS software. The results showed that the use of the PBL model assisted by GeoGebra software can improve students' critical thinking skills on linear programming material.

2. Initial Results

In this section we will graphically solve a few linear programming problems (LP problems) using GeoGebra. Then we will look at the results of the survey carried out with a group of 20 students from the Faculty of computer science at University Goce Delcev Stip, which consists of answering a questionnaire whose questions are related to the advantages and disadvantages of using educational software when graphically solving linear programming problems.

Through the solved examples, we will show few ways to reach the solution with the help of GeoGebra. The examples are given below.

Problem 1. Find graphically the maximum and the minimum of the function $L = x + y$, with the following constraints:

$$\begin{cases} -x+y \geq -3 \\ x \leq 4 \\ x+2y \leq 10 \\ -x+y \leq 2 \\ x \geq 0 \\ y \geq 0 \end{cases}.$$

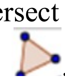
Solution: First step is to draw the bounding lines of the constraints. In the Input bar of GeoGebra we write, one by one, the following equations:

$$\begin{aligned} -x+y &= -3, \\ x &= 4, \\ x+2y &= 10, \\ -x+y &= 2, \\ x &= 0, \\ y &= 0. \end{aligned} \tag{1}$$

As we enter each equation, it will automatically appear in the Algebra window and the corresponding line will be drawn in the Graphics window. Each line can be represented by a different colour. The colour of the equation in the Algebra window will be the same as the colour of the corresponding line in the Graphics window.

The lines in (1) divide the plane into few regions, only one of which is the feasible region. We can discard the regions that are left from the y axis (any point, in any of these regions, does not satisfy the constrain $x \geq 0$), those right from the line $x=4$ and those below the x axis. Then, using the one-point-test, we check which one of the remaining regions (five in total) satisfies all the given constrains. This is the polygon with the vertices in the intersection points of the lines with equations:

- $x=0$ and $y=0$ (i.e., the origin),
- $y=0$ and $-x+y=-3$,
- $-x+y=-3$ and $x=4$,
- $x=4$ and $x+2y=10$,
- $x+2y=10$ and $-x+y=2$, and
- $-x+y=2$ and $x=0$.

We find these points using the Intersect tool of two objects  and then we connect them with the Polygon tool . The result is given in Figure 1.

If we equate the objective function to 0 we'll get $x+y=0$, i.e., $y=-x$, we may perceive the objective function as the line $y=-x$ "moving" from left to right as its value increases (Really?).

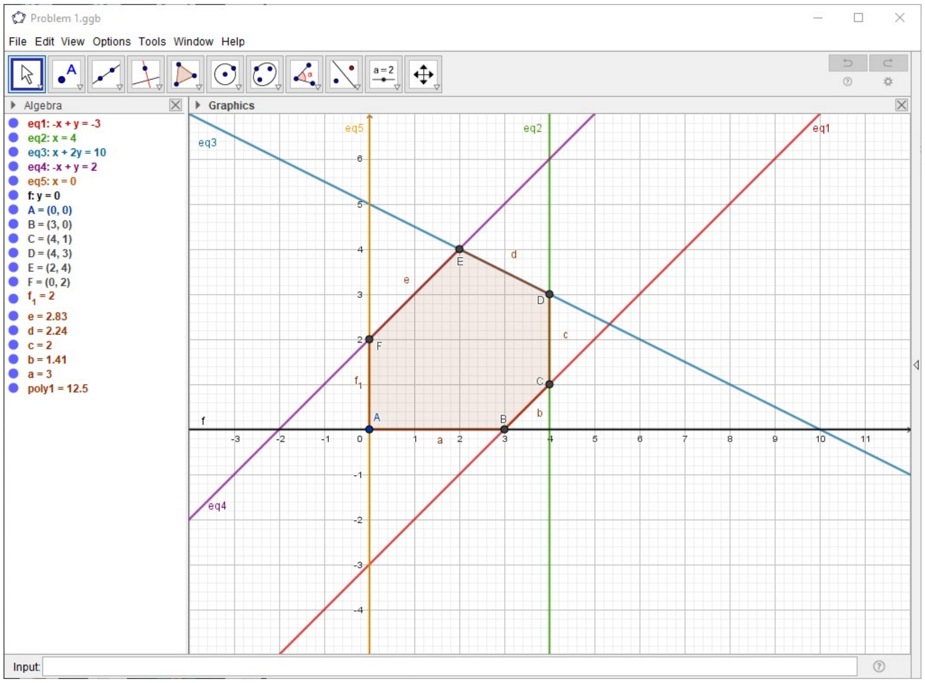


Figure 1: The feasible region for Problem 1 (polygon *ABCDEF*)

So, the objective function will reach its maximum when the line $y = -x$ "passes" through the "last" vertex of the feasible region while increasing its distance from the origin. In this case that will be the point $D(4,3)$. Hence $\max L = 7$.

Similarly, if we are looking for the minimum, it will be reached when the line passes through the origin $(0,0)$ and hence, $\min L = 0$. ■

Remark 2. If the objective function is of form

$$L = ax + yb,$$

most of the teachers and textbooks, suggest that direction in which the objective function increases should be represented by two or more lines obtained by assigning increasing values to the objective function. These lines are usually refered as isoprofit lines (for maximization LP problems), isocost lines (for minimization LP problems), or as "objective function lines". Usually they are also drawn on the graph, as shown on Figure 2. This can be quite helpful. From Figure 2, we can easily conclude that the objective function has a maximum at point $D(4,3)$ and that $\max L = 7$.

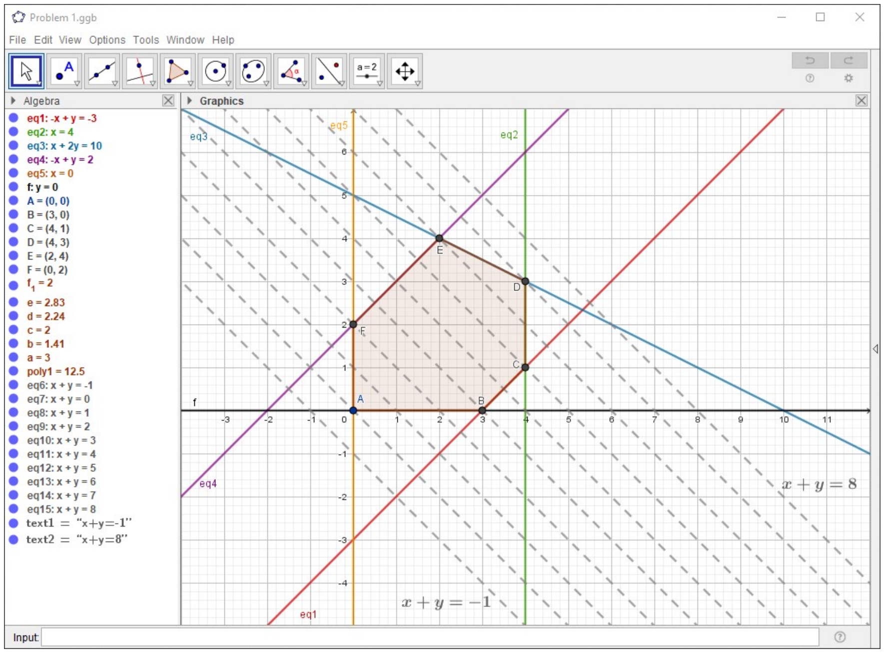


Figure 2: Objective function lines from Problem 1

Problem 3. Find graphically the smallest and largest value of the function $L = x + y$ on the area:

$$\begin{cases} 3x + 2y \geq 6 \\ x - 2y \leq 2 \\ -3x + 2y \leq 6 \\ x \geq 0 \\ y \geq 0 \end{cases}.$$

Solution: As in the previous problem, in the Input bar we write the equations of the bounding lines, one by one,

$$3x + 2y = 6, \quad x - 2y = 2, \quad -3x + 2y = 6, \quad x = 0 \text{ and } y = 0.$$

For this problem we will obtain the feasible region with GeoGebra in a different way. GeoGebra supports graphical representation of a single inequality with two variables and a system of inequalities with two variables as well. For the system of inequalities given in the problem, in the Input bar we write:

$$3x + 2y \geq 6 \wedge x - 2y \leq 2 \wedge -3x + 2y \leq 6 \wedge x \geq 0 \wedge y \geq 0.$$

The symbols $\leq, \geq, \wedge$ are contained in the palette available by clicking on α at the end of the Input bar.

The result obtained with GeoGebra is given in Figure 3. The graphic suggests that the feasible region may be unbounded (but not necessarily!). If, after few clicks with the Zoom out tool we don't get a bounded region, chances are that the region is indeed unbounded. But we need to verify this algebraically. By Figure 3, the feasible region may contain every point of the line $y = x$ which satisfies $x \geq 2$ i.e., the whole ray $R = \{(x, y) : x \geq 2, y = x\}$. It can be easily verified that this is indeed true: all the constrains in the problem are satisfied for every point in R. Since the ray is unbounded, the feasibility region will also be unbounded.

As in the solution of Problem 1, from Figure 3 we can see that the objective function reaches its smallest value $L = 2$ for $x = 2, y = 0$. But the largest value does not exist (there is no "last" point in the feasible region through which the line $y = -x$ "passes" while "moving" from left to right). ■

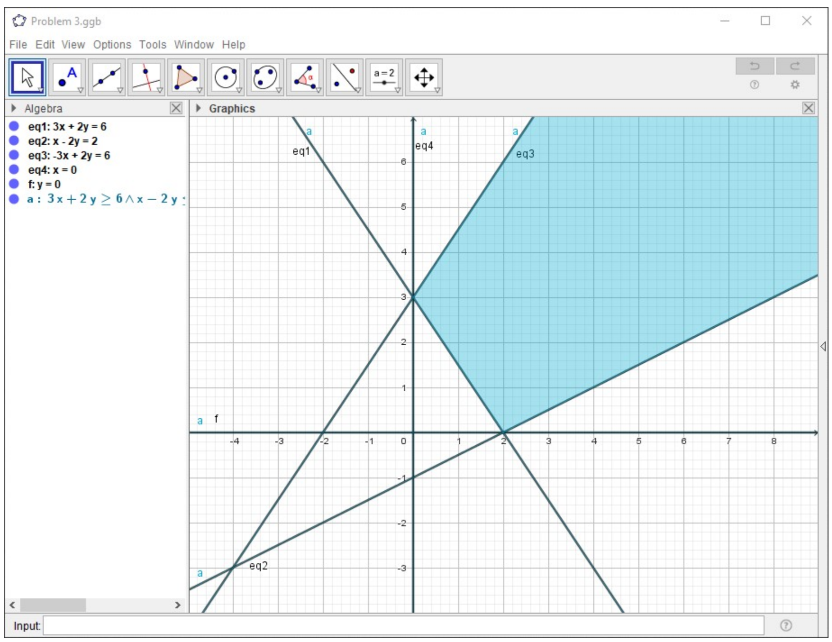


Figure 3: The bounding lines and the feasible region from Problem 3

Problem 4. Find graphically the minimum of the function $L = 2x + 3y$ with constraints:

$$\begin{cases} x + y \geq 2 \\ x + 3y \leq 12 \\ 3x + y \leq 12 \\ x \geq 0 \\ y \geq 0 \end{cases}.$$

Solution: In the Input bar, one by one, we enter the equations of the bounding lines

$$x+y=2,\ x+3y=12,\ 3x+y=12,\ x=0 \text{ and } y=0,$$

and then

$$x+y\geq 2\wedge x+3y\leq 12\wedge 3x+y\leq 12\wedge x\geq 0\wedge y\geq 0.$$

Once the feasibility region is visible in the graphics view, using the intersection tool we find its vertices. The result in GeoGebra is given in Figure 4.

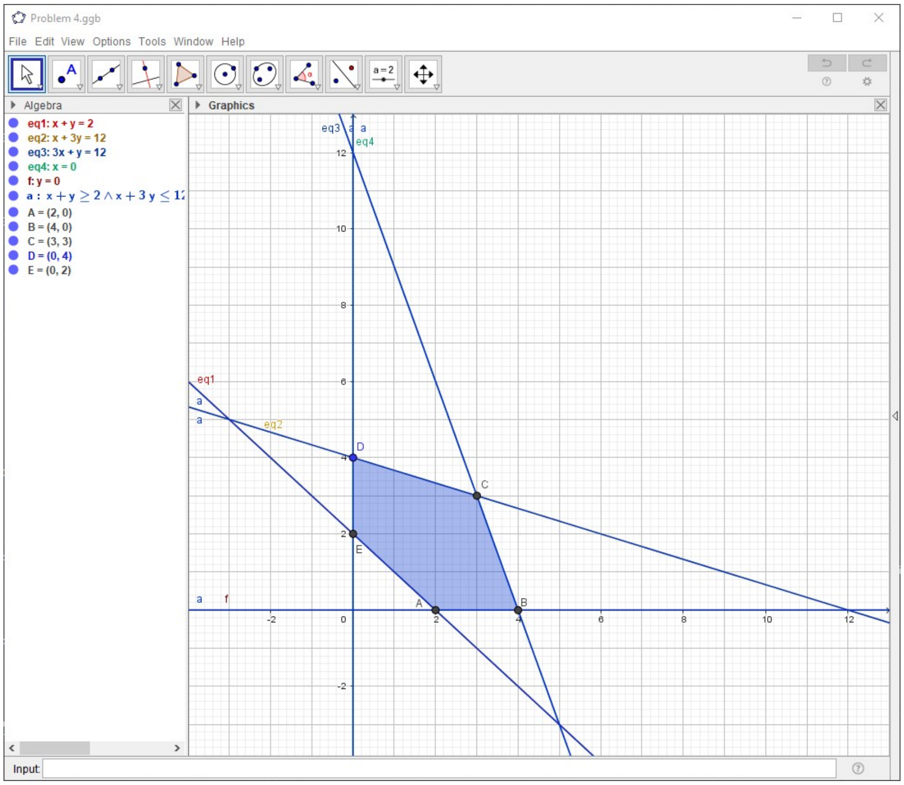


Figure 4: The feasible region from Problem 4

We check the value of the function $L=2x+3y$ at each of the points $A(2,0)$, $B(4,0)$, $C(3,3)$, $D(0,4)$ and $E(0,2)$ by entering in the Input bar, one by one, the following expressions:

$$2x(A)+3y(A),$$
$$2x(B)+3y(B),$$
$$2x(C)+3y(C),$$
$$2x(D)+3y(D),$$
$$2x(E)+3y(E).$$

The values will be displayed in the Algebra window. So, we get:

$$2x(A)+3y(A)=4\,,$$
$$2x(B)+3y(B)=8\,,$$
$$2x(C)+3y(C)=15\,,$$
$$2x(D)+3y(D)=12\,,$$
$$2x(E)+3y(E)=6\,.$$

Hence, the minimum value is $L=4$ which is obtained when $x=2,\ y=0.$ ■

Next, we will give an example of a word problem.

Problem 5. Products of a same kind are manufactured in two production plants A and B. 250 units are produced in plant A and 350 units in plant B. Three stores I, II and III have a demand of 150, 240 and 210 units, respectively. The transportation costs per unit from the production plants to the stores are given in the following table:

Table 1: Transportation costs per unit

Store / *Plant*	I	II	III
A	4	3	5
B	5	6	4

Find a transportation plan so that the total transportation expenses will be the lowest.

Solution: Since the given problem is a balanced transportation problem (i.e., the total amount of units produced in both plants A and B equals the total amount of demanded units by all three stores) we may proceed as follows. Let x denote the number of products transported from plant A to store I and y the number of products transported from plant A to store II. Since the needs of store I are 150 units, it is necessary to bring $(150-x)$ units from B to I. Also, from plant B to store II it is necessary to bring $(240-y)$ units. Further, plant A produces 250 units and we have already allocated $(x+y)$ units. This means that from A to III will be transported $(250-x-y)$ units. To ensure that the demand of store III is met, the number of units that we need to transport from B to III is $210-(250-x-y) = x+y-40$. Thus, we have the following transportation plan:

Table 2: Transportation plan

Store / *Plant*	I	II	III
A	x	y	$250-x-y$
B	$150-x$	$240-y$	$x+y-40$

By Table 1, the total transportation expenses will be:

$$\begin{aligned} L &= 4x + 3y + 5(250 - x - y) + 5(150 - x) + 6(240 - y) + 4(x + y - 40) \\ &= -2x - 4y + 3280. \end{aligned} \tag{2}$$

According to the condition of the problem, it is necessary to find the minimum of the function in (2). But in this example x and y cannot take arbitrary values. The quantity of products cannot be a negative number. Therefore, all the numbers in table 3 are nonnegative, that is:

$$\begin{gathered} x \geq 0 \wedge y \geq 0 \wedge 250 - x - y \geq 0 \wedge 150 - x \geq 0 \wedge \\ 240 - y \geq 0 \wedge x + y - 40 \geq 0 \end{gathered} \tag{3}$$

which means we need to find a minimum of the function in the area given by the system of inequalities (3). That area is shown in Figure 5.

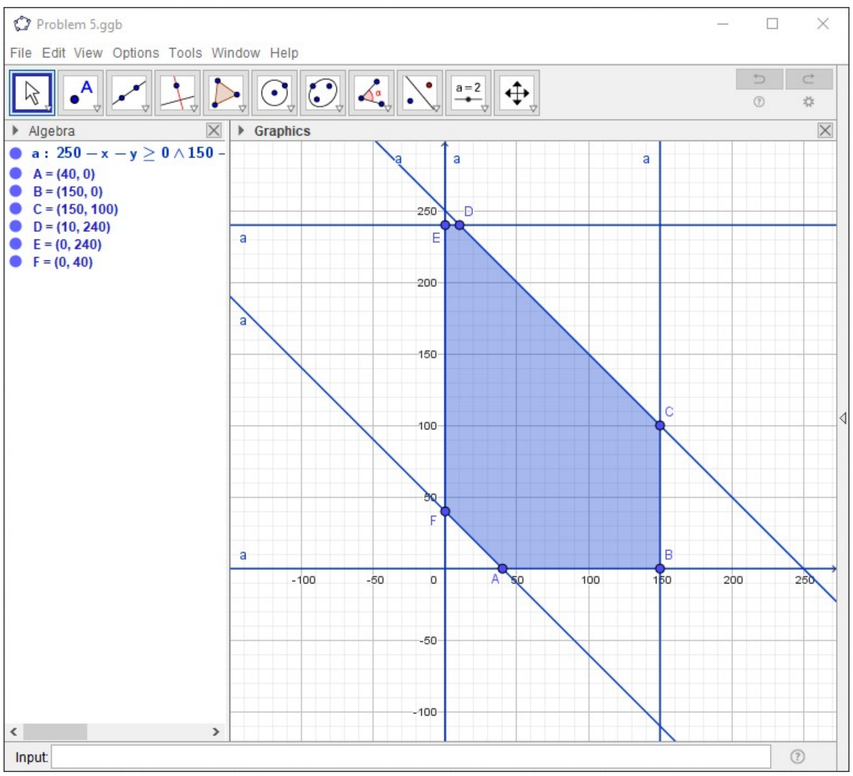


Figure 5: The feasible region from Problem 5

The area is obtained by entering the inequalities in the input field in the GeoGebra window that is in the input field we enter:

$$x \geq 0 \wedge y \geq 0 \wedge 250 - x - y \geq 0 \wedge 150 - x \geq 0 \wedge 240 - y \geq 0 \wedge x + y - 40 \geq 0 .$$

The vertices of the polygon are the points:

$$A(40,0),\quad B(150,0),\quad C(150,100),\quad D(10,240),\quad E(0,240),\quad F(0,40),$$

The function will take the smallest value at one of the vertices of the polygon *ABCDEF*. We check the value of the function in each of the points *A*, *B*, *C*, *D*, *E*, *F* by entering in the Input bar, one by one the following expressions:

$$\begin{gathered}-2x(A)-4y(A)+3280\,,\\ -2x(B)-4y(B)+3280\,,\\ -2x(C)-4y(C)+3280\,,\\ -2x(D)-4y(D)+3280\,,\\ -2x(E)-4y(E)+3280\,,\\ -2x(F)-4y(F)+3280\,.\end{gathered}$$

In the Algebra window we obtain:

$$\begin{gathered}-2x(A)-4y(A)+3280=3200\,,\\ -2x(B)-4y(B)+3280=2980\,,\\ -2x(C)-4y(C)+3280=2580\,,\\ -2x(D)-4y(D)+3280=2300\,,\\ -2x(E)-4y(E)+3280=2320\,,\\ -2x(F)-4y(F)+3280=3120\,.\end{gathered}$$

The minimum value is obtained at point $D(10,240)$. Hence, for the transportation plan we get the following table:

Table 3: Transportation plan

Store / *Plant*	I	II	III
A	10	240	0
B	140	0	210

which means that:

- 10 units of product should be transported from plant A to store I,
- 240 units of product should be transported from point A to point II, etc.

For this transportation plan the value of the total expenses will be 2300. ■

In order to confirm our opinion and firm belief for the benefits of the use of software for learning mathematics, as well solving mathematical problems, especially when applying the graphical solution method, we created a group of 20 students from the Faculty of computer science who voluntarily attended classes where they learned how to use the GeoGebra software while learning different mathematical topics. In these classes, in addition to other mathematical topics, Linear programming was also covered. After the classes, at the next meeting, the students received a questionnaire that they had to answer in a short time. Questionnaire and the results are given on the next page.

Students completed the questionnaire in a very short time. All 20 students answered "yes" to five from ten questions. For the rest five questions most of the students also answered "yes", and just few with "no" or "maybe". The responses could only confirm our opinion that students benefit greatly from using the software. The results from a questionnaire are given in Table 4 and Figure 6.

QUESTIONNAIRE

Question	**Answer**
1.With help of GeoGebra software solving tasks of linear programming is taking less time than manually?	Yes/ no / maybe
2. Did the GeoGebra software help you to permanently remember the solved tasks in your memory?	Yes/ no / maybe
3. When solving linear programming tasks graphically at home, would you use GeoGebra software?	Yes/ no / maybe
4. Does the quick and accurate solution of problems from linear programming graphically with the help of GeoGebra software increase the motivation of students for learning problems from this topic and for learning mathematical problems in general?	Yes/ no / maybe
5. Does GeoGebra educational software is a good choice for solving graphical linear programming tasks?	Yes/ no / maybe
6. Would you continue to follow additional classes in which tasks from various mathematical topics would be solved with the help of GeoGebra software?	Yes/ no / maybe
7. Do you think that your success in math subjects will be higher after using GeoGebra software for problems solving?	Yes/ no / maybe
8. Is it interesting for you solving problems of linear programming graphically using GeoGebra software?	Yes/ no / maybe
9. Would you recommend to other students using the software when learning any math topic?	Yes/ no / maybe
10. Do you think there should be books that explain how to solve graphical problems from linear programming using GeoGebra software?	Yes/ no / maybe

Answers:

Table 4: Results from a questionnaire

Question (#)	yes	no	maybe
1	20	0	0
2	10	5	5
3	15	3	2
4	20	0	0
5	18	2	0
6	12	4	4
7	20	0	0
8	16	3	1
9	20	0	0
10	20	0	0

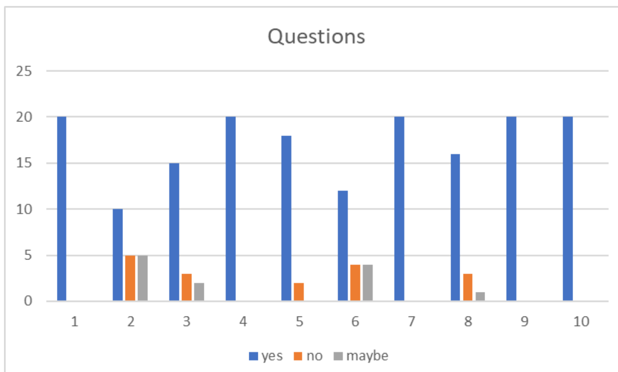


Figure 6: Results from a questionnaire

3. Few Additional Examples

In this section, through few additional examples, we are going to give an updated version of the instructions contained in [8] on how the use GeoGebra for the graphical solution of a given LP problem. Our goal is to take advantage, as much as possible, of GeoGebra's powerful tools to obtain fast and accurate graphical solution for a given LP problem with two decision variables.

The algebraic and the graphical method for solving LP problems with two variables are usually performed simultaneously. Sometimes we can obtain the solution only with one of the methods, but sometimes each method, if performed alone may lead to an incorrect answer. So, we always encourage students to do some blend of these methods.

In general, the algebraic method consists of finding the coordinates of the intersecting points of the bounding lines as solutions of systems of two linear equations with two variables. We have to consider all the systems that we can form from the equations of the bounding lines. Then, we check which of the points that are a unique solution to some of these systems are extreme points (i.e., satisfy all the given constrains – for the LP problem we can immediately discard all points for which we get at least one negative coordinate). Finally, we calculate the values of the objective function at the extreme points and determine which of these values is the smallest and which is largest. But this method can be employed only when the feasible region is bounded. To verify algebraically (only) that the feasible region is indeed bounded usually takes much more time then, for example, drawing a quick sketch on paper or use GeoGebra to draw a *convex* polygon with vertices at the extreme points. Otherwise, we may obtain an incorrect answer. For example, if we use only the algebraic method to get the solution of Problem 3, then we will have to find the solution of $C_5^2 = \frac{5!}{2!\cdot 3!} = 10$ systems of two linear equations with two variables. Each of these systems will have a unique solution. Three of the systems will have the same solution $x = 2$, $y = 0$, i.e., the point $A(2,0)$, which is an extreme point, another three will have the same solution $x = 0$, $y = 3$, i.e., the point $B(0,3)$, which is also an extreme point, while the solutions of the other four systems are not extreme points. Since it is not unusual to obtain that the feasible region is a segment in the plane, which is a bounded convex set, some students proceed with calculating the values of the objective function and conclude that $L = x + y$ has a minimum $L = 2$ at $A(2,0)$ and maximum $L = 3$ at $B(0,3)$.

On the other hand, the accuracy of the solution obtained with the graphical method mainly depends on:

- the accuracy of the graphical representation of the feasibility region,
- the accurate determination of the points at which the objective function has a minimum and/or maximum (usually done by drawing two or more objective function lines).

So, most of the time, if we what to get a correct answer or verify our conclusions from the graphics, we usually calculate the coordinates of the points at which the minimum or/and maximum is attained by solving systems of equations, and after that we calculate the value of the objective function at those points.

The general steps of the graphical method, regardless of whether it is performed on paper or with the help of some software, were covered with the examples in the previous section. The way GeoGebra is used in the solution of Problem 1, is the one that most closely resembles the way we usually perform the graphical solution on paper. We deliberately left one of the most common misconceptions that is found both among teachers and students, and in some textbooks as well. Is about the geometrical interpretation of the objective function. Although this misconception rarely affects the accuracy of the final solution, since it is one of the crucial elements for building a more effective GeoGebra file, we will address this first.

In the solution of Problem 1 we've placed the question "Really?" in small brackets. The question is not about the interpretation of the objective function as a movable line across the plane, but the direction in which it moves.

The objective function of a given LP problem with two variables is a linear function (with two variables) of form:

$$f = f(x, y) = ax + by,$$

where $a, b \in \mathbb{R} \setminus \{0\}$ are given numbers. When we draw the objective function lines (the isoprofit lines or the isocost lines) on the graph, we are actually trying to determine which of the elements from the one parameter family of parallel lines:

$$\{ax + by = f : f \in \mathbb{R}\}, \tag{4}$$

intersects the feasible region (see for example Ch. 23, Example 5.1 in [7]). Since the lines in (4) may be regarded as different positions of a line that moves across the plane, the geometrical interpretation of the objective function as a movable line on the plane is perfectly fine and quite useful when we perform a graphical solution of a given LP problem. But, as the values of objective function increase, we would get the same set of dashed lines on Figure 2 if we perceive the movement of the line from top to bottom, or from southwest to northeast, or in the direction of *any* vector $\vec{v} = (x, y)$ whose coordinates satisfy $x + y \geq 0$. This would indicate that the movement of the line can be interpreted as the "sliding across the plane". However, the direction in which the line moves is not always form left to right, or from bottom to top etc. Hence, the movement is not sliding. The direction in which the line moves is not arbitrary as we usually think or say. It moves:

- in the direction of the vector $\vec{n}_f = (a, b)$, as f increases,
- in the opposite direction of the vector $\vec{n}_f = (a, b)$, as f decreases.

The vector $\vec{n}_f = (a, b)$ is perpendicular to every line in (4) and is called the *direction* (or the *gradient*) *of the objective function* (see for example Ch. 3, Section 3.1 in [4]). So, the line $x + y = 0$ (or $y = -x$) in Problem 1 and Problem 2, actually does not move from left to right as the value of the objective function increases, it moves in the direction of the vector $\vec{n}_L = (1,1)$.

Instead of "sliding across the plane", a more accurate analogy would be a "rolling line" (like the rolling of a cylindrical straight pencil which is initially placed parallel to the objective function lines, the pencil may slide in any direction, and still remain parallel to the objective function lines, but it can roll in only two directions to remain parallel to these lines). Thus, instead of drawing a net of objective function lines, we may simply draw the vector $\vec{n}_f = (a, b)$ and, if necessary, lines that are perpendicular to this vector trough the extreme points that may be the points at which the objective function has minimum and/or maximum.

Now let's see some additional examples.

Problem 6. Find the minimum and the maximum of the function $f = 2x + y$ over X if,

$$X: \begin{cases} -2x + y \le 1 \\ x + 2y \le 9 \\ x \le 5 \\ y \le 3 \\ x \ge 0 \\ y \ge 0 \end{cases}.$$

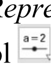
Solution: *Step 1: Determining the feasibility region*. In the Input bar we enter the following expressions (commands), one line at a time:

```
b1:= -2x+y=1
b2:= x+2y=9
b3:= x=5
b4:= y=3
b5:= x=0
b6:= y=0
n1:= -2x+y<=1
n2:= x+2y<=9
n3:= x<=5
n4:= y<=3
n5:= x>=0
n6:= y>=0
X:= n1 && n2 && n3 && n4 && n5 && n6
```

In this way we will assign more meaningful names for the objects instead of the letting GeoGebra to do the naming. After entering these expressions, the Graphic window will be quite cluttered, so we need to turn of the visibility of the individual inequalities `n1`, `n2`, `n3`, `n4`, `n5` and `n6`. The commands for the symbols $\le, \ge, \wedge$ which were used in the previous section are `<=`, `>=`, `&&` respectively. After completing this, from the graphic obtained in the Graphics window, we can see that the feasible region (obtained with the last command) is bounded and determine which pairs of the lines `b1`, `b2`, `b3`, `b4`, `b5` and `b6` have an intersection at a vertex of the feasible region. Then, either by using the Intersect tool, or entering in the Input bar the following commands, one line at a time:

```
A:=Intersect(b5,b6)
B:=Intersect(b6,b3)
C:=Intersect(b3,b2)
D:=Intersect(b2,b4)
E:=Intersect(b4,b1)
F:=Intersect(b1,b5)
```

we determine the coordinates of vertices of the feasible region.

Step 2: Representing the objective function as a movable line. First, with the Slider tool  we create a slider in the Graphics window. A window as in Figure 8 will appear. We'll change the default name from "a" to, let's say "s_f ", which will give the name s_f to the slider in the Algebra window. We'll replace the values -5 and 5 with "larger" ones. Although the coefficients of the objective function and the constrains are relatively small (by absolute values), for cases like this, in our experience the interval for the slider may go from -50 to 50 (and even wider). If this doesn't work well, we can adjust these values later. At this moment will leave the Increment empty (i.e., at its default value 0.1).

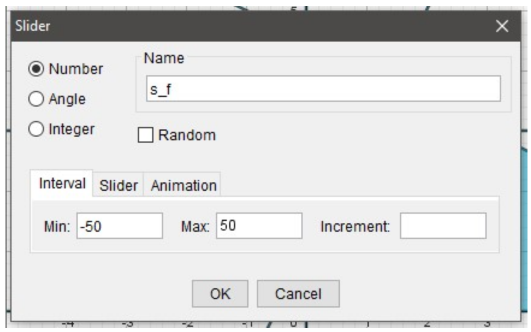


Figure 7. Initial settings for the slider

Next, in the Input bar we enter the following expressions, one line at a time:

```
n_f:= (2,1)
m_f:= x(n_f)*x+y(n_f)*y=s_f
f(x,y)= x(n_f)*x+y(n_f)*y
```

The first expression is the direction of the objective function. Sometimes, to ensure that the direction is visible in the Graphics window (not to small, not to large) we need add to another vector, like

```
p_f:= q*n_f
```

for an appropriate choice of a *positive* value for `q` (see the Problem 8.b). The second expression is the line that will visualize the "movement" of the objective function line in the Graphics window. This line will move:

- in the direction of the objective function, as the value the slider increase,
- opposite of the direction of the objective function, as the value the slider decrease,

while the third expression is the actual objective function. We could've entered `m_f:= 2*x+y=s_f` and `f(x,y)= 2*x+y`, but we want to use the dynamic property of GeoGebra as much as possible (see Remark 7).

Before we proceed, we need to test the slider. For a bounded feasible region, the line m_f should move so that it can pass through and exit the feasible region as we drag the dot on the slider in both directions, and yet, remain visible in the

Graphics window (assuming that its visibility is not turned off in the Algebra window). If the movable line m_f is still intersecting the feasible region or not visible at all, ether when the slider is at its minimum or at its maximum value, then we should replace the values -50 to 50. For this problem, we may adjust the slider from -5 to 20.

After Step 1 and Step 2, by changing the colour of the vector n_f, the slider s_f and the movable line m_f, we'll get something like Figure 8.

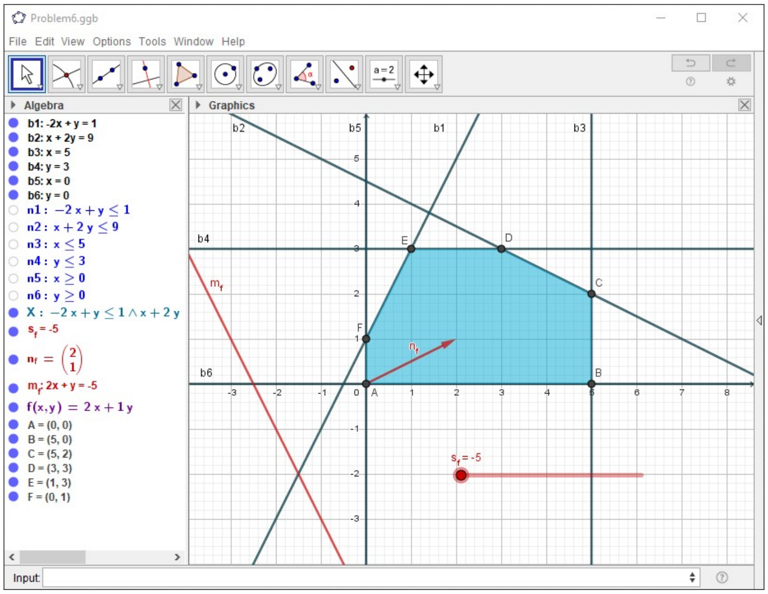


Figure 8: Initial elements for the graphical solution

Next, we need to slowly drag the dot on the slider and pay attention to the values above the slider (or those of s_f and on the right side of the equation in m_f in the Algebra window). As we drag the dot from lowest to highest value of the slider, the order of the vertices of X in which the movable line m_f in Graphics window will pass through are:

$$A\,(\approx 0)\,,\ F\,(\approx 1)\,,\ E\,(\approx 5)\,,\ D\,(\approx 9)\,,\ D\,(\approx 10)\ \text{ and } \ C\,(\approx 12)\,.$$

The values in the parenthesis are the approximate values that will appear above the slider.

Note: Since the coordinates of these points and the coefficient of the objective function are all integers, we may obtain the exact values while dragging the dot on the slider if we set 1 as the Increment value of the slider.

At this point, having in mind that a linear function will attain is minimum and maximum on a bounded convex polyhedral set at its extreme points, we already have all the information that we need to give an answer to the problem. The objective function attains its:

- minimum at $A(0,0)$ and $\min_X f = 2\cdot 0 + 0 = 0$,

- maximum at $C(5,2)$ and $\max_X f = 2\cdot 5 + 2 = 12$.

But, let the GeoGebra confirm the results. In the Input bar we enter the following expressions, one line at a time:

```
ValueAtA:=f(A)
ValueAtB:=f(B)
ValueAtC:=f(C)
ValueAtD:=f(D)
ValueAtE:=f(E)
ValueAtF:=f(F)
```

The Algebra window will be updated with the corresponding objects as shown in Figure 9. ■

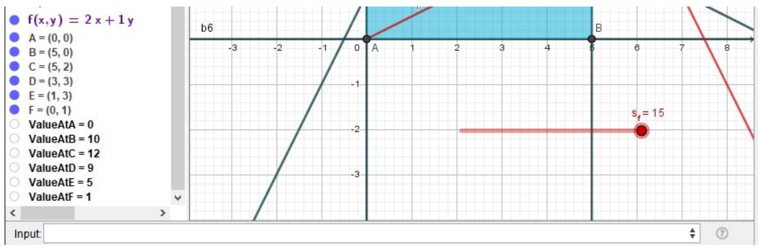


Figure 9: Values of the objective function at the extreme points

Remark 7: After entering the expressions for the bounding lines, the inequalities for the constrains and the feasible region, we'll get the same results as in Section 2. The advantages of the approach described in the solution of Problem 6 are, at least, two folded. First, we may easily encounter LP problems, especially when working with real-life problems, without any feasible solution. This will occur whenever two or more of half planes determined with the constrains don't intersect. For example, if:

$$X: \begin{cases} x+y\le 4 \\ x+y\ge 8 \\ x\ge 0 \\ y\ge 0 \end{cases},$$

and if we directly enter in the Input bar

$$x+y\le 4 \ \wedge\ x+y\ge 8 \ \wedge\ x\ge 0 \ \wedge\ y\ge 0,$$

then nothing will be displayed in this Graphics window, which will not be the case if we first enter separately the inequalities of the constrains (and then the appropriate command for their intersection like the one for the set X in the solution of Problem 6).

Second, we what to use the dynamic feature of GeoGebra as much as possible. We can easily make any changes or corrections if needed. Small changes in the coefficients of the objective function, or the constrains, or replacing one or more of the inequality signs with the reverse ones, may have huge impact on the solution of the problem. Let's see how it works.

We are going to use the GeoGebra file created for Problem 6 without the suggested modification of the Increment of the slider. We've saved this file under the name Problem6.ggb, made two copies of it and renamed them as:
Problem6(modified for Problem8a).ggb, and
Problem6(modified for ForProblem8b).ggb.

Problem 8. a) Find the maximum of the objective function in Problem 6 over the sets

$$Y:\begin{cases}-2x+y\le 1\\ 3x+y\le 17\\ x\le 5\\ y\le 3\\ x\ge 0\\ y\ge 0\end{cases},\quad \text{and}\quad Z:\begin{cases}-2x+y\le 1\\ x\le 5\\ y\le 3\\ x\ge 0\\ y\ge 0\end{cases}.$$

b) Find the maximum of $f=5x+10y$ over the set X given in Problem 6.

Solution: a) Each of the sets of constrains Y and Z can be derived from X: Y by replacing the second constrain in X with $3x+y\le 17$, Z by completely removing the second constrain in X.

Now let's open the file Problem6(modified for Problem8a).ggb and continue were we left in the solution of Problem 6.

In the Algebra window, first by double clicking on the object `b1` we will adjust the coefficients of $x+2y=9$ to obtain $3x+y=17$, then by double clicking on the object `n1`, we will adjust the coefficients of $x+2y\le 9$ to obtain $3x+y\le 17$.

And that is all.

At least for now.

As we can see form Figure 10, everything is adjusted at once. But now we have something different. From the changes in the last six objects in the Algebra window, which are made automatically, we see that the maximum of the function is no longer at $C(5,2)$, but in D. Moreover, due to the rounding, the first coordinate of D is not the exact one (from the information displayed in the Algebra window the coordinates of this point are $D(4.67,3)$). The maximum value of the objective function is also not the exact one, but an approximate value of it (12.33). If we want to get the exact coordinates of D and the exact value of the maximum, we need to solve the system of two linear equations formed by the equalities b2 and b4. We can do this on paper or employ GeoGebra do this for us. GeoGebra has two commands for solving systems of linear equations, `Solve` and `Solutions`, but they are available only through the CAS module. We will use the first command. So, let's open the CAS window.

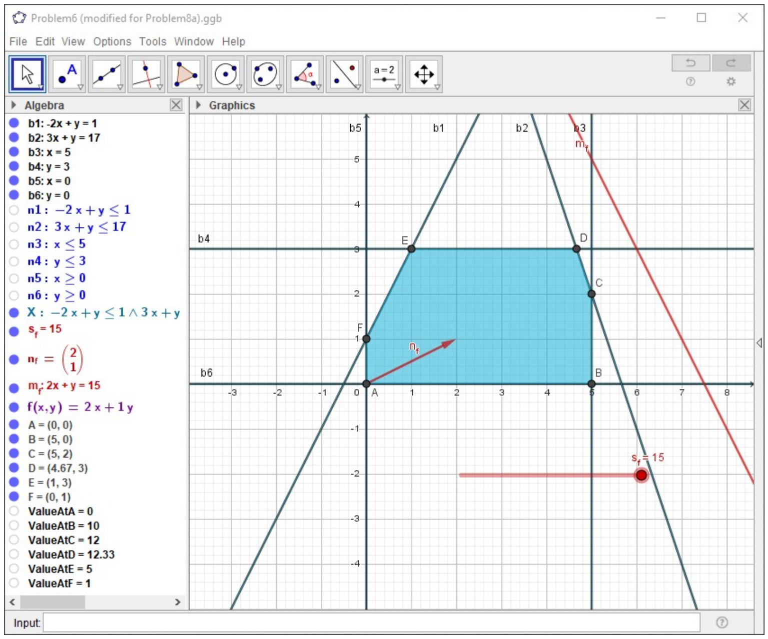


Figure 10: The results after changing one constrain from Problem 6

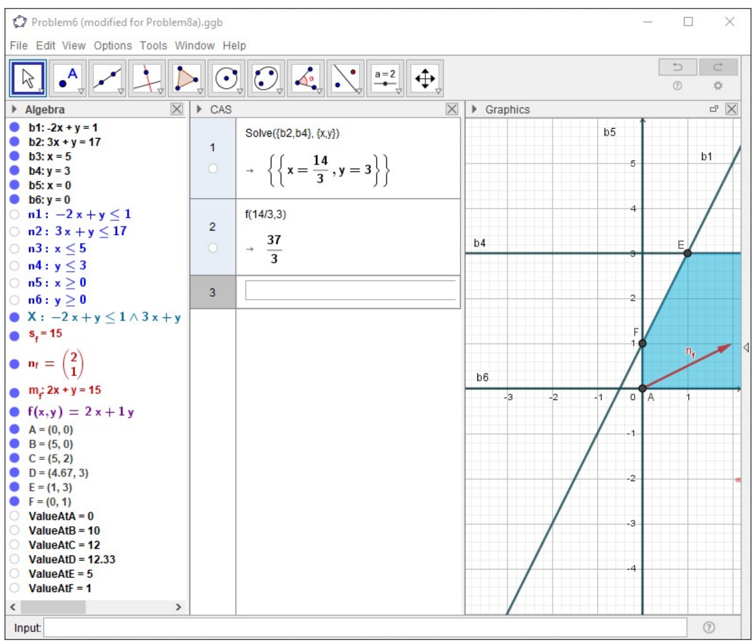


Figure 11: Use of the CAS module in GeoGebra for Problem 8.a).

Since we already have all the equations of the bounding lines, and we've obtained the point D as in intersection point of the lines `b2` and `b4`, in the Input bar *in the CAS window* (see Figure 11) we enter:

```
Solve({b2,b4},{x,y})
```

The result will be displayed bellow in form of two equations, $\left\{\left\{x=\frac{14}{3}, y=3\right\}\right\}$. No rounding of the first coordinate! Now that we have the exact coordinates of D, since the Algebraic windows already contains an expression for calculating the values of the objective function, in the next Input bar *in the CAS window* we enter:

```
f(14/3,3)
```

The result displayed bellow will be $\frac{37}{3}$. Hence, $f = 2x + y$ has a maximum value $\frac{37}{3}$ at point $D\left(\frac{14}{3},3\right)$.

Now let's find the maximum of $f = 2x + y$ over Z. We can save the above changes, close the file, and work on a copy of it (if we what to preserve the changes), or continue and lose all the results we've just obtained for the set Y.

For the set Z we must proceed cautiously. The best is to follow these steps in exact order, except maybe interchanging step 3 and 4. All actions are on the objects in the Algebra window.

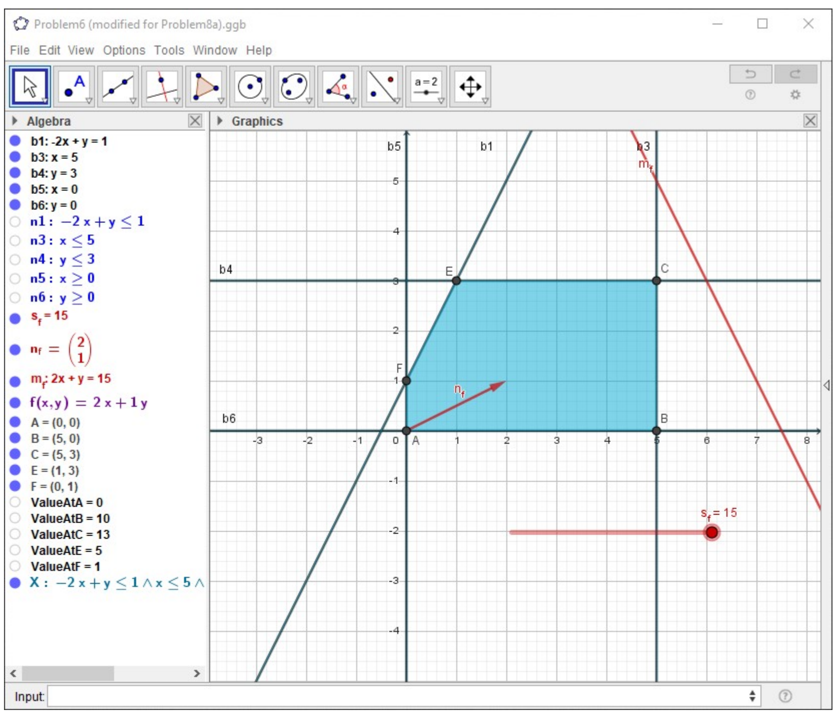


Figure 12: Result in GeoGebra after changes for the set Z in Problem 8.a)

1. double click on X and in the new window delete: ∧ n2(x,y),
2. delete the inequality n2,
3. double click on C, and the new window replace b2 with b4,
4. double click on D, and the new window replace b2 with b3,
5. delete one of the points C or D,
6. delete b2.

The result should look the image in Figure 12. This time we have: the maximum of $f = 2x + y$ over Z is 13, and it's attained at point $C(5,3)$.

b) Now we'll work on file Problem6(modified for ForProblem8b).ggb. We need to find the maximum value of the function $f = 5x + 10y$ over the set X defined in Problem 6. For this, the only change that we have to make is on direction of the objective function n_f and observe how everything adjusts, both in Algebraic and in Graphics windows after that. By double clicking on the vector n_f in the Algebra window we repalce its coordinates (2,1) with (5,10). This vector is quite large in "size" relative to the other objects, but we must not make any further changes, since this will have effects on the values of the objective function. Instead, we'll turn off the visibility of the vector n_f and, to keep the direction of the movement of the line m_f in the Graphical windows, we enter in the Input bar:

```
p_f:= 0.5*n_f
```

and then, if necessary, change its colour and line style.

That's all.
At least for now.
The result should look like the image in Figure 13.

Now look at the results we obtained. There are two different points at which we have largest value of 45 of the objective function: $C(5,2)$ and $D(3,3)$ Both points lie on a same objective function line:

$$5x + 10y = 45.$$

For every point on this line, hence for every point of the segment with end points at C and D (which is included in the feasible region), the value of the objective function is 45.

So, the maximum of $f = 5x + 10y$ over X is attained at infinitely many points, each point on the segment with end points at C and D. ■

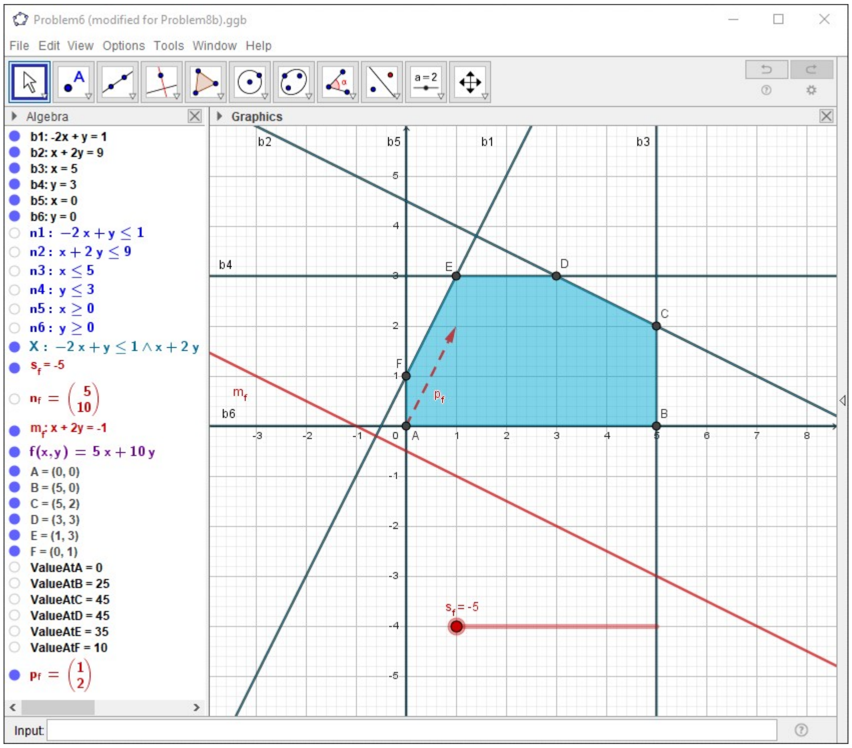


Figure 13: Result after changes for Problem 8.b).

Some additional remarks before we finish this section.

Remark 9. In the solution of the part a) in the Problem 8 we briefly described how we can use GeoGebra for calculation of the coordinates of one of the extreme points. Since we had previously entered all the equations of the bounding lines, we may repeat this procedure for every other possible combination of `b1`, `b2`, `b3`, `b4`, `b5` and `b6` (in total $C_6^2 = \frac{6!}{2! \cdot 4!} = 15$), and thus finish the first step in the algebraic solution to the LP problem.

Remark 10. In the solution of the part b) in Problem 8, we gave an example of a function that attains its maximum value at infinitely many points of the feasible region. This situation, which is highly probable whenever one or more of the bounding lines is also an isoprofit/isocost/objective-function-line, may occur even in real life problems. Let's look again at the Problem 5 from Section 2. Table 2 in the solution of this problem depends only on the number of units produced at A and B, and the quantities demanded by the stores (i.e., with the unknowns x and y it is totally independent from the transportation costs per unit). Let's change one of values in Table 1. We'll increase the transportation cost per unit from A to II. Instead of 3 we'll take 5. So, we will consider the following table for the transportation costs per unit:

Table 5: Modified transportation costs per unit for Problem 5

Store / *Plant*	I	II	III
A	4	5	5
B	5	6	4

In this case, the objective function will be $L = -2x - 2y + 3280$. The result obtained with GeoGebra according to the instructions in this section is given in Figure 14.

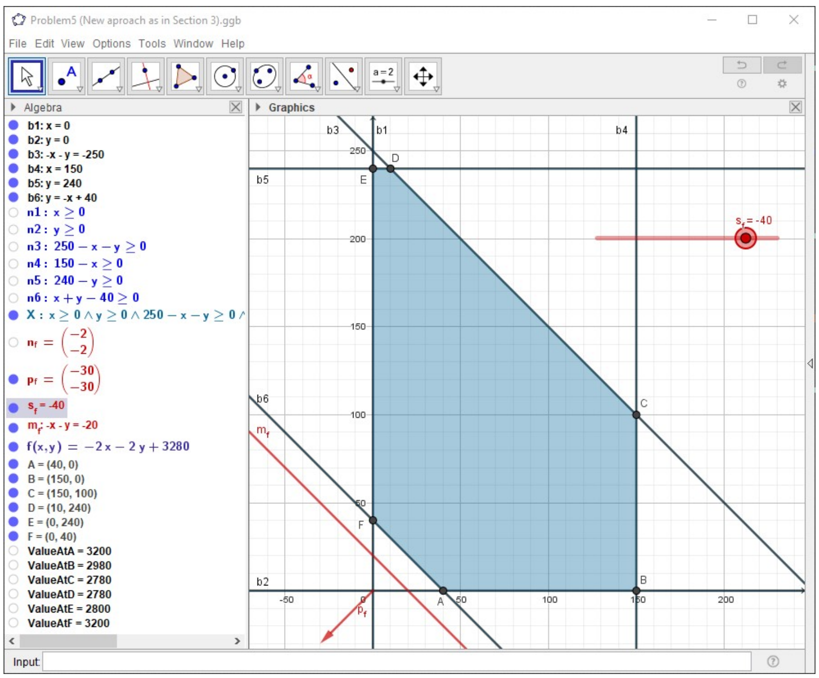


Figure 14: GeoGebra file for modified Problem 5

Note: If we create a GeoGebra file as described in this section, then we must add a vector `p_f:= 30*n_f` (or some other value for `q` above 25 so the this vector is visible in the Graphics view) and set the values for the slider around from about -700 to 100 and set the Incrment value as 10. Observe the direction of the movement of m_f (it seems opposite of "from left to right" as we drag the dot on the slider from left to right i.e., when the s_f increases). The equations in the Algebra window are entered as described in this section, but GeoGebra sometimes automatically overwrites the equalities. For example, the equation $250 - x - y = 0$ is automatically overwritten as $-x - y = -250$.

From Figure 14, we have that the minimum of $L = -2x - 2y + 3280$ is 2780, and is attained both at point $C(150,100)$ and point $D(10,240)$. This means that the minimum is attained on the segment with end points at C and D i.e., at infinitely many points. But this time we have a special case of LP problem, an integer programming problem, which adds another constrain upon the solution. The values of x and y must also be integers (we don't transport $3\frac{1}{2}$, 2.8 or 0.3 units).

Hence, we don't have infinitely many solutions for the problem. But we've still end up with "too many". How many? Exactly 141 (140 plus Table 3). This means that, in addition to Table 3, we can create another 140 different tables, for which the total cost of transportation will be the same (2780). Enough to fill approximately another 18 pages of this paper. So, we end up here.

4. CONCLUSION

Graphically solving problems in any mathematics topic on a sheet of paper often ends up with an incorrect solution. The result is usually incorrect because students make mistakes when drawing, the drawing can by unprecise (and precision is often very important), manual drawing takes a lot of time, to create an accurate drawing students need to prepare appropriate tools, etc. In this paper we offer a method to increase the interest in mathematics with using learning software which will help for greater curiosity and increased motivation to work and solve problems. For graphical solution of linear programming problems, we've used GeoGebra. With this software students can check if their solution is correct, to get a solution in advance which will guide them to correctly solve the problem on paper, to get solution in short time, etc. Thus, students will be motivated to study mathematics and to achieve better results. This is confirmed by the results of the previously mentioned questionnaire, which was answered by group of 20 students. We encourage, both teachers and students, to use free software, as GeoGebra, while solving the math problems. It is a software that has been used for a long time in education around the world and research shows that its use gives better results.

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РАЗВОЈ НА ЕЛАСТИЧНОСТ НА МИСЛЕЊЕТО КАЈ УЧЕНИЦИТЕ ВО ПОЧЕТНОТО ОБРАЗОВАНИЕ

Методи Главче[1)], Ристо Малчески[2)]

Abstract. Предмет на разработка на овој труд е значењето на наоѓање на повеќе начини за решавање на една иста задача за развојот на еластичноста на мислењето кај учениците во почетното образование. Во трудот е даден краток преглед на квалитетите на мислењето и се разработени три примери погодни за развој на елаатичноста на мислењето.

Клучни зборови. Мислење, квалитети на мислењето, математичка задача

1. Вовед

Наставата по математика претпочита реализирање на повеќе општи и посебни цели, меѓу кои како посебна цел е подобрување на квалитетите на мислењето, како на пример *еластич-носта на мислењето* која се карактеризира со умеењето лесно да се премине од еден кон друг начин на решавање на проблемот, да се наоѓаат нови начини за решавање на проблемите при промена на условите, да се преструктуира системот од сопствени знаења, што ќе овозможи усвојување нови знаења. Покрај еластичноста, од посебна важност е кај учениците да се разви-ва *длабочината на мислењето* која се карактеризира со умеењето да се проникнува во суш-тината на изучуваните факти, да се согледа нивната врска со други факти, да се моделираат различни ситуации, да се согледа како тие модели можат да се применат во практиката итн.

Претходно споменатите квалитети на мислењето не е можно да се унапредуваат без разви-вање на *целесообразноста на мислењето*, која претставува стремеж да се оствари разумен избор на методи и средства за решавање на некој проблем, при што постојано се ориентираме кон целта поставена во проблемот и кон наоѓањето најкратки патишта за остварување на таа цел. Докажувањето теореми и воопшто решавањето задачи во наставата е незаменливо средство за развивање на умеењето за избор на средства (дефиниции, аксиоми, теореми) за постигнување на дадена цел. Понатаму, целесообразноста на мислењето овозможува појавување на уште едно негово својство, а тоа е *рационалноста*. Ова својство се карактеризира со економичност во однос на времето и на средствата за решавање на даден проблем. Рационалноста на мислењето е тесно поврзана со *шаблонизацијата* на истото. Имено, за да се постигне рационалност во мислењето се користат алгоритми и теореми, кои како готови шаблони се применуваат во практиката. Освен тоа, рационалноста се должи на фактот дека алгоритмите и теоремите се однесуваат на цели класи објекти.

Понатаму, решавањето на задачи е најдобар начин за развој на *широчината на мислењето*, која се карактеризира со способноста да се опфатат проблемите во целост, да се прошири примената на добиените резултати итн. Затоа, рационалноста на мислењето е во тесна врска со широчината на мислењето. Сето ова е непосредно поврзано со критичноста на мислењето, која се карактеризира со тоа што различните мислења не се прифаќаат без доволно аргументи, туку тие да подлежат на проценка. Јасно, критичноста на мислењето е еден од квалитетите, кои човештвото го довеле до неопходноста од убедување во точноста на тврдењето и наоѓање објективни критериуми за оценка на нивната вистинитост.

2. Развојот на еластичноста на мислењето во функција на развој на остана-тите квалитети на мислењето кај учениците во почетното образование

Претходно изнесеното непосредно укажува дека квалитетите на мислењето се заемно поврзани, па затоа развивањето на еден квалитете не е можно без притоа да се развиваат и останатите квалитети на мислењето. Меѓутоа, согледувајќи ја суштината на еластичноста, широчината, длабочината, шаблонизацијата, рационалноста, целесообразноста и критичноста на мислењето, лесно може да се заклучи дека кај учениците во почетното образование најлесно можеме да ја развиваме еластичноста на мислењето. Токму затоа во следните разгледувања ќе презентираме три примери за кои ќе бидат дадени повеќе начини за нивно решавање, што како што рековме е основен метод за развивање на еластичноста на мислењето.

Пример 1. Горјан како шифра за заклучување на својот мобилен телефон поставил парен четирицифрен број. За да не го заборави бројот, тој запишал:

- сите цифри се различни, а збирот на сите цифри е 15,
- цифрата на единиците е три пати помала од цифрата на илјадитите,
- цифрата на десетките е помала од цифрата на единиците.

Определи ја шифрата која Горјан ја поставил за отклучување на својот мобилен.

Решение. *Прв начин.* Ако a,b,c,d се последователно различни цифри на илјадитите, стотките, десетките и единиците на бараниот четирицифрен број $\overline{abcd}$, тогаш

$$a+b+c+d=15.$$

Бидејќи бараниот број е парен, цифрата d на единиците може да е 0, 2, 4, 6 или 8. Цифрата не единиците е три пати помала од цифрата на илјадитите, па затоа $a=3d<10$, што значи дека можни се два случаја и тоа:

- ако $d=0$, тогаш $a=3\cdot 0=0$, што не е можно бидејќи цифрите се различни и бројот е четирицифрен,
- ако $d=2$, тогаш $a=3\cdot 2=6$.

Бидејќи $d=2$ и $a=6$, а цифрата на десетките е помала од цифрата на единиците, т.е. $c<d$, заклучуваме дека $c=0$ или $c=1$. Сега имаме два случаја:

- ако $c=0$, тогаш $b=15-6-0-2=7$, па бараниот број е 6702,
- ако $c=1$, тогаш $b=15-6-1-2=6$ и тоа не е можно бидејќи цифрите на бараниот број се различни.

Втор начин. Од вториот услов следува дека цифрите на единиците и илјадитите може да бидат паровите 3 и 9, 2 и 6, 1 и 3, соодветно. Понатаму, бидејќи бараниот број е парен добиваме дека цифрата на единиците е 2, а цифрата на илјадитите е 6. Сега, заради третиот услов цифрата на десетките може да е 0 или 1. Бидејќи цифрите мора да се различни бараните број е некој од броевите:

6102, 6302, 6402, 6502, 6702, 6802, 6902,
6012, 6312, 6412, 6512, 6712, 6812, 6912.

Но, меѓу овие броеви само кај бројот 6702 збирот на цифрите е еднаков на 15, па затоа бараниот број е 6702.

Трет начин. Од вториот услов следува дека цифрите на единиците и илјадитите може да бидат паровите 3 и 9, 2 и 6, 1 и 3, соодветно. Понатаму, бидејќи бараниот број е парен добиваме дека цифрата на единиците е 2, а цифрата на илјадитите е 6. Сега, заради третиот услов цифрата на десетките може да е 0 или 1. Ако цифрата на единиците е 0, тогаш заради условот за збирот на цифрите добиваме дека цифрата на стотките е 7, па едно решение е 6702. Ако цифрата на десетките е 1, тогаш заради условот за збирот на цифрите добиваме дека цифрата на стотките е 6, и се добива бројот 6612, кој не е решение бидејќи има две еднакви цифри. Конечно, шифрата на телефонот на Горјан е 6702.

Пример 2. Тројца пријатели Раде, Ласте и Марко тренираат маратон. Тие имаат зелено, сино и црвено капче, а носат патики број 43, 44 и 45. Чето пати нешто забораваат во соблекувалната, а денес во шкавчето за заборавени работи имало зелено капче и патики број 43. Кој денеска заборавил работи ако се знае дека:

- Ласте нема сино капче,
- оној што има патики број 44 има зелено капче,
- Раде нема патики број 43,
- Ласте нема патики број 44,
- оној што има патики број 43 нема црвено капче.

Решение. *Прв начин.* Знаеме дека Ласте нема сино капче. Бидејќи Ласте нема ниту патики број 44, заклучуваме дека тој нема зелено капче. Значи, Ласте има црвено капче. Лицето што има црвено капче нема патики број 43, па затоа Ласте нема патики број 43. Но, Ласте нема и патики број 44, па затоа тој има патики број 45. Според тоа, Раде нема патики број 43 и 45, па заклучуваме дека Раде има патики број 44. Но, лицето со патики број 44 има зелено капче, па заклучуваме дека Раде има зелено капче. Конечно, Марко има патики број 43 и сино капче.

Од претходните разгледувања следува дека Раде и Марко заборавиле работи во соблеку-валната.

Втор начин. Со З да го означиме зеленото капче, со Ц црвеното капче и со С синото капче. Понатаму, со 43, 44 и 45 да ги означиме патиките број 43, број 44 и број 45, соодветно. Ќе ја пополниме табелата со знаците $+$ и $-$ на следниов начин: ако лицето поседува некој предмет на соодветноот место во табелата ќе ставиме знак $+$, а ако не го поседува тој предмет ќе ставиме знак $-$. Оваа постапка го прати заклучувањето како и во првиот начин на решавање.

	43	44	45	З	С	Ц
Раде						–
Ласте		–		–	–	+
Марко						–
З						
С						
Ц						

	43	44	45	З	С	Ц
Раде			–			–
Ласте	–	–	+	–	–	+
Марко			–			–
З			–			
С			–			
Ц	–	–	+			

	43	44	45	З	С	Ц
Раде	–	+	–			–
Ласте	–	–	+	–	–	+
Марко	+	–	–			–
З			–			
С			–			
Ц	–	–	+			

	43	44	45	З	С	Ц
Раде	–	+	–	+	–	–
Ласте	–	–	+	–	–	+
Марко	+	–	–	–		–
З	–	+	–			
С		–	–			
Ц	–	–	+			

	43	44	45	З	С	Ц
Раде	–	+	–	+	–	–
Ласте	–	–	+	–	–	+
Марко	+	–	–	–	+	–
З	–	+	–			
С	+	–	–			
Ц	–	–	+			

Бидејќи денес се заборавени зелено капче и патики број 43, заклучуваме дека Раде и Марко заборавиле работи во соблекувалната.

Трет начин. Со З да го означиме зеленото капче, со Ц црвеното капче и со С синото капче. Понатаму, со 43, 44 и 45 да ги означиме патиките број 43, број 44 и број 45, соодветно. Ќе ја пополниме табелата со знаците $+$ и $-$ на следниов начин: ако лицето поседува некој предмет на соодветноот место во табелата ќе ставиме знак $+$, а ако не го поседува тој предмет ќе ставиме знак $-$.

Првата таблица е пополнета според условите кои се дадени во задачата.

	43	44	45	З	С	Ц
Раде	–					
Ласте		–			–	
Марко						
З	–	+	–			
С		–				

Ц	–	–	

Од табелата се гледа дека сино капче има лицето кое носи патки број 43 и црвено капќе има лицето кое носи патики број 45. Така ја добиваме следнава табела.

	43	44	45	З	С	Ц
Раде	–					
Ласте		–			–	
Марко						
З	–	+	–			
С	+	–	–			
Ц	–	–	+			

Сега, сино капче носи лицето кое носи патики број 43, а Раде не носи патики број 43, па затоа Раде нема сино капче. Значи, сино капче има Марко. Зелено капе има лицето кое носи патики број 44. Марко има сино, а не зелено капче, па затоа Марко не носи патики број 44.

	43	44	45	З	С	Ц
Раде	–				–	
Ласте		–			–	
Марко		–		–	+	–
З	–	+	–			
С	+	–	–			
Ц	–	–	+			

Раде носи патики број 44 и има зелено капче.

	43	44	45	З	С	Ц
Раде	–	+	–	+	–	–
Ласте		–		–	–	
Марко		–		–	+	–
З	–	+	–			
С	+	–	–			
Ц	–	–	+			

Сега уште може да се види дека Ласте има црвено капче и носи патики број 45, Марко носи патики број 43.

	43	44	45	З	С	Ц
Раде	–	+	–	+	–	–
Ласте	–	–	+	–	–	+
Марко	+	–	–	–	+	–
З	–	+	–			
С	+	–	–			
Ц	–	–	+			

Бидејќи денес се заборавени зелено капче и патики број 43, заклучуваме дека Раде и Марко заборавиле работи во соблекувалната.

Четврт начин. Прво ќе ги спариме капчето и патиките на ист сопственик. Познато е дека тој што има патики број 44 има зелено капче. Исто така е познато дека тој што има патики број 43 нема црвено капче. Значи, сопственикот на патиките број 43 нема ниту зелено ниту црвено капче, па затоа тој има сино капче. Конечно, тој што има патики број 45 има црвено капче.

Сега, бидејќи Ласте нема сино капче и нема патики број 44, тој мора да има црвено капче и патики број 45. Раде нема патики број 43, па затоа тој има патики број 44 и зелено капче, а Марко го има преостанатиот пар патики број 43 и сино капче.

Бидејќи денес се заборавени зелено капче и патики број 43, заклучуваме дека Раде и Марко заборавиле работи во соблекувалната.

Пример 3. На шаховски турнир учествувале 10 шахисти. Секој шахист играл со секој од преостанатите шахисти по една партија.

а) Колку партии изиграл секој играч?

б) Колку партии се вкупно одиграни на турнирот?

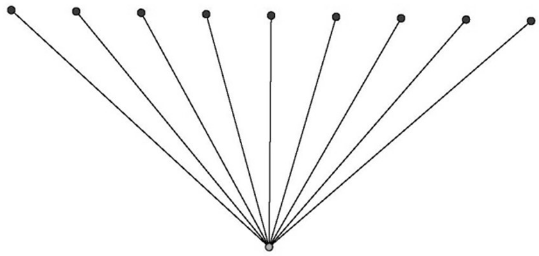
Решение. а) *Прв начин*. Играчите да ги означиме со броевите од 1 до 10. Бидејќи изборот на играчот не влијае на бројот на партиите кои ги игра еден играч, ќе ги испишеме можностите за еден од играчите, на пример за играчот 1. Имаме

1 и 2, 1 и 3, 1 и 4, 1 и 5, 1 и 6, 1 и 7, 1 и 8, 1 и 9, 1 и 10.

Бидејќи е сеедно кој од играчите сме го разгледувале, заклучуваме дека секој играч одиграл по 9 партии шах.

Втор начин. Бидејќи секој играч мора да игра со секој од преостанатите играчи, да разгледмае еден играч, на пример првиот. Првиот играч мора да игра со секој од преостанатите $10-1=9$ играчи, т.е. тој игра 9 патии. Бидејќи е сеедно кој од играчите сме го разгледувале, заклучуваме дека секој играч одиграл по 9 партии шах.

Трет начин. Секој од десетте играчи да го означиме со точка и графички, со отсечки, да ги прикажеме сите партии кои тој ги одиграл (цр-теж десно). Вкупниот број партии е еднаков на бројот на отсечките со кои сме поврзале една точка (играч) со преостанатите точки (играчи), што значи дека секој играч одиграл точно 9 партии шах.



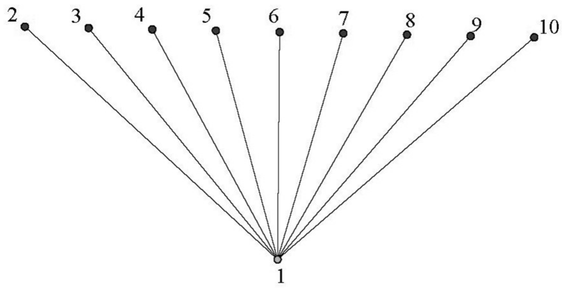
б) *Прв начин*. Да ги запишеме сите можни партии кои ќе се изиграат на трнирот, при што ќе внимаваме партиите кои ги играат исти играчи да не ги броеви два пати.

1 и 2	1 и 3	1 и 4	1 и 5	1 и 6	1 и 7	1 и 8	1 и 9	1 и 10	9 партии
2 и 3	2 и 4	2 и 5	2 и 6	2 и 7	2 и 8	2 и 9	2 и 10		8 партии
3 и 4	3 и 5	3 и 6	3 и 7	3 и 8	3 и 9	3 и 10			7 партии
4 и 5	4 и 6	4 и 7	4 и 8	4 и 9	4 и 10				6 партии
5 и 6	5 и 7	5 и 8	5 и 9	5 и 10					5 партии
6 и 7	6 и 8	6 и 9	6 и 10						4 партии
7 и 8	7 и 9	7 и 10							3 партии
8 и 9	8 и 10								2 партии
9 и 10									1 партија

Конечно, ако ги собереме броевите на партиите во последната колона добиваме дека на турнирот се изиграни вкупно $9+8+7+6+5+4+3+2+1=45$ партии шах.

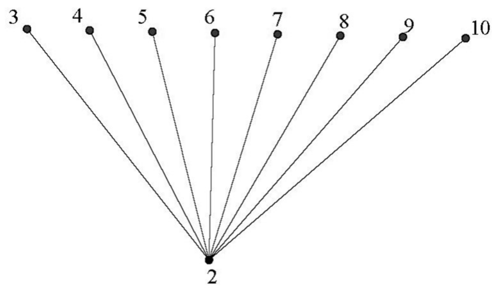
Втор начин. Да ги собереме последователно сите партии кои ги изиграле играчите почнувајќи од првиот до последниот играч. Првиот играч изиграл 9 партии. Вториот играч исто така изиграл 9 партии, но веќе ја броевме партијата која ја изиграл со првиот играч, па така добиваме 8 нови партии. Третиот играч изиграл 9 партии, но веќе ги броеви партиите кои ги изиграл со првите двајца играчи, па така добиваме 7 нови партии. Понатаму, заклучуваме дека четвртиот играч изиграл 6 нови партии, петтииот играч изиграл 5 нови партии, шестиот изиграл 6 нови партии, седмиот 3 нови партии, осмиот 2 нови партии и деветтиот 1 нова партија и тоа таа со десеттиот играч. За десеттиот играч се броени сите партии кои ги изиграл. Значи, вкупно се изиграни $9+8+7+6+5+4+3+2+1=45$ партии шах.

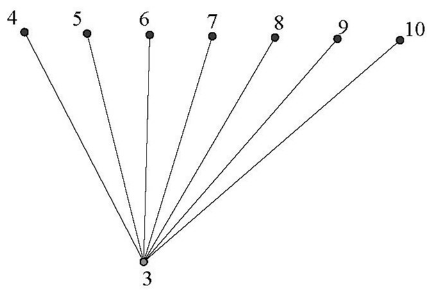
Трет начин. Да ги прикажеме графички сите партии кои ги изиграле играчите. За попрегледно нека секој играч го означиме со една точка , а точките ги означиме со броевите од 1 до 10. Тогаш бројот на партиите е еднаков на бројот на нацртаните отсечки чии крајни точки се точките со кои се означени играчите.



Првиот играч изиграл 9 партии кои се прикажани на цртежот десно.

Вториот играч изиграл 9 партии, но таа со првиот играч веќе е прикажана, па затоа имаме 8 нови партии (цртеж долу лево). Третиот играч изиграл 9 партии, но оние со првиот и вториот играч се веќе прикажани, па затоа имаме 7 нови партии (цртеж долу десно).



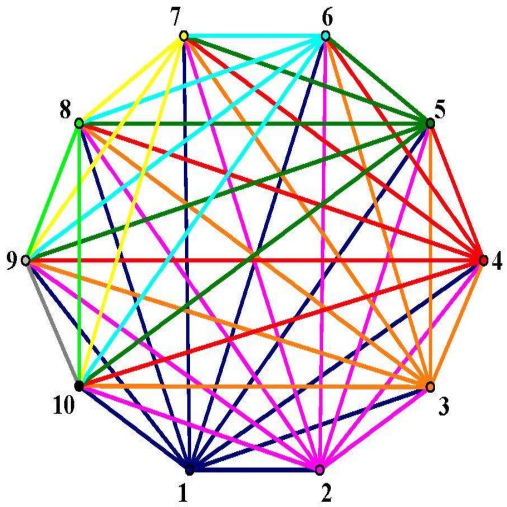


Со аналогни постапки добиваме дека за чет-вртиот играч треба да додадеме 6 нови партии, за петтиот играч 5 нови партии, за шестиот играч 4 нови партии, за седмиот играш 3 нови партии, за осмиот 2 нови партии и за деветтиот 1 нова партија. Конечно, вкупно се изиграни

$$9+8+7+6+5+4+3+2+1=45$$

партии шах.

Четврт начин. Ако играчите ги претставиме со точки распоредени на кружница, тогаш бројот на партиите шах кои се изиграни на турнирот можеме да го добиеме со пребројување на отсеч-ките прикажани на цртежот десно.



3. Заклучок

Развивањето на квалитетите на мислењето е важна задача на наставата по математика. Притоа, заради недостатокот на теориските знаења и воопшто малиот број предзнаења се добива впечаток дека кај учениците од почетното образование скоро и да е невозможно да се развиваат определени квалитети на мислењето, како што се длабочината, целесообразноста и рационалноста на мислењето. Меѓутоа ако се земе предвид дека основната алатка за равивање на еластичноста на мислењето е наоѓањето на повеќе начини за решавање на една иста задача, претходно разгледаните и многу други примери укажуваат дека не постојат никакви пречки за развој на еластичноста на мислењето кај учениците од почетното образование. Понатаму, ако се земе предвид дека при развивањето на еден од квалитетите на мислењето ние во определена мерка ги развиваме и преостанатите квалитети на мислењето, можеме слободно да кажеме дека со развивање на еластичноста на мислењето ние всушност ги развиваме сите квалитети на мислењето. Ова посебно се однесува на целесообразноста и рационалноста на мислењето, бидејќи наоѓањето на повеќе начини за решавање на една иста задача овозможува стекнување умеења за разумен избор на методи и средства за решавање на слични задачи и наоѓање на најкратки патишта за остварување на поставената цел, што е тесно поврзано со економичноста во однос на времето и средствата за решавање на даден проблем.

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ПИСА ТЕСТИРАЊЕТО ВО СРЕДНИТЕ СТРУЧНИ УЧИЛИШТА

Ана Димовска, Томи Димовски

Абстракт. Во оваа статија правиме детална анализа на ПИСА тестирања во други земји и правиме компарација со ПИСА тестирање спроведено во средно стручно училиште во нашата земја. Добиваме резултати за квалитетот на образовниот процес со и без соодветна подготовка за овој вид тестирање, како и разбирање на математиката воопшто.

1. Вовед

Програмата за меѓународно оценување на учениците (ПИСА) е најголемото истражување во светот во образованието, кое е за прв пат спроведено од Организацијата за економска соработка и развој (ОЕЦД) кон крајот на 1990-тите. Негова цел е собирање на меѓународно споредливи податоци за знаењата и компетенциите на учениците на крајот на основното образование на возраст од 15 години, во три подрачја: јазична писменост, математичка писменост и природни науки. Основната идеја на тестирањето ПИСА е испитување на подготвеноста на младите за целосно и активно учество во општеството. Во подрачјето математичка писменост основни критериуми за оценување се: читање, интерпретирање и решавање на даден проблем, со организирање, толкување на дадени информации и избирање на метод за решавање.

Со резултатите од ПИСА тестирањето се добиваат одговори на следниве прашања:

- Колку добро училиштето ги подготвува учениците за живот?
- Дали учениците се способни да анализираат, логично да расудуваат и ефективно да ги пренесат своите идеи?
- Дали учениците се способни да учат и да стекнуваат нови вештини во текот на нивниот живот?
- Дали ќе знаат учениците како да ги решат проблемите со кои никогаш не се сретнале?
- Дали ќе можат учениците да се справат со брзите промени, да го препознаат и искористат потенцијалот на новите технологии?

Земјите учеснички ги користат резултатите и податоците собрани од тестирањето ПИСА при донесување одлуки со цел да го подобрат квалитетот и ефикасноста на нивните образовни системи.

2. ПИСА ТЕСТИРАЊЕТО И РЕЗУЛТАТИТЕ НА НАШАТА ЗЕМЈА

Република Северна Македонија по четврти пат учествува на меѓународнтото тестирање Писа со 7850 средношколци, а оваа година првпат ќе се мери и креативното размислување на учениците.

Во периодот од 20.2.2022 до 10.4.2022 е реализирана обука со професорите по математика кои во учебната 2022/2023 година изведуваат настава во прва година средно образование. Целта на обуката е информирање и оспособување на професорите за подготовка на учениците за ПИСА тестирањето. Обуката е спроведена од советници од Бирото за развој на образованието и Државниот испитен центар.

Моделот на задачи од ПИСА треба да овозможи наставата по математика да се планира врз основа на Пијажеовата теорија на когнитивниот развој, која бара на децата да им се овозможи да учат сами. Наставата по математика не треба да биде насочена кон пренесување и усвојување на готови знаења, туку кон активно стекнување на нови знаења. При решавање на задачите по моделот на ПИСА потребно е да се присутни следните принципи.

- Користење на различни техники/стратегии на поучување во наставата по математика во кои учениците самостојно го читаат, интерпретираат и решаваат дадениот проблем со организирање и толкување на дадени информации и избирање на метод за решавање.
- Барање учениците да ги образложуваат своите математички размислувања.
- Прифаќање и охрабрување на различни постапки, методи и техники на решавање, кои ги користат учениците.

Од официјалната страна на ОЕЦД можеме да ги превземеме податоците од резултатите за нашата држава и второрангираната Сингапур.

Северна Македонија (ПИСА 2018)

• Во просек, 15-годишниците добиваат 394 поени по математика во споредба со просечните 489 поени во земјите на ОЕЦД. Девојчињата имаат подобри резултати од момчињата со статистички значајна разлика од 7 поени (просек на ОЕЦД: 5 поени повеќе за момчињата).

Сингапур (ПИСА 2018)

• Во просек, 15-годишниците добиваат 569 поени по математика во споредба со просечните 489 поени во земјите на ОЕЦД. Момчињата имаат подобри резултати од девојчињата со нестатистички значајна разлика од 4 поени (просек на ОЕЦД: 5 поени повисок за момчињата).

Значително подобрите резултати на Сингапур во однос на нашата држава во 2018 година се должат на повеќе фактори. Воведувањето на Кембриџ програмата во образовниот процес по предметот математика очигледно не дава резултати. Спиралното учење на материјалот и меморирањето на фактите не ја развиваат интуицијата на ученикот при решавање на практични проблеми. Незаинтересираноста и немотивираноста на професорите за спроведување и менување на наставните програми како би се приближиле кон трендовите и барањето на практични решенија на секојдневните проблеми доведува до некомпатибилност на знаењата кои ги поседуваат нашите ученици и практичните задачи од ПИСА тестирањата.

3. Истражување

Како професор во средно стручно училиште, решив да спроведам истражување, колку ќе се подобри заентерисираноста на учениците, доколку на часовите се промени концептот на задачите и се вметнат применливи задачи од типот на ПИСА тестирањето. Секојдневните потешкотии на моите часови за да го привлечам вниманието на незаентерисираните ученици за предметот математика ми беа мотивација, а и како главен координатор од моето училиште бев директно инволвирана во ПИСА тестирањето.

Истражувањето го спроведов за темата Пропорционалност на величините. Како последна тема која се изучува во прва година средно стручно образование. Оваа тема е една од најпотребните за натамошното образование на моите ученици. Истражувањето го спроведов во струката лични услуги, сектор фризер, во два класа.

ТЕМАТСКО ПЛАНИРАЊЕ
Наслов на тематска целина: ПРОПОРЦИОНАЛНОСТ НА ВЕЛИЧИНИ. ПРОЦЕНТНА СМЕТКА

Реден број	Наставна единица	Време на реализација
83.	Размери и пропорции	IV-3н
84.	Вежби. Задачи од размер и пропорции	IV-3н
85.	Права и обратна пропорционалност	IV-3н
86.	Вежби. Задачи од права и обратна пропорционалност	IV-4н
87.	Просто тројно правило	IV-4н
88.	Вежби. Задачи од просто тројно правило	IV-4н
89.	Процентна сметка	V-1н
90.	Вежби. Задачи од процентна сметка	V-1н
91.	Промилна сметка	V-1н
92.	Вежби. Задачи од промилна сметка	V-2н
93.	Вежби. Задачи од процентна и промилна сметка	V-2н
94.	Повторување на темата	V-3н

95.	Подготовка за писмена работа	V-3н
96.	Четврта писмена работа	V-3н
97.	Анализа на писмена работа	V-4н
98.	Повторување за пропорционалност	V-4н
99.	Повторување на темата	V-4н
100.	Годишно повторување	VI-1н
101.	Годишно повторување	VI-1н
102.	Годишно повторување	VI-1н

Во I_1 клас темата ја предадов според наставната програма по математика во средно стручно образование, но на секој час вметнав и применливи задачи по примерот од ПИСА. Во I_2 клас темата ја предадов според наставната програма по математика во средно стручно образование. На крајот од темата на сите ученици им беше даден ист тест во кој освен вообичаените задачи од програмата беа дадени и задачи по примерот од ПИСА.

Пример за час со задачи од ПИСА
Вежби: Пропорционалност на величините

Задача 1. Одреди го непознатиот член x во пропорциите
а) 3:2=x:4; б) (5-x):(5+x)=1:2.

Задача 2. Една работа може да се заврши од 20 работници за 288 часа. Колку работници се потребни за истата работа да се заврши за 16 дена?

Задача 3. Сума од 18 000 денари треба да се раздели на три дела во однос 4:3:2. Колку изнесува секој дел?

Задача 4. Првата фаза за да се избели косата со водород пероксид е подготовка на растворот. За таа цел ни се потребни перхидрол и вода. Концентрацијата зависи од структурата на косата на муштериите, а исто толку важна е и посакуваната нијанса. За порозна тенка коса, концентрацијата ќе биде пониска (3-6%-тен раствор), а за погуста и силна коса малку повисока (8-12%-тен раствор). За обична употреба мешавина од 6-12%-тен раствор.

Во табелата се дадени односите на перхидрол и вода за 3,6,9 и 12%-тен раствор.

%-тен ратвор	перхидрол : вода

3	1:9
6	2:8
9	3:7
12	4:6

Посакуваната количина на смесата зависи од должината на косата. По правило 50 ml раствор едоволен за средна коса, а 100 ml за долга коса. За подобрување на ефикасноста на производот, 5 капки амонијак се додаваат на 50 ml раствор. За да го задебелите составот, можете да додадете 6 ml алкален шампон или течен сапун во него.

Прашање 1/4

Колку ml вода треба Гориан да измеша со 250 ml перхидрол за да добие 3%-тен раствор?

Прашање 2/4

Во едно фризерско студио има 3 муштерики кои сакаат да ја осветлат својата коса. Две од нив имаат средна коса, а третата муштерика има долга коса. Колку ml вода и перхидрол се потребни за да се направи 9%-тен раствор за да се осветли косата на овие муштерики?

Прашање 3/4

Едно студио месечно употребува 15l од 6%-тен раствор на перхидрол и вода за потребите на нивните муштерии. Колку ml перхидрол студиото троши месечно? Колку капки амонијак се потребни за да се подобри ефикасноста на растворот? Колку ml алкален шампон се потребни за да се задебели составот?

Пример за Тест
Прашања

1. Што е размер?
2. Што е пропорција?
3. Кое е основното својство на пропорцијата?
4. Што е продолжена пропорција?

Задачи
Задача 1. Заокружи ги количниците кои не се размери и образложи го својот одговор.

а) 7:21 __________________________________
б) 35 : 5 kg __________________________________
в) $\pi : \sqrt{2}$ __________________________________
г) 3l : 3cm __________________________________

Задача 2. Во паралелката I_8 се одржал избор за претседател на паралелката. Во најтесен избор биле избрани учениците Дамјан и Јован. Со освоени 25 гласови Дамјан го победил Јован во однос 5:3.

Прашање 1/3

Колку гласови добил Јован?

Одговор ____

Прикажи ја постапката за решавање____________________________________

____

Прашање 2/3

Колку вкупно ученици броела паралелката?

Одговор ____

Прашање 3/3

Колку гласови треба да изгуби Дамјан на сметка на Јован за да победи Јован со 4 пати повеќе гласови?

Одговор ____

Прикажи ја постапката за решавање ____________________________________

____

Задача 3. Храната која ја јадеме го снабдува нашето тело со енергија. За да бидат здрави и во добра форма на луѓето им е потребна избалансирана храна со здрави хранливи состојки. Во табелата подолу се дадени нутритивните вредности на една зеленчукова супа од домати.

Нутритивни факти	Количина/ сервирање
Сервирање ½ чаша (120ml) Калории 90 Калории од масти 0	Масти 0 g
	Холестерол 0 g
	Содиум 710 mg
	Јаглехидрати 20 g
	Шеќери 15 g
	Протеини 2 g

Прашање 1/4

Колку ml има во $3\frac{1}{3}$ чаши супа?

Одговор ____

Прикажи ја постапката за решавање ____________________________________

____

Прашање 2/4

Конзервата има 2,5 сервирања од половина чаша. Ако за секое лице е потребно 1 чаша за оброкот од супа, колку конзерви супа ќе бидат потребни за 8 лица?
Одговор ____________________________________
Образложи го својот одговор____
____

Прашање 3/4
За здрав оброк идеално е потребно односот на јаглехидратите спрема шеќерите да е 3:2, протеините спрема содиумот 5:4, а на јаглехидратите спрема содиумот 30:1. Кој е идеалниот однос на протеини, содиум, јаглехидрати и шеќери во еден оброк од здрава супа?
Одговор ____________________________________
Прикажи ја постапката за решавање _________________________________
____

Прашање 4/4
Дали оваа супа е идеалниот оброк - супа?
Одговор ____________________________________
Прикажи ја постапката за решавање _________________________________
____

Задача 4. Во едно училиште има 750 ученици и 225 ученици се со слаби оценки. Колку ученици се со слаби оценки изразени во проценти?

Задача 5. При превоз на 400 000 kg јаболка, 20‰ од нив се неупотребливи. Колку тони јаболка се неупотребливи?

Задача 6. Сума од 8 000 денари треба да се подели на три дела во однос 5:1:3. Колку изнесува секој дел?

4. Заклучок

На крајот од спроведеното истражување добиени се следниоте резултати.

Клас	просек пред ПИСА тест	Просек по ПИСА тест
I_1	2,28	2,73
I_2	2,46	1,83

Можеме да заклучиме дека учениците доколку во програмата немаат вметнато задачи од ПИСА тестирањето имаат послаби резултати, во однос на оние кои се сретнале со задачи од тој тип. Сметам дека ПИСА тестирањето е несоодветно за нашето образование, бидејќи концептот на

Кембриџ програмата не ги припрема за решавање на ваков тип на задачи. Најпрво за да може да се спроведе вакво тестирање треба да се промени концептот на работа на часовите. Да се вметнат соодветни програми за да можеме да го спроведеме и да можеме да бидеме оценети по тој концепт.

Слабиот резултат на ПИСА тестот може во голема мера да се должи на тоа како предаваме математика. Земјите како Сингапур кои се истакнуваат во математиката предаваат помалку математички теми, но ги учат секоја подлабоко. Земјите со високи перформанси, исто така, имаат тенденција да предаваат теми последователно, наместо да се враќаат на истите теми секоја година. На пример, во Сингапур, користат принцип на учење математика сличен на нашиот стар концепт на работа. Тие започнуваат со основните операции, како разбирање на својствата на броеви, додека учениците не ја разберат таа тема длабоко и темелно, а потоа ќе преминат на посложени концепти. Тие ретко се враќаат на претходните теми во следните години, онака како што кај нас се работи сега по Кембриџ програмата.

Спроведувањето на ПИСА тестирењето, кое се базира на решавање на практични проблеми може да го подобри нашиот образовен систем, да го направи пофункционален, попрактичен и поинтересен за учениците. Адаптирањето на програмата и обучувањето на професорите и учениците за решавање практични проблеми, несомнено ќе го подобри резултатот на нашата земја.

Судир на интереси

Авторите изјавија дека нема судир на интереси.

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COSINE AND COTANGENT THEOREMS FOR A QUADRILATERAL, TWO NEW FORMULAS FOR ITS AREA AND THEIR APPLICATIONS

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Abstract. Here we show new relationships between elements of a convex quadrilateral, which generalize the cosine and the so-called cotangent theorems for a triangle. We named the new ones cosine and cotangent theorems for a quadrilateral. We derive by them new formulae for the area of any quadrilateral, which help to find various relationships in a triangle and a quadrilateral (a Carnot theorem for a triangle and the Brahmagupta's theorem for the area of an inscribed quadrilateral are generalized, as examples). The newly discovered formulas and dependencies find application in deriving various properties and regularities of the triangle and the quadrilateral.

1. INTRODUCTION

In the last time there were discovered many noticeable points in an arbitrary convex quadrilateral (see the reference list at the end of this work). Part of them were defined analogously to some noticeable points of a triangle. It became clear, as we will see, that besides the properties of remarkable points, some popular theorems for a triangle can be transferred to a quadrilateral (as the so-called cosine and cotangent theorems). Via the obtained cosine and cotangent theorems for a quadrilateral we proved unknown till now formulas for it's area. As two applications of derived here formulas and dependencies, we generalize Carnot theorem for a triangle and Brahmagupta's theorem for calculating the area of an inscribed quadrilateral.

2. COSINE AND COTANGENT THEOREMS FOR A QUADRILATERAL.

Before we formulate and prove the cosine and the cotangent theorems for a quadrilateral, let us remind and prove the cotangent theorem for triangle, as it is less popular, and then use it.

Theorem 1 (Cotangent theorem for a triangle). The side lengths of $\triangle ABC$ are $AB = a$, $BC = b$ and $CA = c$, $\measuredangle ACB = \gamma$ and S is the triangle's area. This relationship is valid $$c^2 = a^2 + b^2 - 4S.\cot\gamma \qquad (1)$$

Proof 1. According to the cosine theorem for $\triangle ABC$ we have (**Fig. 1**):

$$c^2 = a^2 + b^2 - 2ab.\cos\gamma .$$

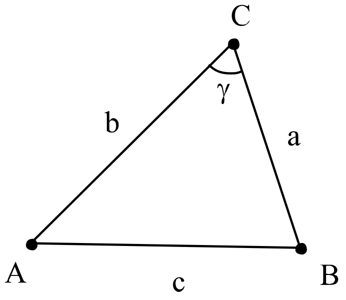

Figure 1. Shows objects from **Theorem 1**.

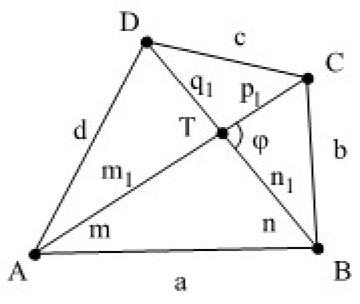

Figure 2. Shows objects from **Theorem 2, 3 and 4**.

From the other side, there holds the equation $S = \frac{1}{2} ab.\sin\gamma$. From here we derive:

$$c^2 = a^2 + b^2 - 4\frac{ab}{2}\cdot\sin\gamma.\cot\gamma = a^2 + b^2 - 4S.\cot\gamma .$$

Thus the equation (1) is proved.
Now we are able formulate and prove the cosine and cotangent theorems for a quadrilateral:
Theorem 2. (Cosine theorem for a quadrilateral). Denote the side lengths *AB*, *BC*, *CD* and *DA* in a quadrilateral *ABCD* with *a*, *b*, *c* and *d*, *m* and *n* – the lengths of the diagonals *AC* and *BD*, and φ – the measure of the angle between the diagonals, opposite to *BC* (**Fig. 2**). Then:

$$b^2 + d^2 = a^2 + c^2 - 2mn.\cos\varphi \qquad (2)$$

Proof 2. Let the diagonals *AC*, *BD* intersect at point *T* and $AT = m_1$, $BT = n_1$, $CT = p_1$, $DT = q_1$. Applying the cosine theorem to ΔABT, ΔBCT, ΔCDT and ΔDAT, we get respectively:

$$\left| \begin{array}{l} a^2 = m_1^2 + n_1^2 - 2m_1n_1.\cos\left(180^\circ - \varphi\right) \\ b^2 = n_1^2 + p_1^2 - 2n_1p_1.\cos\varphi \\ c^2 = p_1^2 + q_1^2 - 2p_1q_1.\cos\left(180^\circ - \varphi\right) \\ d^2 = q_1^2 + m_1^2 - 2q_1m_1.\cos\varphi \end{array} \right. \qquad (3)$$

We add the first with the third equations of (3), and the second with the forth, and get:

$$a^2 + c^2 = m_1^2 + n_1^2 + p_1^2 + q_1^2 + 2m_1n_1.\cos\varphi + 2p_1q_1.\cos\varphi$$

$$b^2 + d^2 = n_1^2 + p_1^2 + q_1^2 + m_1^2 - 2n_1p_1.\cos\varphi - 2q_1m_1.\cos\varphi$$

From the last two ones there follows the equation:

$$a^2 + c^2 - 2m_1n_1.\cos\varphi - 2p_1q_1.\cos\varphi = b^2 + d^2 + 2n_1p_1.\cos\varphi + 2q_1m_1.\cos\varphi ,$$

which can be transformed this way:

$$b^2 + d^2 = a^2 + c^2 - 2\left(m_1n_1 + n_1p_1 + p_1q_1 + q_1m_1\right).\cos\varphi .$$

As $m_1n_1 + n_1p_1 + p_1q_1 + q_1m_1 = \left(m_1 + p_1\right)\left(n_1 + q_1\right) = mn$, it leads to (2), which we wanted to prove.
Note 1: It's easy to guess, that in the boundary case, when the quadrilateral *ABCD* distorts in a ΔABC, i.e. when $D \to A$, then $d = 0$, $c = m$, $\varphi = \sphericalangle CAB$, $n = a$ and (2) gives then the relationship $b^2 = a^2 + m^2 - 2am.\cos\sphericalangle CAB$, which is the cosine theorem for ΔABC. This fact legitimates the usage of the term "cosine theorem" for this dependency.
The cotangent theorem for a quadrilateral is derived by the cosine theorem for it in the same way, as in the triangle.

Theorem 3 (Cotangent theorem for a quadrilateral). Let *ABCD* be a quadrilateral of side lengths *AB* = *a*, *BC* = *b*, *CD* = *c* and *DA* = *d*, and area *S*. If the angle between the diagonals, which is opposite to the side *BC*, is φ, then:

$$b^2+d^2=a^2+c^2-4S.\cot\varphi \tag{4}$$

Proof 3. Let the lengths of the diagonals *AC* and *BD*, be *m* and *n* resp. (**Fig. 2**). According to the proved cosine theorem for a quadrilateral, we have:

$$b^2+d^2=a^2+c^2-2mn.\cos\varphi.$$

Therefore, having in mind the formula $S=\frac{1}{2}mn.\sin\varphi$ for a quadrilateral's area, we get:

$$b^2+d^2=a^2+c^2-4\frac{mn}{2}\cdot\sin\varphi.\cot\varphi=a^2+c^2-4S.\cot\varphi.$$

Thus the equation (4) is proved.

Note 2. It's easy to guess, that in the boundary case, when the quadrilateral *ABCD* becomes $\triangle ABC$, i.e. if $D\to A$, then $d=0$, $c=m$, $\varphi=\sphericalangle CAB$ and the dependency (4) transforms to the dependency $b^2=a^2+m^2-4S.\cot\sphericalangle CAB$, i.e. in the cotangent theorem for the $\triangle ABC$. This legitimates the term "cotangent theorem", which we give to this relationship.

2. NEW FORMULAS FOR THE AREA OF AN ARBITRARY QUADRILATERAL.

From the proven relationship (4), which we've called cotangent theorem for a quadrilateral, in the case if $\varphi\neq 90^\circ$, the quadrilateral's area *S* can be expressed through the lengths of the sides and the tangent of the angle between the diagonals of the quadrilateral. We thus get the following unknown up to now formula for the area of an arbitrary quadrilateral:

$$S=\frac{1}{4}\left(a^2+c^2-b^2-d^2\right).\tan\varphi,\quad \text{where } \varphi\neq 90^\circ \tag{5}$$

Let us underline, that in this formula φ is those angle between the diagonals *AC* and *BD*, which lies opposite to the side of length *b*. With the help of the cosine theorem for a quadrilateral, a second new formula for its face is derived, by which it is expressed by the lengths of the sides and the diagonals of the quadrilateral.

Theorem 4. *ABCD* is a convex quadrilateral with side lengths *AB* = *a*, *BC* = *b*, *CD* = *c*, *DA* = *d* and diagonals' lengths *AC* = *m* and *BD* = *n* (**Fig. 2**). The area *S* of the quadrilateral is expressed by these magnitudes through the formula:

$$S=\frac{1}{4}\sqrt{4m^2n^2-\left(a^2+c^2-b^2-d^2\right)^2} \tag{6}$$

Proof 4. Via the cosine theorem for the quadrilateral *ABCD* we have:

$$b^2+d^2=a^2+c^2-2mn.\cos\varphi.$$

$$\Rightarrow \cos\varphi = \frac{a^2+c^2-b^2-d^2}{2mn}, \quad \sin^2\varphi = \frac{4m^2n^2-\left(a^2+c^2-b^2-d^2\right)^2}{4m^2n^2} \text{ and as } S = \frac{1}{2}mn.\sin\varphi,$$

from the last equation we get (6), which we had to prove.

The just obtained formulas for area of a convex quadrilateral, and the cosine and the cotangent theorems for it, have important applications. With their help, for example, a series of inequalities connecting the lengths of the sides and the diagonals of any convex quadrilateral, as well as other important relationships between the lengths of the sides and the diagonals of the quadrilateral, are derived (see [1], [2] for more). Here we will apply the derived cotangent theorem for the quadrilateral and the second derived formula for its area to generalize two classical theorems of geometry.

3. A GENERALIZATION OF THE ABOVE CARNOT THEOREM.

The French engineer Lasar Carnot (1753 – 1823) has proved the following:

Theorem 5 (of Carnot). $\triangle ABC$ is an arbitrary one and l_1, l_2 and l_3 are the perpendiculars from arbitrary points A_1, B_1 and C_1 on *BC*, *CA* and *AB*, to the same sides (**Fig. 3**). The lines l_1, l_2 and l_3 meet at a single point if and only if there holds the equation:

$$CA_1^2 - BA_1^2 + AB_1^2 - CB_1^2 + BC_1^2 - AC_1^2 = 0 \qquad (7)$$

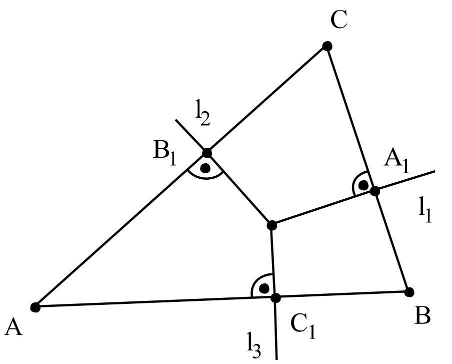

Figure 3. Shows objects in **Theorem 5.**

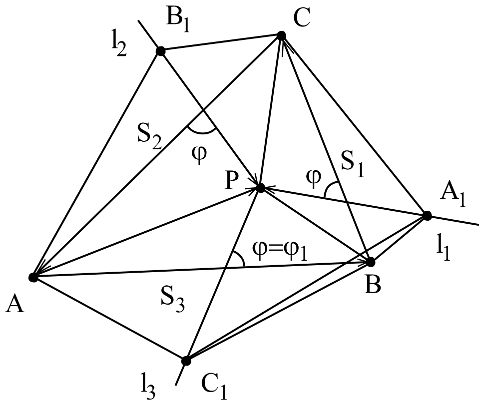

Figure 4. Shows objects in **Theorem 6.**

We will generalize the Theorem 5 by cancelling the condition the points A_1, B_1 and C_1 to lie either on the sides *BC*, *CA* and *AB* of $\triangle ABC$, or on the lines *BC*, *CA* and *AB*, and by replacing the perpendiculars l_1, l_2 and l_3 to these sides with lines, sloped to *BC*, *CA* and *AB* at the same angle φ:

Theorem 6 (Generalization of the Carnot theorem). $\triangle ABC$ is a positively oriented one (**Fig. 4**). A_1 is an arbitrary point either on the semi plane along the line *BC*, which do not include the triangle, or on the line *BC*. The points B_1 and C_1 satisfy the same conditions with respect to the lines *CA* and *AB*. The sloped lines $l_1^{\rightarrow}$, $l_2^{\rightarrow}$ and $l_3^{\rightarrow}$, respective to the sides *BC*, *CA* and *AB* of the $\triangle ABC$, pass through the points A_1, B_1 and C_1, and form angles of equal measure φ with the positive directions. If *S* is the area of the hexagon $AC_1BA_1CB_1$, which is not necessarily convex, then the lines l_1, l_2 and l_3 meet at a single point if and only if:

$$CA_1^2 - BA_1^2 + AB_1^2 - CB_1^2 + BC_1^2 - AC_1^2 = 4S.\cot\varphi \tag{8}$$

Proof 6. 1) Let firs assume that the $l_1^{\rightarrow}$, $l_2^{\rightarrow}$ и $l_3^{\rightarrow}$ meet at a single point *P* (**Fig. 4**). Denote S_1, S_2, S_3 the areas of the covering quadrilaterals BA_1CP, CB_1AP, AC_1BP of the hexagon $BA_1CB_1AC_1$. Via the cotangent theorem we get from these quadrilaterals resp.:

$$BA_1^2 + CP^2 = CA_1^2 + BP^2 - 4S_1.\cot\varphi\,,$$
$$CB_1^2 + AP^2 = AB_1^2 + CP^2 - 4S_2.\cot\varphi\,,$$
$$AC_1^2 + BP^2 = BC_1^2 + AP^2 - 4S_3.\cot\varphi\,.$$

We add the last equalities term-by-term; as $S_1 + S_2 + S_3 = S$, therefore:

$$BA_1^2 + CP^2 + CB_1^2 + AP^2 + AC_1^2 + BP^2 = CA_1^2 + BP^2 + AB_1^2 + CP^2 + BC_1^2 + AP^2 - 4S.\cot\varphi$$

The last equation is easily simplified to (8). Thus we proved, that *if* the lines $l_1^{\rightarrow}$, $l_2^{\rightarrow}$ and $l_3^{\rightarrow}$, sloped at angle φ, meet at one point, *then* (8) holds.

2) Now we'll prove the inverse implication, i.e. that if (8) is true, then the lines $l_1^{\rightarrow}$, $l_2^{\rightarrow}$ and $l_3^{\rightarrow}$, sloped at angle φ, meet at one point. Denote *P* the common point of $l_1^{\rightarrow}$ and $l_2^{\rightarrow}$. It's sufficient to prove that the ray $C_1P^{\rightarrow}$ coincides with $l_3^{\rightarrow}$, which passes through the point C_1 under angle φ, i.e. that it forms angle φ with the positive direction of the side *AB*. Otherwise, we have to prove, that the angle in the quadrilateral AC_1BP, which form the diagonals C_1P and *AB*, which lies opposing the side *BP*, equals φ. Denote this angle φ_1. According to the cotangent theorem, from the quadrilaterals BA_1CP, CB_1AP and AC_1BP, we get respectively:

$$BA_1^2 + CP^2 = CA_1^2 + BP^2 - 4S_1.\cot\varphi\,,$$
$$CB_1^2 + AP^2 = AB_1^2 + CP^2 - 4S_2.\cot\varphi\,,$$
$$AC_1^2 + BP^2 = BC_1^2 + AP^2 - 4S_3.\cot\varphi_1\,.$$

Adding these equations term by term, we get:

$$\left(BA_1^2 + CP^2\right) + \left(CB_1^2 + AP^2\right) + \left(AC_1^2 + BP^2\right) =$$
$$= \left(CA_1^2 + BP^2\right) + \left(AB_1^2 + CP^2\right) + \left(BC_1^2 + AP^2\right) - 4\left(S_1 + S_2\right).\cot\varphi - 4S_3.\cot\varphi_1$$

and after simplification:

$$CA_1^2 - BA_1^2 + AB_1^2 - CB_1^2 + BC_1^2 - AC_1^2 = 4\left(S_1 + S_2\right).\cot\varphi + 4S_3.\cot\varphi_1\,.$$

From the other side, we assume that (8) holds, which can be represented thus:

$$CA_1^2 - BA_1^2 + AB_1^2 - CB_1^2 + BC_1^2 - AC_1^2 = 4\left(S_1 + S_2\right).\cot\varphi + 4S_3.\cot\varphi .$$

From the last two equations $4S_3.\cot\varphi_1 = 4S_3.\cot\varphi$, i.e. $\varphi = \varphi_1$, and the theorem is proved.

4. A GENERALIZATION OF BRAHMAGUPTA'S THEOREM.

From the new formula for area of a convex quadrilateral (formula (6)) we get in particular the famous Brahmagupta's formula (7 century AD) for area of an inscribed quadrilateral. As for such quadrilateral we have $mn = ac + bd$ (according to the Ptolemy theorem), after replacing in (6) we get (fig. 2):

$$S = \frac{1}{4}\sqrt{4\left(ac+bd\right)^2 - \left(a^2+c^2-b^2-d^2\right)^2} =$$

$$= \frac{1}{4}\sqrt{\left(2ac+2bd+a^2+c^2-b^2-d^2\right)\left(2ac+2bd+b^2+d^2-a^2-c^2\right)} =$$

$$= \frac{1}{4}\sqrt{\left[\left(a+c\right)^2-\left(b-d\right)^2\right]\left[\left(b+d\right)^2-\left(a-c\right)^2\right]} =$$

$$= \frac{1}{4}\sqrt{\left(a+c+b-d\right)\left(a+c-b+d\right)\left(b+d+a-c\right)\left(b+d-a+c\right)}$$

By setting $p = \dfrac{a+b+c+d}{2}$, we get $S = \sqrt{\left(p-a\right)\left(p-b\right)\left(p-c\right)\left(p-d\right)}$, which is the Brahmagupta's formula for the area of an inscribed quadrilateral.

We see, that formula (8) for the area of a convex quadrilateral generalizes the Brahmagupta's formula for the area of an inscribed quadrilateral.

5. CONCLUSIONS.

The above-proven dependencies in an arbitrary quadrilateral and the formulas for its area serve to derive various other relationships in it. We will consider them in further articles.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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WORKING WITH MATHEMATICALLY GIFTED STUDENTS AGED 18 TO 19

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Abstract. The aims and objectives of all educational systems state that working with gifted students is of special social interest. At the same time, the care for these students comes down to organizing competitions and preparing the students for several days before the competitions. We believe that this approach does not even remotely meet the needs of gifted students, therefore, in this paper we have made an attempt to develop a curriculum for working with mathematically gifted students aged 18-19.

1. INTRODUCTION

Papers [1], [2] and [3] provide the curricula for working with mathematically gifted students from first, second and third year in the secondary education. This paper is actually a continuation of the abovementioned papers and it will provide an integral program for working with mathematically gifted students aged 18-19 years, that is, for students in the fourth year of secondary education. We deem that the preparation of such a curriculum that should necessarily be accompanied by appropriate books and collections of problems complementary to the curriculum will complete the work with mathematically gifted students in secondary education thus filling the existing gap in this field. In particular, such an approach will contribute to turn the declarative support of these students into real support, since the organization of competitions and the selective awarding of scholarships (scholarships are awarded to only a few of the best placed students in the competitions) is not enough to say that there is a serious social interest in the development of these children.

As we have already stated, this paper is in a way a continuation of the abovementioned papers. In addition, based on the experience of the authors, but also the experience of the countries in the immediate and wider surrounding, an attempt was made for part of the topic Generating functions to give an example of a system of problems that would determine the level that students should reach at this age.

2. CURRICULUM FOR WORKING WITH MATHEMATICALLY GIFTED STUDENTS AGED 18-19

In this section, we will present a curriculum for working with mathematically gifted students aged 18-19, that is, for students in the fourth year of secondary education. The offered curriculum actually builds on the respective teaching curriculum that was previously prepared for students in secondary education and is presented in papers [1], [2] and [3]. During the preparation of the curriculum, the method of concentric circles was used, which means that part of the contents that were adopted in the previous years at a certain level are expanded and extended. This curriculum should be implemented continuously, and not only in periods when students are preparing for certain math competitions.

The knowledge obtained while working with mathematically gifted students given in papers [4] and [6] was used during the development of this curriculum. The aims of the curriculum for students aged 18-19 are the following:

- To develop students' qualities of thinking such as: flexibility, stereotyping, width, rationality, depth and criticality,

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Key Words: gifted students, curricula, aims and objectives of the curriculum

- The student to apply the scientific methods: observation, comparison, experiment, analysis, synthesis, classification, systematization and the axiomatic method,
- The student to apply the types of conclusions: induction, deduction and analogy, whereby it is of particular importance to present suitable examples from which the student will realize that the analogy conclusion is not always true,
- The student to adopt the prescribed contents in the field of functions of one real variable and to enable them to apply the same when solving appropriate problems,
- The student to adopt the prescribed contents in the field of differential and integral calculus of a function of one real variable and to be able to apply the acquired knowledge in problem solving,
- The student to adopt the prescribed contents in the field of inequalities and to be able to apply the acquired knowledge in problem solving,
- The student to adopt the prescribed contents in the field of combinatorics and to be able to apply the acquired knowledge in problem solving,
- The student to adopt the prescribed contents in the field of generating functions and to be able to apply the acquired knowledge in problem solving,
- The student to adopt the prescribed contents in the field of graph theory and to be able to apply the acquired knowledge in problem solving.

In order to achieve the aforementioned aims, it is necessary to adopt the following contents:

Analysis **(4 classes per week – 144 classes per year).** *Functions of one real variable*: basic properties of real functions, even, odd, periodic, monotone and bounded functions, elementary real functions (graphs), classification of real functions, parametrically defined functions and functions defined in polar coordinates, limit of a function at a point, the limits: $\lim\limits_{x\to 0}\frac{\sin x}{x}$ and $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x$, continuous function at a point and of a set, elementary properties of continuous functions, properties of continuous functions on closed intervals.

Differential calculus of a function of one real variable: notion of derivative, basic properties and derivatives of elementary functions, derivative of inverse, composite and implicit function, higher order derivatives, calculating sums using derivatives, basic theorems of differential calculus (Fermat, Rolle, Lagrange and Cauchy), L'Hôpital's rule, equation of a tangent, angle between two curves, Maclaurin and Taylor formula, monotonicity and local extrema of a function, application, convexity and concavity of a function, points of inflection, asymptotes of a curve, construction of a function graph.

Integral calculus of a function of one variable: notion of primitive function and indefinite integral, change of variables, partial integration, integration of rational functions, the notion of definite integral, basic properties of a definite integral, connection between definite and indefinite integral, change of variables and partial integration for a definite integral, area of a plane figure, arc length of a plane curve.

Inequalities: proving inequalities using monotonicity of a function, proving inequalities using extrema of a function, the inequalities of Popoviciu, Jensen, Bernoulli, Young, Jordan, Hölder, Minkowski, Karamata, Schur, Muirhead, Petrovic, Nesbitt, Hadwiger-Finsler, weighted inequalities of the means, inequalities of the means of order s and order r, notion of symmetric inequality, symmetric inequalities with three variables, normalization procedure and application of differential calculus in proving symmetric inequalities.

***Selected contents from discrete mathematics* (3 classes per week – 108 classes per year).** *Combinatorics:* partition of a number, ordered partition of a number, partition of a set, derangements, games and strategies, problems with coloring, covering and dissecting, weight and acquaintance problems, double counting and Hall's theorem.

Generating functions: concept of generating function, operations with generating functions, generating functions and differential equations, Hadamard product for rational generating functions, application of generating functions in the theory of enumeration, generating functions and partitions, exponential generating functions, harmonic numbers, sums of powers of natural numbers, Bernoulli polynomials and Bernoulli numbers, Catalan numbers, the snake oil method.

Graph theory: notion of graph, isomorphic graphs, matrix representation of graphs, types of graphs, subgraphs, degree of a vertex, regular graph, graph operations, trails and cycles, connectivity, trees, cyclomatic number of a graph, cutset, cuts, properties of adjacency and incidence matrices, Eulerian and Hamiltonian graphs, planar graphs and characterization of planar graphs. Matching in graphs. The notion of matching. System of distinct representatives. Perfect matching theorem. Coloring graphs. Chromatic number of a graph. Graphs with large chromatic number.

3. EXAMPLE OF SYSTEM OF PROBLEMS FOR SECTION "GENERATING FUNCTIONS"

In order to realize the suggested curriculum for working with gifted 18-19 years old students, it is necessary to make appropriate teaching aids, that is to say, textbooks that must be accompanies by appropriate books with collections of problems. Hereinafter, we will present a system of problems that we deem is suitable for studying the section Generating functions and which tasks are selected from the books [5], [7] and [8].

1. Determine the generating function of:

   a) binomial coefficient of $n=$th order $\binom{n}{k}$, $k=0,1,2,...,n$,

   b) sequence $a_k=(-1)^k$, $k=0,1,2,3,...$

2. Let $g_1(x)$ be the generating function of the sequence $\{a_i\}_{i=0}^{\infty}$ and $g_2(x)$ be the generating function of the sequence $\{b_i\}_{i=0}^{\infty}$. Prove that $g(x)=g_1(x)+g_2(x)$ is a generating function of the sequence $\{c_i\}_{i=0}^{\infty}$, where $c_i=a_i+b_i$, $i=0,1,2,3,...$.

3. Let $g(x)$ be the generating function of the sequence $\{a_i\}_{i=0}^{\infty}$ and c be a constant. Prove that $f(x)=cg(x)$ is the generating function of the sequence $\{ca_i\}_{i=0}^{\infty}$.

4. Let $g_1(x)$ be the generating function of the sequence $\{a_i\}_{i=0}^{\infty}$, $g_2(x)$ be the generating function of the sequence $\{b_i\}_{i=0}^{\infty}$ and $\alpha,\beta\in\mathbb{R}$. Prove that $g(x)=\alpha g_1(x)+\beta g_2(x)$ is a generating function of the sequence $\{c_i\}_{i=0}^{\infty}$, where $c_i=\alpha a_i+\beta b_i$, $i=0,1,2,3,...$.

5. Let $g(x)$ be the generating function of the sequence $\{a_i\}_{i=0}^{\infty}$. Prove that $x^n g(x)$ is a generating function of the sequence $\{b_i\}_{i=0}^{\infty}$, where $b_k=0, k=0,1,2,...,n-1$ and $b_k=a_{k-n}, k\geq n$.

6. Determine the generating function of the sequence $a_k=1$, $k=0,1,2,3,...$.

7. Let $g(x)$ be the generating function of the sequence $\{a_i\}_{i=0}^{\infty}$. Prove that

$$\frac{g(x)-a_0-a_1x-a_2x^2-...-a_{n-1}x^{n-1}}{x^n}$$

is the generating function of the sequence $a_n, a_{n+1}, a_{n+2}, \ldots$.

8. Let $g(x)$ be the generating function of the sequence $\{a_i\}_{i=0}^{\infty}$. Prove that $g(cx)$ id the generating function of the sequence $\{c^i a_i\}_{i=0}^{\infty}$.

9. a) Let $g_1(x)$ be the generating function of the sequence $\{a_i\}_{i=0}^{\infty}$ and $g_2(x)$ be the generating function of the sequence $\{b_i\}_{i=0}^{\infty}$. Prove that $g(x) = g_1(x)g_2(x)$ is the generating function of the sequence $\{c_i\}_{i=0}^{\infty}$ where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_{n-1} b_1 + a_n b_0,$$

b) Let $g_1(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$ be the generating function such that $g_1(0) = a_0 \neq 0$. Prove that there exists a generating function

$$g_2(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots$$

such that $g_1(x)g_2(x) = 1$. Hence we write $g_2(x) = \frac{1}{g_1(x)}$.

10. For the sequence $\{a_n\}_{n=0}^{\infty}$ determine the generating function of the partial sums of the series $\sum_{i=0}^{\infty} a_i$.

11. If $g(x)$ is a generating function for the sequence $\{a_n\}_{n=0}^{\infty}$, then $g'(x)$ is the generating function for the sequence $\{na_n\}_{n=1}^{\infty}$. Prove it!

12. If $g(x)$ is a generating function for the sequence $\{a_n\}_{n=0}^{\infty}$, then $\int_0^x g(t)dt$ is the generating function for the sequence $0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \ldots$. Prove it!

13. Determine the generating functions of the sequences $\{n+1\}_{n=0}^{\infty}$ and $0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$.

14. The numbers $H_0 = 0$ and $H_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k}$, for $k = 1, 2, \ldots$ are called *harmonic numbers* and they are equal to the partial sums of the harmonic series $\sum_{k=0}^{\infty} \frac{1}{k+1}$.

a) Prove that for each natural number m, the following is valid $H_{2^m} \geq 1 + \frac{m}{2}$.

b) Determine the generating function for the harmonic series.

15. Prove that for each natural number m the following is valid $H_{2^m} \leq 1 + m$.

16. Prove that for $n \geq 2$ the following is valid $\sum_{k=2}^{n} \frac{1}{k(k-1)} H_k = 2 - \frac{H_{n+1}}{n} - \frac{1}{n+1}$.

17. Determine the generating function of the sequence $a_n = \frac{1}{(n+1)(n+2)}$, $n = 0,1,2,\ldots$.

18. Prove that for every $m \geq 1$ the following is valid

$$\frac{1}{(1-ax)^m} = 1 + \binom{m}{1} ax + \binom{m+1}{2} a^2 x^2 + \binom{m+2}{3} a^3 x^3 + \ldots + \binom{m+n-1}{n} a^n x^n + \ldots$$

19. Determine the generating function for the sequence:

a) $a_n = \frac{1}{(n+5)!}$, $n = 0,1,2,\ldots$,

b) $a_n = \frac{n^2+n+1}{n!}$, $n = 0,1,2,\ldots$,

20. Determine the generating function for the number b_n of integers in the interval from 0 to $10^m - 1$ whose sum of digits is equal to n.

21. Solve the differential equation $a_0 = 5$, $a_k = a_{k-1} + 3$, за $k \geq 1$.

22. Using generating functions derive the formula for the general term of the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$, for $n \geq 0$.

23. Let the sequence $\{a_n\}_{n=0}^{\infty}$ satisfy the linear differential equation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_m a_{n-m}$$

from $m-$th order with constant coefficients $c_1, c_2, \ldots, c_m$ and starting conditions $a_0, a_1, \ldots, a_m$.

Prove that the generating function $g(x)$ of this sequence is of the kind $g(x) = \frac{P_{m-1}(x)}{Q_m(x)}$, where $Q_m(x)$ is a polynomial of $m-$th degree, and the degree of the polynomial $P_{m-1}(x)$ is smaller or equal to $m-1$.

24. Let the generating function

$$g(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{m-1} x^{m-1} + a_m x^m + \ldots \qquad (1)$$

be rational, that is, if $g(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are coprime polynomials. Prove that, starting from a number n the sequence $\{a_n\}_{n=0}^{\infty}$ satisfies the linear differential equation

$$a_{n+m} = c_1 a_{n+m-1} + c_2 a_{n+m-2} + \ldots + c_m a_n, \qquad (2)$$

where m is the degree of the polynomial $Q(x)$, and $c_1, c_2, \ldots, c_m$ are some constants.

25. Solve the differential equation $a_0 = 1$, $a_1 = 4$, $a_k = a_{k-1} + 6a_{k-2}$, $k \geq 2$.

26. The sequence $\{a_n\}_{n=0}^{\infty}$ is given with a recurrence relation $a_0 = 2, a_1 = 7,\ a_{n+2} = 4a_{n+1} - 4a_n + 3^n,\ n \geq 0$. Determine the explicit formula for a_n.
27. Solve the differential equation $a_0 = 1, \quad a_k = 3a_{k-1} + 4^k, k \geq 1$
28. Solve the differential equation $a_0 = 3, \quad a_k = 2a_{k-1} + k, k \geq 1$.
29. For every natural number n, let

$$S_n = 1 + \tfrac{1}{2} + \ldots + \tfrac{1}{n},$$
$$T_n = S_1 + S_2 + \ldots + S_n,$$
$$U_n = \tfrac{T_1}{2} + \tfrac{T_2}{3} + \ldots + \tfrac{T_n}{n+1}.$$

For every n determine the constants a_n, b_n, c_n, d_n such that

$$T_n = a_n S_{n+1} + b_n \text{ and } U_n = c_n S_{n+1} + d_n.$$

30. Prove that the generating function for the sequence $\{a_i\}_{i=0}^{\infty}$ is rational if and only if there exist numbers $q_1, q_2, \ldots, q_k$ and polynomials $p_1(t), p_2(t), \ldots, p_k(t)$ such that starting from some n the following is valid

$$a_n = p_1(t)q_1^n + p_2(t)q_2^n + \ldots + p_k(t)q_k^n. \tag{1}$$

The expression on the right-hand side from (1) is called *quasipolynomial* from the variable n.
31. *Hadamard product* for the generating functions

$$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \text{ and } h(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots$$

is called the generating function

$$f(x) = a_0b_0 + a_1b_1x + a_2b_2x^2 + a_3b_3x^3 + \ldots.$$

Prove that the Hadamard product for two rational generating functions is a rational generating function.
32. Let $g(s) = \frac{P_1(s)}{Q_1(s)}$ and $h(s) = \frac{P_2(s)}{Q_2(s)}$ be rational generating functions, given with coprime fractions and let $(g * h)(s) = \frac{P(s)}{Q(s)}$ be their Hadamard product, written as a fraction in simplest form. What can be said about the polynomial $Q(s)$, if we know the polynomials $Q_1(s)$ and $Q_2(s)$?
33. Determine the number of solutions of the equation $e_a + e_b + e_c + e_d = r$, where $0 \leq r \leq 6$, $e_a, e_b, e_c, e_d \in \mathbb{Z}$ and $0 \leq e_a \leq 1, 0 \leq e_b \leq 1, 0 \leq e_c \leq 2, 0 \leq e_d \leq 2$.

34. Let A be a set that contains 1 object of type a, 1 object of type b, 2 objects of type c and 2 objects of type d. Determine the number of ways in which we can choose 4 objects from the given 4 types in the set A.
35. A box contains 4 red, 5 blue and 2 green balls.
a) In how many different ways can 7 balls be chosen from the box?
b) In how many different ways can 7 balls be chosen but there must be 1 red and 2 blue balls?
36. Let us assume that there are 3 red, 8 green, 9 orange and 2 white balls in a box. In how many ways can we choose 12 balls if we have to choose at least one red ball, an even number of green balls and an odd number of orange balls?
37. Determine the coefficient in front of x^{24} in the series $(x^3+x^4+x^5+x^6+...)^4$.
38. In how many ways can 12 objects be chosen from 5 types of objects, if there are at most 2 objects from the first three types, and unlimited number of objects from the remaining two types?
39. In how many ways can 20 objects be selected, if objects from the first type can only be selected in packages of 5, of the second type only in packages of 3 objects each, of the third type can only be selected 4 at most, of the fourth type at least 3 objects and at most 2 objects from the fifth type?
40. Find the generating function whose $n-$th coefficient gives the number of non-negative solutions of the equation $e_1+4e_2+5e_3+3e_4=n$.
41. Prove that the function $\frac{1}{(1-x)(1-x^2)(1-x^3)...(1-x^k)...}$ is a generating function whose $n-$th coefficient gives the number of ways of placing of n same objects into n same boxes, so that some boxes can remain empty, that is, gives the number of partitions of the number n to n or less partitions (non-negative integer summands).
42. Using generating functions determine the number of ways to partition the number n as the sum of n or fewer distinct numbers.
43. Prove that the number of ways to partition the natural number n, as the sum of n or fewer distinct natural numbers is equal to the number of ways to partition the natural number n as the sum of n or fewer odd natural numbers.
44. a) For the natural number n let f_n be the number of subsets of the set $\{1,2,...,n\}$ which do not contain a pair of consecutive numbers. Determine the recurrence (differential) equation that these numbers satisfy, and then find the numbers.
b) For the natural numbers n and k let $f_{n,k}$ be the number of $k-$subsets of the set $\{1,2,...,n\}$ that do not contain a pair of consecutive numbers. Determine the recurrence equation that these number satisfy, and then determine the appropriate generating function and the numbers.

45. The function $h(x)=\sum\limits_{k=0}^{\infty}\frac{a_k}{k!}x^k$ is called *exponential generating function* for the sequence $\{a_k\}_{k=0}^{\infty}$. Prove that if f and g are exponential generating function for the sequences $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$, then
a) the function $f(x)+g(x)$ is exponential generating function for the sequence $\{a_k+b_k\}_{k=0}^{\infty}$.
b) the function $f(x)g(x)$ is exponential generating function for the sequence

$$c_n=\sum_{k=0}^{n}\binom{n}{k}a_kb_{n-k},\ n=0,1,2,3,\dots,$$

which is called *binomial convulsion of the sequences* $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$.
46. Determine the exponential generating function of the sequence $a_k=\frac{n!}{(n-k)!},\ k=0,1,2,\dots,n$.
47. Let us assume that we have unlimited number of white, black, green and blue balls. Determine the number of ways in which we can choose and rearrange 2 white, 4 black, 3 green and 3 blue balls.
48. Determine the exponential generating function that can be used to find the number of ways in which n persons can be accommodated into 3 rooms with at least 2 but not more than 9 persons.
49. Determine the exponential generating function for the sequence of variations with repetition from n elements from k -th class.
50. Determine the number of accommodating n guests in three halls, so that in the first hall there must be at least one guest, in the second hall there must be an odd number of guests, and in the third hall there must be an even number of guests.
51. Using the exponential generating functions prove that the number of ways in which n distinct objects can be put in k different boxes and no box remains empty and и $1\le k\le n$ is $A_k^{(n)}=\sum\limits_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$.
52. Prove that the number of ways in which n distinct objects can be put in k same boxes so that no box remains empty and $1\le k\le n$ is

$$S_k^{(n)}=\frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n,$$

where $S_k^{(n)}$, $0\le k\le n$ are Stirling numbers of the second kind.

53. $2n$ points are given on a circle. In how many ways can these points be partitioned into n pairs so that between these $n-$chords determined by these pairs of points there are no two that intersect?

54. For the string of zeros and ones with length $2n$ ($2n$-string over alphabet $\{0,1\}$) we shall say it is *balanced* if it contains n zeros and n ones. For the balanced $2n$-string over alphabet $\{0,1\}$ we shall say it is good if none of its initial parts have more zeros than ones. Otherwise we will say that $2n$-string is bad.

    Prove that the number of good $2n$-strings over alphabet $\{0,1\}$ is $C_n = \frac{1}{n+1}\binom{2n}{n}$.

55. There are $2n$ persons standing in queue in front of the ticket office. Each of them wants to buy a ticket that costs 50 denars. Among the people in the queue, exactly n persons have 50 denars, whereas the rest have one banknote of 100 denars. At the beginning, the cash desk is empty. What is the number of customer arrangements so that the salesperson can return the change to every person buying a ticket?

56. Find the number of sequences with length $2n$: $a_1, a_2, ..., a_{2n}$ with elements from the set $\{-1,1\}$ such that $\sum_{k=1}^{2n} a_k = 0$ and $\sum_{k=1}^{m} a_k \geq 0$ for $1 \leq m < 2n$.

57. Find the number of sequences with length n whose elements are integers $a_1, a_2, ..., a_n$ such that $1 \leq a_1 \leq a_2 \leq ... \leq a_n$ and $a_1 \leq 1, a_2 \leq 2, ..., a_n \leq n$.

58. The rook has to pass from the lower left to the upper right corner square of a chessboard with dimensions $n \times n$. The rook can only move from left to right and from bottom to top. How many different paths are there to achieve the goal if at no time can the rook be placed above the diagonal that connects the starting and ending square?

59. *An anti-Pascal* triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

    Is there an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1+2+...+2018$?

60. Using the Snake oil method, prove that:

    a) $\sum_k \binom{n}{k}\binom{k}{j} x^k = \binom{n}{j} x^j (1+x)^{n-j}$, for each $n \geq 0$, б) $\sum_{k}\binom{2n+1}{2k}\binom{m+k}{2n}=\binom{2m+1}{2n}$,

в) $\sum_{k=0}^{n}\binom{2n}{2k}\binom{2k}{k}2^{2n-2k}=\binom{4n}{2n}$

4. CONCLUSION

Previously, we provided a curriculum for working with gifted students aged 18-19 and for one of the topics contained in it we presented a system of problems that we believe is sufficient to achieve the aims of the curriculum from the given topic. Among other things, we believe that the acquisition of the theoretical knowledge envisaged by this curriculum, supported by appropriate collections of problems, will enable:

- Formation of students' qualities of thinking at an enviable level,
- Students to apply the types of inferences correctly: induction, deduction and analogy,
- Students apply the scientific methods correctly: observation, comparison, experiment, analysis, synthesis, classification, systematization and the axiomatic method, and
- Students to acquire the necessary knowledge needed for their future development, that is, for the successful continuation of their academic career at the best universities.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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ANALYSIS OF STUDENT ACHIEVEMENTS IN TEACHING COMPLEX NUMBERS USING GEOGEBRA SOFTWARE

Dragana Nedić, Gordana Jotanović, Tijana Paunović, Aleksandar Kršić

Abstract. The research deals with the analysis of students' achievements at the level of complex numbers in a "traditional" way and teaching enriched with multimedia software. The paper focuses on teaching complex numbers using the GeoGebra/HotPotatoes multimedia software. This approach to teaching aims to increase student activity and engagement, but also to raise the teaching process to a higher level of achievements and motivate students to work and learn more independently.

1. Introduction

Today, the technology is used more and more, it is necessary to adapt the teaching process accordingly. An important role has to be given to linkage between images and certain conditions, in order for the students to develop their knowledge [1], [2]. Multimedia approach in teaching of mathematics can be very useful in explanation of mathematical ideas, abstract terms and evaluation of knowledge.

In secondary schools of the Republika Srpska (Bosnia and Herzegovina), the curriculum for mathematics deals with the subject of complex numbers. Secondary school is the first time students are informed about complex numbers. As the name suggests, this topic is quite complex and demanding, since these are not the kind of numbers students face in their daily life. Through our extensive experience, we have come to a conclusion that students have a hard time adopting the lessons from this topic and, thus, lack motivation to further explore the application of complex numbers. In order to make the term more familiar and give a geometrical interpretation of a complex number, an idea was born to step back from the classical approach to this topic and use mathematical software called GeoGebra. This software is useful for visualisation of the term of a complex number, as well as the presentation of all operations over the field of complex numbers. Given that the software is simple to use, free and available in Serbian language, it makes it much more advantageous compared to other similar software. Therefore, GeoGebra software can be applied in different forms of mathematics teaching [3].

Key words. Complex number, Gauss plane, imaginary unit, geometric interpretation.

2. Teaching complex numbers in a "traditional" manner

In order to cover the topic it is necessary to have 5 lessons during which the students are informed about: the term of a complex number, set of complex numbers, and presentation of complex numbers in a Gauss plane, arithmetic operations and their application. The experience has shown that the emphasis in teaching complex numbers is often put on mechanical adoption of the procedures for solving problems from this topic. The biggest problem that occurs during the traditional way of treating this subject is the lack of deeper understanding of the geometrical interpretation of a complex number. This is the reason why such an approach has to be changed with a new one, which requires the use of GeoGebra.

3. Teaching complex numbers in Geogebra software

Teaching mathematics in GeoGebra is interesting because it assesses the level and method of application and the unique characteristics of teaching mathematics and computer science courses with the aim of improving the general context of learning and improving the digital competencies of students, that is, it uses the advantages of this teaching system compared to the traditional system [4]. GeoGebra is mathematical software that successfully links geometry, algebra, analysis and other areas. GeoGebra software is often used in the teaching of mathematics [5]-[9]. It is suitable for presentation and better understanding of mathematical content. GeoGebra has three ways of representation of mathematical objects: graphical, algebraic and table presentation. All three ways of representation of an object are dynamically linked and are automatically adjusted to each change that occurs in any of the representations, regardless of the way the object was created. The power of visualisation can be used as a mean for development of theoretical meaning of geometrical terms. GeoGebra can be installed on a computer, or can be used in an on-line mode [10]. Using GeoGebra software teaching and evaluation processes in addition to mathematical knowledge, they also include students' ICT (Information and Communications Technology) knowledge and skills [11].

3.1. The term of a complex number

During the first lesson, by solving an equation (1):

$$x^2 + a = 0 \tag{1}$$

The students are encouraged to conclude that given equations cannot always have a solution in the set of real numbers. This is the reason for the introduction of an *imaginary unit* and an algebraic form of a complex number (2).

$$z = a + bi, \qquad a, b \in \mathbb{R} \tag{2}$$

Each complex number has its geometrical interpretation. Just as all real numbers can be represented by an infinite straight line, in the same way the area of real and imaginary numbers can be represented by an infinite plane.

Number $z = a + bi$ can be represented in the Gauss plane as a point with coordinates *(a,b)* where the first coordinate of the ordered pair is the abscissa *a* (real part of the number) and the second coordinate of the ordered pair is the ordinate *b* (imaginary part of the number). It is possible to show this procedure in GeoGebra (Figure 1).

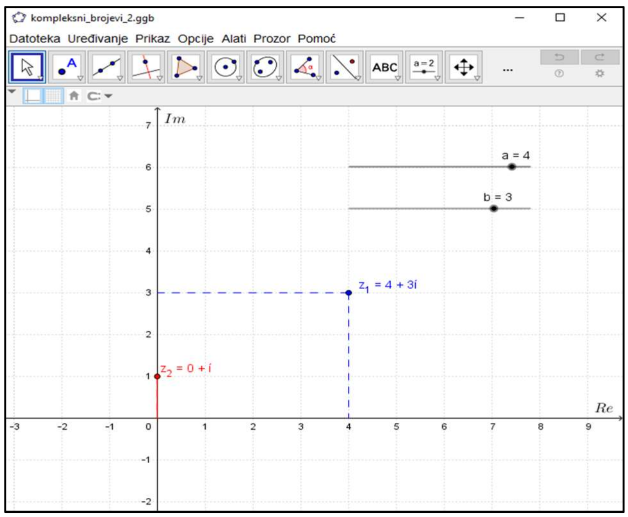


Figure 1. Display of complex number in the Gauss plane

By moving the sliders *a* and *b* the complex number in the Gauss plane is changed, so the students can see many examples in a short period of time and, if the class is held in a computer classroom, which is desirable in this case, they can explore for themselves.

3.2. Complex conjugate of a number

As a complex number is represented by a point in the Gauss plane, the distance between the observed point and the origin is defined as a modulus of the complex number and is calculated by the formula (3).

$$|z| = \sqrt{a^2 + b^2} \tag{3}$$

The following term is defined as a complex conjugate. If $z = a + bi$ is denoted as a complex number, then its complex conjugate is in formula (4):

$$\bar{z} = a - bi \tag{4}$$

Complex conjugate of a complex number is made when the sign of the imaginary part of the complex number is changed [12]. GeoGebra is suitable software for the representation of a complex number and it's conjugate.

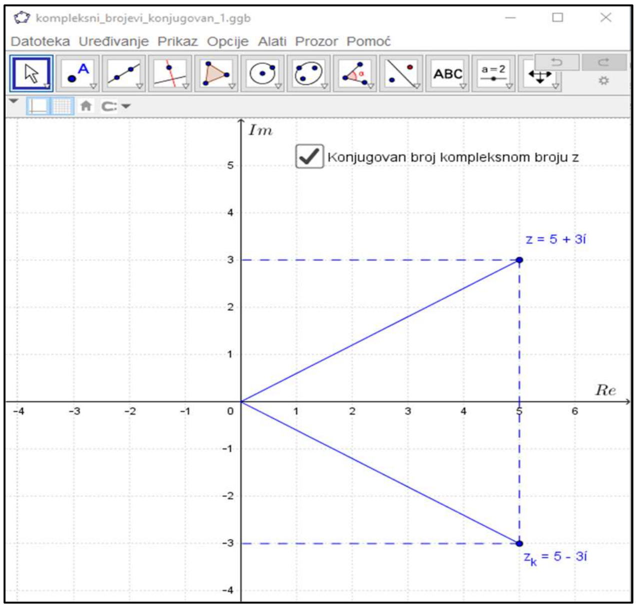


Figure 2. Display of a complex conjugate

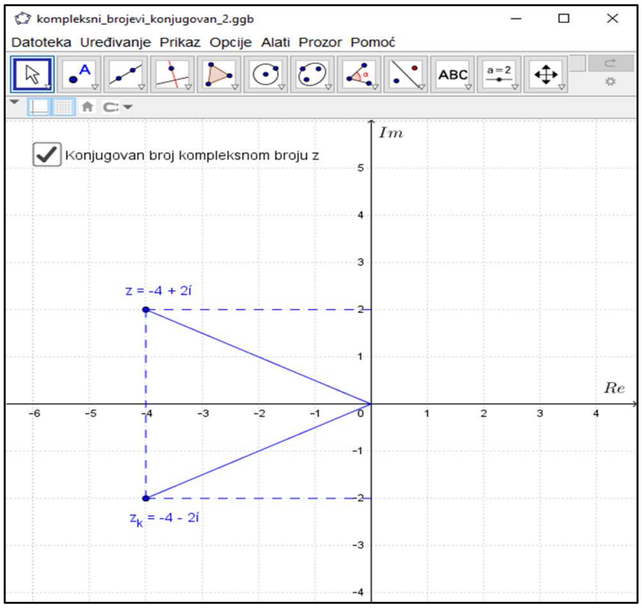


Figure 3. Display of complex conjugate

This is a simple way to show students *a conjugate of a complex number*. By showing students the geometric representation of a complex number and its conjugate, they can clearly see that their moduli are the same. This means that

$$|z| = |\bar{z}|$$

The advantage of GeoGebra is that, by moving point *z* we automatically get a new$\bar{z}$. This enables students to see many examples in a short time and notice the properties of a complex number and its conjugate (Figure 2. and Figure 3.)

3.3. Operations with complex numbers

After the introduction of algebraic and geometric form of a complex number, it is necessary to define binary operations over the field of complex numbers $\mathbb{C}$.

Let $z_1 = a + bi$ and $z_2 = c + di \quad a, b, c, d \in \mathbb{R}$
be complex numbers. Arithmetic operations are defined in the following way:

$$z_1 + z_2 = (a + c) + (b + d)i$$
$$z_1 - z_2 = (a - c) + (b - d)i$$
$$z_1 \cdot z_2 = (ac - bd) + (ad + bc)i$$
$$\frac{z_1}{z_2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i, \qquad z_2 \neq 0$$

The field of complex numbers is closed under all these operations, which means that the result of an operation between complex numbers is a complex number [13].

GeoGebra has also proven to be suitable for the processing and interpretation of the operations over the field of complex numbers. It is easy for the student to see in a Gauss plane that the sum, difference, product and quotient of complex numbers is a complex number (Figure 4.) It is important to stress that, besides being able to see the *geometric interpretation* of the newly formed complex number (including the modulus), by manipulating the points in the plane, it is possible to present these operations for different values of complex numbers. Also, it is important to mention that this software enables us to simplify complicated algebraic expressions consisted of several arithmetic operations and present their values geometrically.

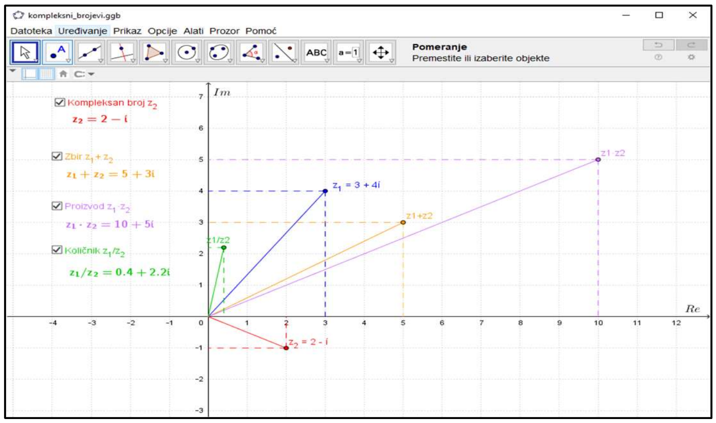


Figure 4. Operations with complex numbers

This feature helps the users find the solution easier and see its visualisation.

3.4. Power of the imaginary unit

Imaginary unit is the solution of the equation (5).

$$x^2 + 1 = 0 \tag{5}$$

It is denoted by i and its value is $i^2 = -1$. Geometric representation of the imaginary unit is point (0,1) [14]. Value of the imaginary unit can be checked in the following way:

$$i^2 = (0,1)(0,1) = (-1,0) = -1$$

It is obvious that:

$i^{4k} = 1$
$i^{4k+1} = i$
$i^{4k+2} = -1$
$i^{4k+3} = -i$, $k = 0,1,2,\ldots$

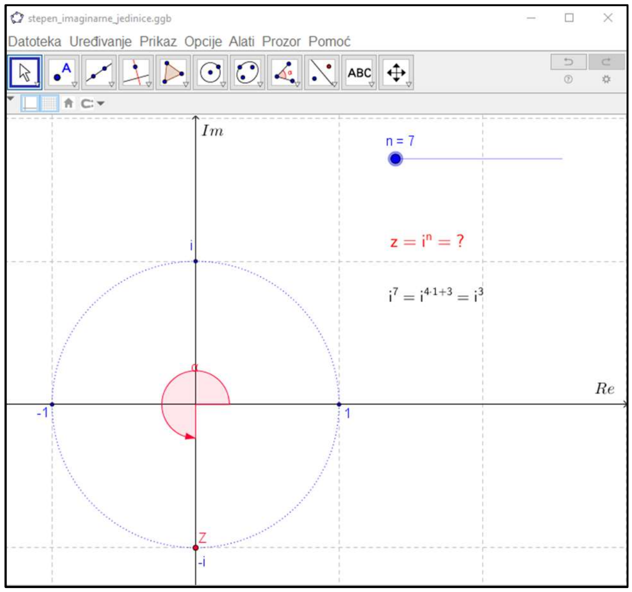


Figure 5. The n-th – power of number i

Figure 5. shows a graphical representation of the power of the imaginary unit. Movement of the slider easily represents the link between the power of the imaginary unit and the position of the corresponding point on the unit circle. By changing the exponent, the point is rotated in a positive direction by the angle of $\frac{\pi}{2}$, meaning that there are 4 different values of the power of the imaginary unit. This interpretation in GeoGebra enables the students to understand cyclical repetition of the imaginary unit power value.

3.5. Performance tasks

Example 1

The following complex numbers are given:
$z_1 = 3 - 4$ and $z_2 = 2 - i$.
Find the modulus of the complex number $\frac{z_1 \cdot z_2}{\bar{z}_1}$.

Solutions:

1. $z_1 \cdot z_2 = 2 - 11i$
2. $\bar{z}_1 = 3 + 4i$
3. $\frac{z_1 \cdot z_2}{\bar{z}_1} = -1{,}52 - 1{,}64i$
4. $\left|\frac{z_1 \cdot z_2}{\bar{z}_1}\right| = \sqrt{5} \approx 2{,}24$

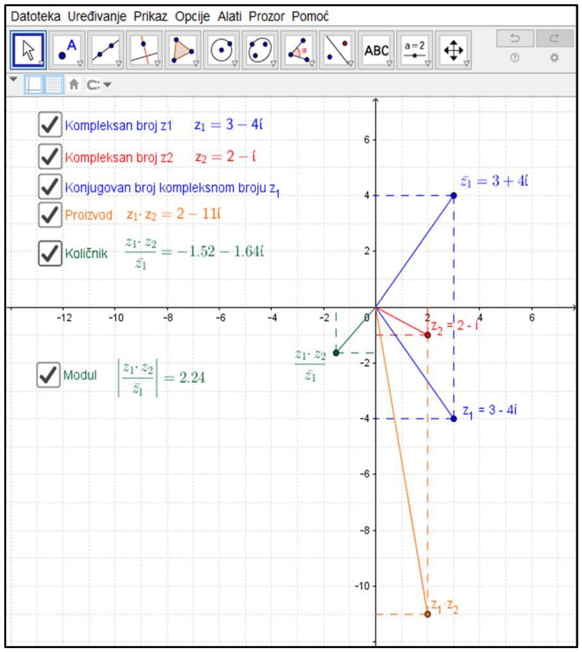


Figure 6. Graphical representation of Example 1

Figure 6 shows the steps in solving the given example. Therefore, the image shows complex numbers $z_1$andz_2. Their product, complex conjugate of z_1 quotient of the product and conjugate. Finally, the modulus of the complex number is clearly visible. Classical way of solving this task does not provide a possibility of visual representation of each step. GeoGebra saves time and enables us to check the solution itself.

Example 2

The following complex numbers are given:

$z_1 = 1 - i$ and $z_2 = -1 - i$. Find the value of expression$\left(\frac{z_1}{z_2}\right)^{2018}$.

Solution:

$$\frac{z_1}{z_2} = i$$

$$\left(\frac{z_1}{z_2}\right)^{2018} = i^{2018} = i^{4 \cdot 504 + 2} =$$
$$= (i^4)^{504} i^2 = 1 \cdot (-1) = -1$$

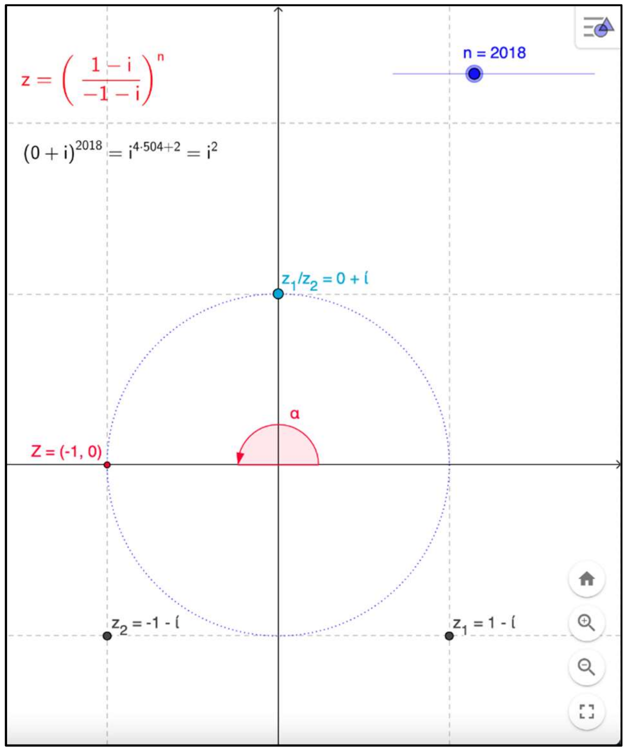


Figure 7. Graphical representation of Example 2

Figure 7. shows graphical solution of the example. Each solution step is shown with its geometric and algebraic interpretation, which helps the user understand the operations with complex numbers better. Besides the solution of this example, it is possible to use the slider and change the value of the exponent and get a number of similar examples.

3.6. GeoGebra CAS view

CAS (Computer Algebra System) view enables the users to work with algebraic expressions, functions, equations, matrices, numbers and datasets. In a simple way, this GeoGebra view enables solving equations, factoring of polynomials, differential and integral calculus. Since we have already seen that complex numbers can be represented in this software, all the above mentioned problems can be solved over the field of complex numbers with the use of CAS window.

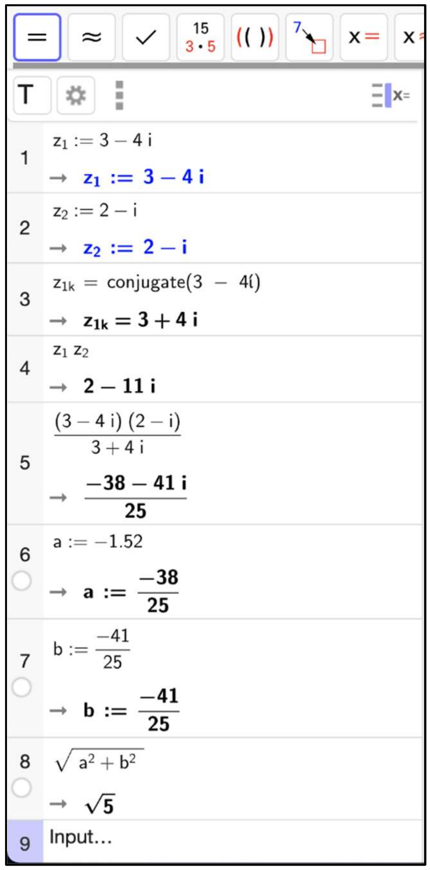


Figure 8. CAS view of Example 1

This GeoGebra option enables the users to find the solution of a problem in short time, or to check an existing solution. It is very important to highlight that the use of CAS view for the complex numbers requires for the imaginary unit to be defined.

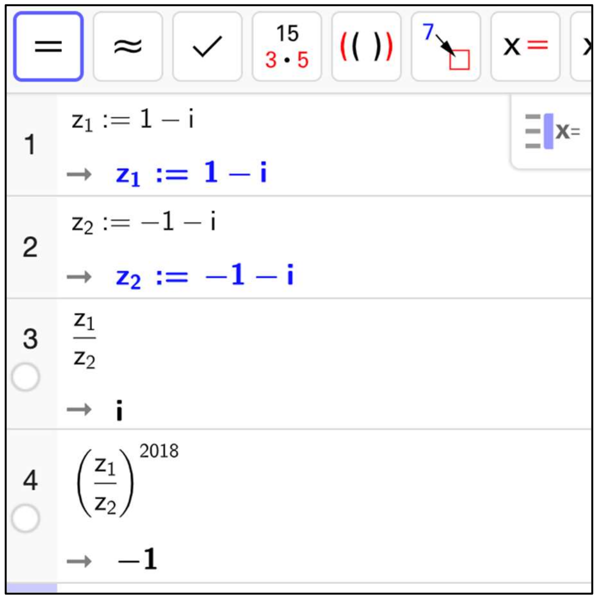


Figure 9. CAS view of Example 2

GeoGebra does not recognize i as the imaginary unit (Figure 8. and Figure 9.).

4. Results and discussion

The purpose of this research is to analyse the achievements of students who use the GeoGebra/HotPotatoes multimedia software. Teaching mathematics in this case is done by complex numbers. The research was conducted in the Secondary School of Economics in Doboj, Republika Srpska (Bosnia and Herzegovina), among the students of the second year, profile of economic technician, 220 students participated in the research.

The survey was conducted in the following chronological order:

1. Two classes were selected by the following criteria. The number of students in both classes was equal; the average grade in mathematics was approximately the same.
2. In Class A GeoGebra software was used during the elaboration of lesson of complex numbers. Class B processed the topic in the traditional way.
3. Upon the completion of the topic elaboration, both classes were tested. Students had 45 minutes to solve 10 performance tasks from the topic. The tasks included both, theoretical and arithmetic problems. Both classes did the test with the same problems, in the same period of time, on the same day (Figure 10a and Figure 10b.).
4. The test was presented electronically, using HotPotatoes software. Students solved the problems on a sheet of paper and then entered the final answers and solutions into the on-line test.

5. Upon entering their answers in HotPotatoes, the students had an opportunity to see whether the answer was correct, which gave them an opportunity to self-evaluate their test, prior to the evaluation made by their teacher.
6. Upon the evaluation of the test, the students were presented with automatically generated results of each task, as well as the final test score.

The advantage of this on-line test is that the students can compare their self-evaluation with the teacher's one.

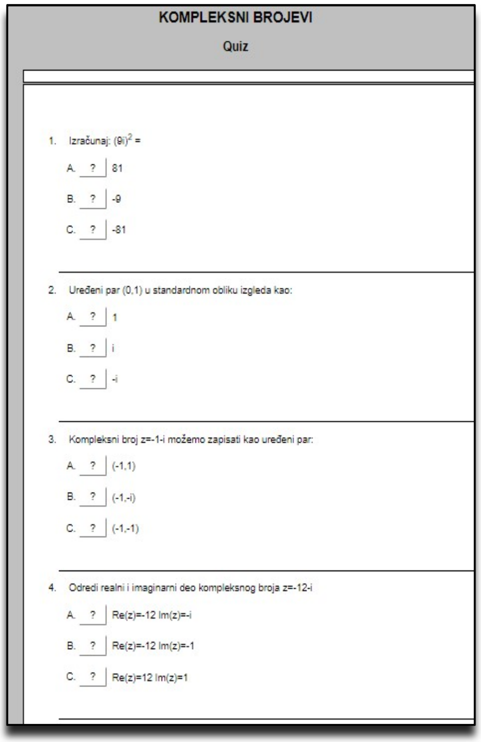
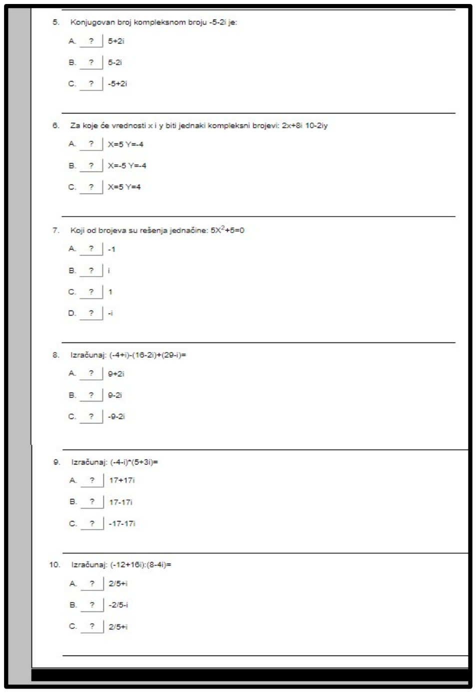


Figure 10a. Review of the first page of the test to check knowledge of complex numbers



5. Konjugovan broj kompleksnom broju -5-2i je:
A. ? 5+2i
B. ? 5-2i
C. ? -5+2i

6. Za koje će vrednosti x i y biti jednaki kompleksni brojevi: 2x+8i 10-2iy
A. ? X=5 Y=-4
B. ? X=-5 Y=-4
C. ? X=5 Y=4

7. Koji od brojeva su rešenja jednačine: $5X^2+5=0$
A. ? -1
B. ? i
C. ? 1
D. ? -i

8. Izračunaj: (-4+i)-(16-2i)+(29-i)=
A. ? 9+2i
B. ? 9-2i
C. ? -9-2i

9. Izračunaj: (-4-i)*(5+3i)=
A. ? 17+17i
B. ? 17-17i
C. ? -17-17i

10. Izračunaj: (-12+16i):(8-4i)=
A. ? 2/5+i
B. ? -2/5-i
C. ? 2/5+i

Figure 10b. Review of the second page of the test to check knowledge of complex numbers

A detailed analysis of the tests made by the class that used GeoGebra and the class that processed the topic in a traditional way produced the following results, shown in Figure 11.

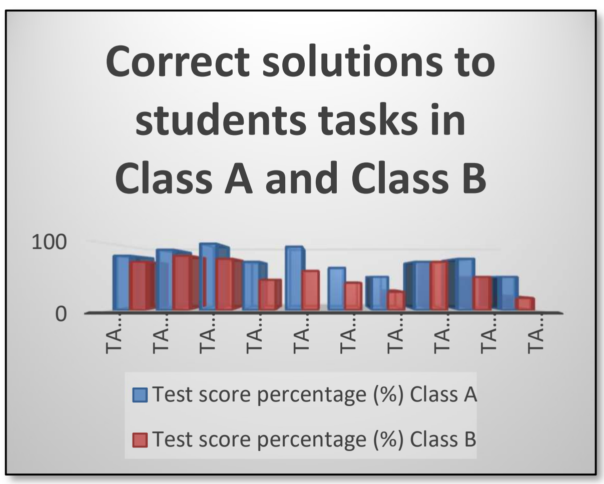


Figure 11. Test results for checking knowledge in the area of complex numbers

Out of possible 220 correct answers, Class A produced 166 (75.45%), while Class B produced 120 (54.54%) correct answers. We can conclude that Class A produced 20. 91% better results than Class B. Also, if we have a look at the results of individual tasks, we can see that in the majority of tasks Class A produced better results. If we analyse the content of the test, the tasks that produce bigger deviation are related to the determination of the real and imaginary part of a complex number (TASK 4), complex conjugate of a complex number (TASK 5) and division of complex numbers (TASK 10). Tasks that produced similar results in both classes were related to the determination of the power of the imaginary unit (TASK 1), different forms of the presentation of a complex number (TASK 2) and addition and subtraction of complex numbers (TASK 8). Also, Figure 11. shows a graph with the results of on-line test made by Class A and Class B. Results of the Class A are marked in blue and the results of the Class B in red. Columns of the histogram are marked with the number of students who gave the correct answer and they are shown for each task in the test

5. Conclusions

Analysis of the students' achievements at the level of complex numbers in a "traditional" way and teaching enriched with multimedia software. The paper focuses on learning complex numbers using the GeoGebra/HotPotatoes multimedia software. This approach to teaching aims to increase student achievement, raise the teaching process to a higher level of efficiency, and motivate students to work and learn more. Students tested in Class A ("traditional" method) and Class B (GeoGebra/HotPotatoes multimedia software) have the following results: out of a possible 220 correct answers, students in Class A gave 166 (75.45%) and Class B 120 (54.54%) correct answers. We can conclude that Class A had a 20.91% better result than Class B. Also, if we look at the results of individual tasks, we see that in most tasks Class A achieved better results. If we analyses the content of the test, the tasks that produce a greater deviation refer to the determination of the real and imaginary part of a complex number (TASK 4), the complex conjugate of a complex number (TASK 5) and the division of complex numbers. (TASK 10). The tasks that obtained similar results in both classes were related to determining the potential of an imaginary unit (1st TASK), different forms of complex number representation (2nd TASK) and addition and subtraction of complex numbers (8th TASK).

Finally, we can conclude that the use of multimedia in teaching GeoGebra/HotPotatoes facilitates the process of knowledge transfer and enables students to actively participate, present their proposals for solving problems, research and gain self-confidence. With this approach to teaching, the teacher overcomes the limitations of classical teaching. The teacher offers a creative and interesting approach to teaching, and is obliged to design and adapt the teaching material to suit the students as best as possible. It is necessary to use performance tasks as much as possible, and to develop students' ability to apply different techniques in solving these tasks. GeoGebra is a powerful tool that enables the fulfilment of all set goals in modern mathematics teaching, and is very easy to use by both teachers and students.

Competing interests

Authors/co-authors Dragana Nedić, Gordana Jotanović, Tijana Paunović and Aleksandar Kršić declare that there are no competing interests in the paper entitled "ANALYSIS OF STUDENT ACHIEVEMENTS IN TEACHING COMPLEX NUMBERS USING GEOGEBRA SOFTWARE".

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GAMES IN MATHEMATICS INSTRUCTION

Marina Manić[1], Eugen Ljajko[2], Marina Tošić[2]

Abstract. The paper emphasizes various aspects of mathematical games usage in mathematics instruction. Different mathematical games are discussed in accordance to the Principles of educationally-rich mathematical games. Two of them – Bingo and Tangram – are included in a primary school mathematics instruction and their impact in the instruction was assessed. The feedback we obtained indicates increased students' interest in the subject areas and motivation to learn mathematics. Guidelines for developing more effective game variations and instruction techniques are given.

2010 *Mathematics Subject Classification.* Primary: 97D40 Secondary: 97U10.
Key words and phrases. Principles of educationally-rich mathematical games, Bingo, Tangram, Motivation.

1. INTRODUCTION

Having in mind the age of primary school students and their needs, introducing games in the instruction process comes as a very useful tool in many different aspects: from encouraging and motivating students to improving their understanding of the subject material and even academic achievements. According to the current Serbian primary school mathematics curricula, there are not specified ways of student motivation. Therefore, teachers have to use other sources of ideas for improving the instruction process. One of them is designing, modifying or simply just using pre-designed mathematical games.

Mathematical games are used to improve interest in mathematics in general [1], as well as in some specific fields of mathematics: in algebra – [2], geometry – [3], etc.

Furthermore, a well nuanced use of mathematical games can bring positive effects in various aspects of the instruction process itself. For example, Prensky in [4] records cognitive changes in student behaviour in a game-based learning. Introducing games into mathematics instruction can encourage student-student and student-teacher communication and collaboration [5], thus alleviating development of students' communicational skills [6]. This can be very helpful both in developing a positive learning atmosphere and improving students' mathematical skills and achievements, which is proven in some researches, for example [7], [8]. Mathematical games help students overcome ambivalent feelings they may have about mathematical concepts and subject areas [9], which is, of course, a good starting point for motivating students.

On the other hand, if poorly planned, organized or implemented, mathematical games can bring negative effects into mathematics instruction. If used in isolation, games are less effective at supporting learning retention compared with other engaging, student centred, but more mathematically explicit activities [10]. For that reason, special attention should be paid at the entire process of preparation and implementation of mathematical games in the instruction process. The games that teacher plans to use in the instruction should be assessed in regard of their educational potential, which will be discussed in the next chapter.

2. GAMES AS AN ADDITION TO MATHEMATICS INSTRUCTION

When assessing whether a game should be used in the instruction process, several aspects should be discussed. Among them are:

- the level of integration and interconnections between the game and mathematics,
- the level of effort students need to invest in learning the game and interpreting it in the context of the subject material,
- the danger of developing unwanted patterns of behavior like competition among students and neglecting the importance of the subject material,
- finding as many areas as possible where the game could be successfully introduced and used.

In this regard, very useful guidelines are given by Russo et. al. In [11], they introduce a set of criteria that can help a teacher in determining if a game is appropriate for the subject material and the students he/she is working with. The criteria are known as Principles of educationally-rich mathematical games [11] and deal with five areas crucial for success of the game usage in the instruction:

1. **Students are engaged**. This can be easily recognized. The teacher can notice if the students are engaged and connect tasks given within the game with the ones concerning mathematics. Such a behavior indicates that the game meets the first criterion. On the opposite, a poorly planned introduction of a game, for example excessively repeating the game just for the sake of playing it can lead to boredom [12].
2. Mathematical games should appropriately balance students' **skill** and **luck**. Activities based on luck only cannot be expected to help in improving quality of the instruction process or any of its aspects [8], and contrary – games that emphasize skills bring better students into the focus, which can be demotivating for students that are less skilled.
3. Exploring important **mathematical concepts** and practicing important skills **should be central** to the game strategy and gameplay. One should have in mind that the very nature of most of the games – competitiveness can distract students' attention from mathematical content. This, further, means that a mathematical game should have a clearly defined mathematical purpose and align with the planned mathematical goals [13].

4. **Flexibility for learning and teaching**. Though the time invested in learning rules and regulation of a game can in no circumstances be regarded as wasted, it would be very important if an already introduced game can be used for more different topics. If so, it can lead to building students' positive learning habits.
5. Mathematical games should provide opportunities for **fostering home-school connection**. If a game has this quality, it helps students widen the field of their mathematical thinking and activities to their homes.

Taking into consideration the Five principles of educationally-rich mathematical games, one can easily assess the value of a game and decide whether it is worthy to introduce it into the instruction process or not. Even if a game does not meet all the principles the teacher can modify it in order to enrich its pedagogical value. We assessed two mathematical games – Bingo and Tangram.

Bingo – students are given 5x3 tables with "randomly" inscribed numbers and a set of problems, their solutions being some of the numbers inscribed in the tables. Students solve the problems and mark their solutions in the table, Table 1. The winner is the one who marks all numbers in a column. Some examples of the problems given to students are:

Solve the equation for x: $\frac{x}{9} = 5$.

Calculate the value of the expression $|a-b|-|c|$ *for* $a = 3$, $b = -2$, $c = -5$.

Table 1: Some of the numbers are solutions of the problems

30	63	3
47	2	51
9	45	12
11	8	6
7	0	21

We evaluated the game's compliance with the Principles as follows:

Students are easily engaged – mostly complies with the principle 1. Though the game requires more skill than luck, it is obvious that the teacher can "adjust" the luck using different layouts of numbers in the tables – partially complies with the principle 2. Mathematics is in the core of the game, but students' focus can turn to competition – partially complies with the principle 3. Flexible both in regard of concepts and learners, provides opportunity for fostering home-school connection – complies with the principles 4 and 5.

Tangram – seven boards of skill. The seven pieces can be assembled to form a square: 2 large right triangles, 1 medium right triangle, 2 small right triangles, 1 square and 1 rhomboid, Figure 1.

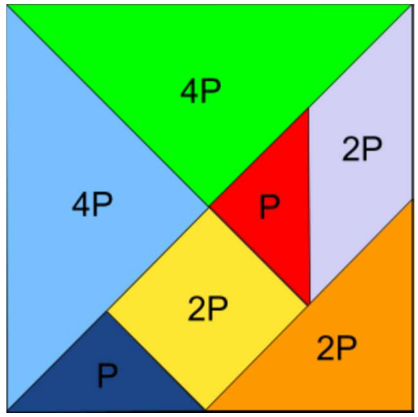

Figure 1: Tangram – seven boards of skill

Out of the pieces almost unlimited number of figures can be constructed. However, we were interested in construction of basic geometric figures – squares, rectangles, rhomboids, triangles, trapezoids. We also evaluated this game's compliance with the Principles as follows:

Students are easily engaged – complies with the principle 1. It requires more skill than luck – complies with the principle 2. Mathematics is in the core of the game – complies with the principle 3. Partially flexible in regard of concepts – mostly complies with the principle 4. Provides opportunity for fostering home-school connection – complies with the principle 5.

As a result of our assessment we came to a conclusion that these two games are appropriate for the classroom usage.

3. Students' activities with mathematical games

In May and June 2021. we organized mathematics instruction including mathematical games in "Dušan Radović" primary school in Pirot, Serbia. There were 53 students in the fifth grade (11 years old) and 55 students in the sixth grade (12 years old). Due to the limitations caused by the COVID-19 pandemics, each of the classes was divided into two groups – one of them attending the instruction in the school and the other one attending online classes. The groups interchanged weekly and a group attended either 3 classes weekly in school and one online, or vice versa.

We introduced the Bingo game in the fifth grade in the instruction on Fractions. After learning the rules of the game, students accepted it as an appropriate means in learning other topics.

In the sixth grade we used the Tangram game in the instruction on Quadrilateral and Area of triangle and quadrilateral. The students were usually given task to form a certain geometric object and explain possible transformations that lead to a simpler calculation of its area. Although the game seems to be applicable in a narrow subject area, we diversified it introducing different restrictions on number of pieces allowed or requesting different approaches.

After the games were introduced into the instruction process, we noticed several changes in the students' behaviour and learning. Students of the group exposed to the games were more active and focused on the topic. They also used to ask questions more frequently than their counterparts form the group that attended online classes, which is in line with observations in [5] and [6]. Though we noticed competitiveness among the students, it was easy to steer it in a direction that ensured faster learning and better understanding of the topic.

We also noticed that students exposed to mathematical games showed better understanding and faster learning of the topics, no matter how abstract they appeared to the students. This was observed both for understanding concepts of algebraic and geometric nature, which only confirms what was stated in [2] and [3] respectively. Another aspect of this type of instruction was an evident increase in students' motivation for learning mathematics.

All these encouraged us to continue with game-based instruction even in other areas. The fact that it was easier to imbed the Bingo game into different mathematical subjects just confirms that assessing a game's educational value according to the Principles is a right way to design, organize and implement a game-based mathematics instruction.

4. CONCLUSIONS

In order to determine if a game is worth to be introduced into the instruction, the teacher has to assess its educational and mathematical value.

It is recommended that the game complies with the Principles of educationally-rich mathematical games.

Our experience shows that such games are easily accepted among the students and make a significant impact in improving several aspects of the mathematics instruction – improves students' concentration, communication and cooperation skills, understanding subject material, etc.

On the other hand, teachers should be aware of the rise of competition among students and turning their attention off the mathematical concepts and procedures, which can altogether hinder students' mathematical development.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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АЛГЕБАРСКИ РАЦИОНАЛНИ ИЗРАЗИ ВО ОСНОВНО И СРЕДНО ОБРАЗОВАНИЕ - ПРОБЛЕМИ И НЕДОСТАТОЦИ

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Апстракт. Алгебарските рационални изрази се применуваат во природните и техничките науки, како и во геометријата, техниката, но и во различни реални ситуации. Затоа, важноста за нивно детално изучување и совладување е огромна. Нивното изучување започнува во 7мо одделение од основното образование и продолжува во 8мо и 9то одделение согласно Кембриџ програмата, за потоа да се заокружи во прва година средно образование. Понудениот материјал за основно и средно образование треба да биде усогласен со цел да не предизвика појава на проблеми кај учениците при усвојување на поимите, меморирање и нивна примена. Затоа, во овој труд ќе биде направена анализа на усогласеноста на темата во овие два степени на образование.

1. Вовед

Образованието е основа и важен елемент во мозаикот наречен општество, негов двигател и мотор за развој, просперитет и слобода. “Образованието е најмоќното оружје што можете да го употребите за да го промените светот”, рекол Нелсон Мандела. Затоа, обврска е на секој поединец и на секоја државна институција да придонесе во негово менување со цел подобрување и развивање. Исто така е многу важно и особено сензитивно разработувањето, менувањето, адаптирањето и усогласувањето на наставните програми по сите наставни предмети вклучувајќи ја и математиката во сите степени на образование во рамките на една држава.

Сведоци бевме на менување на наставните програми како по сите наставни предмети, така и по предметот математика во основно образование. Во овој процес исклучително важно е да се внимава на нивно усогласување со наставните програми во средно образование. Нивното не усогласување може да предизвика бројни проблеми кај учениците при усвојување на поимите и нивна примена подоцна во понатамошното образование.

Сведоци сме исто така дека програмата Кембрич ([1], [2], [3]) е спирална наставна програма, каде учениците почнуваат со изучување на едно подрачје во првото полугодие, но во второто полугодие, како и во двете полугодија од понатамошните одделенија се навраќаат на истата изучувајќи

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Клучни зборови и фрази. алгебарски рационални изрази, полиноми, мономи, алгебарски дропки.

ја на повисоко ниво. Така е и со алгебарските рационални изрази, кои учениците започнуваат да ги изучуваат во рамки на подрачјето Алгебра и решавање проблеми во првото полугодие од 7мо одделение и во второто полугодие, но и во двете полугодија од 8мо и 9то одделение се навраќаат на

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истите изучувајќи ги на повисоко ниво. Оваа спиралност повлекува некои размислувања, прашања и дилеми, кои бараат поопсежен пристап и дебата. Нормално, една цел е една група на ученици што не го совладале или делумно го совладале овој материјал во 7мо одделение, и подоцна им се дава можност да се навратат и да го совладаат истиот. Затоа се наметнува и следното прашање:

- Што е со одличните и талентирани ученици?

На нив оваа спиралност во повисоките одделенија им предизвикува досада и им делува немотивирачки за учење. Тие губат драгоцено време и наместо да изучуваат нови работи, нивното знаење стагнира на истиот материјал, кое повлекува изостанување на компонентите развој и напредок.

Јасно, на ова прашање се надоврзуваат и следните прашања:

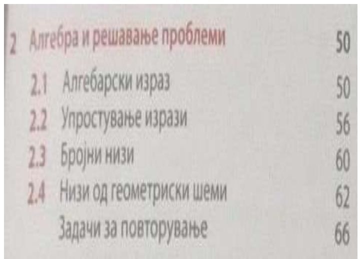
- Што е со подлабинското и подетално изучување на подрачјето и во колкава мера изостанува?
- Дали постои усогласеност помеѓу поимите и обемот на нивно изучување со наредниот степен на образование-средното образование?
- Одредени недостатоци и неусогласености како влијаат на стекнатото знаење на учениците подоцна низ останатиот процес на учење и образование?
- Како резултат на веќе кажаното, со какви проблеми се среќаваат учениците и како можат истите да се отстранат или ублажат?

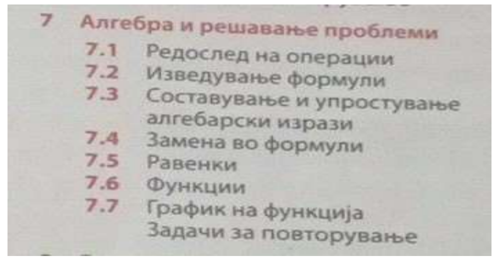
На сите овие прашања и дилеми ќе се обидеме да одговориме преку анализа на она што се изучува по Кембрич програмата за алгебарските рационални изрази во 7мо, 8мо и 9то одделение и она што се изучува и заокружува во прва година средно образование, независно дали се работи за гимназиско или стручно образование, знаејќи дека и кај двете наставни програми тие се застапена подеднакво.

2. Алгебарски рационални изрази по Кембрич програмата

Како што веќе спомнавме, алгебарските рационални изрази учениците започнуваат да ги изучуваат во склоп на подрачјето Алгебра и решавање проблеми во прво полугодие од 7мо одделение и во второ полугодие, како и во двете полугодија од 8мо и 9то одделение се навраќаат на нив изучувајќи ги на повисоко ниво и од друг аспект. Наставните планови за овие три одделенија се изработени од Бирото за развој на образование во 2016 година.

Наставната содржина од 7мо одделение за подрачјето Алгебра и решавање проблеми за прво и второ полугодие е дадена на слика 1, а) и слика 1, б) соодветно.

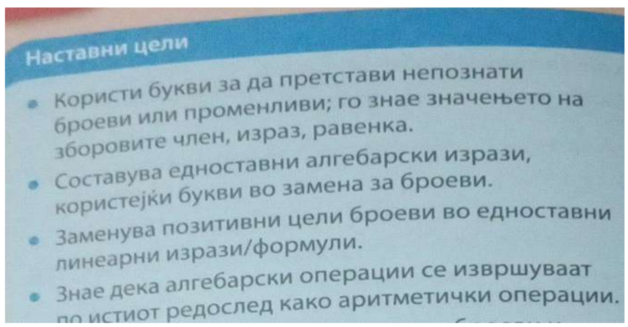




а) прво полугодие **б)** второ полугодие

Слика 1: Содржина за Алгебра и решавање проблеми во 7мо одд.

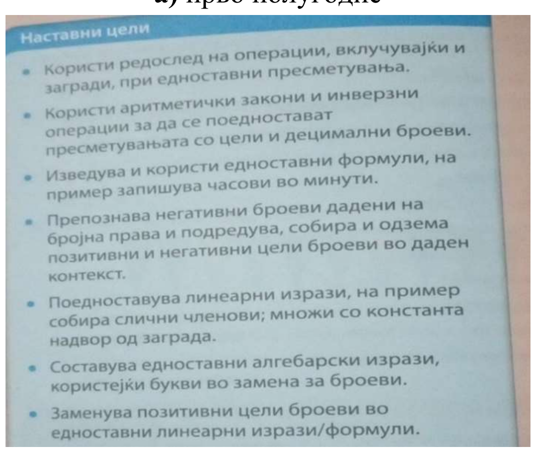
Наставните цели од 7мо одделение за подрачјето Алгебра и решавање проблеми за прво и второ полугодие се дадени на слика 2, а) и слика 2, б) соодветно.



Наставни цели

- Користи букви за да претстави непознати броеви или променливи; го знае значењето на зборовите член, израз, равенка.
- Составува едноставни алгебарски изрази, користејќи букви во замена за броеви.
- Заменува позитивни цели броеви во едноставни линеарни изрази/формули.
- Знае дека алгебарски операции се извршуваат по истиот редослед како аритметички операции.

а) прво полугодие



Наставни цели

- Користи редослед на операции, вклучувајќи и загради, при едноставни пресметувања.
- Користи аритметички закони и инверзни операции за да се поедностават пресметувањата со цели и децимални броеви.
- Изведува и користи едноставни формули, на пример запишува часови во минути.
- Препознава негативни броеви дадени на бројна права и подредува, собира и одзема позитивни и негативни цели броеви во даден контекст.
- Поедноставува линеарни изрази, на пример собира слични членови; множи со константа надвор од заграда.
- Составува едноставни алгебарски изрази, користејќи букви во замена за броеви.
- Заменува позитивни цели броеви во едноставни линеарни изрази/формули.

б) второ полугодие

Слика 2: Наставни цели за Алгебра и решавање проблеми во 7мо одд.

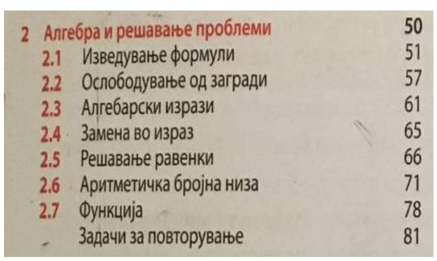
Согласно содржината и наставните цели на програмата, како и разгледувањето на учебникот се забележува следното:

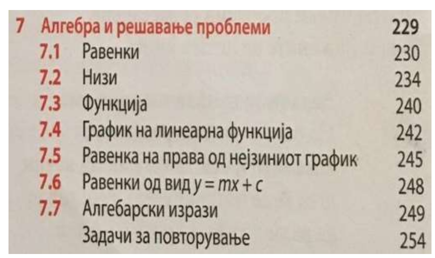
- Во првото полугодие, воведувањето е преку изведување на формули со користење на реални ситуации и геометриски фигури во рамнина;
- Се посветува внимание на ослободување од загради преку групирање на слични членови и множење со позитивен број;
- Во второто полугодие повторно на истото подрачје имаме навраќање сосема кратко.

Недостатоците кои ги воочивме се:

- Именувањето е со Алгебарски изрази, но никаде не се прецизира поимот Алгебарски рационални изрази, иако цело време се работи со нив;
- Не постои воведување на поим моном и полином, кои се цели рационални изрази, а цело време се работи со нив;
- Ослободувањето од загради е само множење со позитивен број;

Наставната содржина од 8мо одделение за подрачјето Алгебра и решавање проблеми за прво и второ полугодие е дадена на слика 3, а) и слика 3, б) соодветно.





а) прво полугодие **б)** второ полугодие

Слика 3: Содржина за Алгебра и решавање проблеми во 8мо одд.

Наставните цели од 8мо одделение за подрачјето Алгебра и решавање проблеми за прво и второ полугодие се дадени на слика 4, а) и слика 4, б) соодветно.

Наставни цели

- Изведува и користи едноставни формули, на пример, претвора Целзиусови (°C) степени во Фаренхајтови (°F).
- Знае дека буквите имаат различни улоги во равенките, формулите и функциите; ги знае значењата на поимите: формула и функција.
- Знае дека алгебарските операции (вклучувајќи и загради) се извршуваат по истиот редослед како аритметичките операции; користи запишување на степени со степенов показател позитивен цел број.
- Упростува или трансформира линеарни изрази со коефициенти цели броеви; собира слични членови; множи со член надвор од заграда.
- Составува линеарни изрази.

а) прво полугодие

Наставни цели

- Изведува и користи едноставни формули, на пример, претвора Целзиусови (°C) степени во Фаренхајтови (°F).
- Знае дека буквите имаат различни улоги во равенките, формулите и функциите; ги знае значењата на поимите: формула и функција.
- Знае дека алгебарските операции (вклучувајќи и загради) се извршуваат по истиот редослед како аритметичките операции; користи запишување на степени со степенов показател позитивен цел број.
- Упростува или трансформира линеарни изрази со коефициенти цели броеви; собира слични членови; множи со член надвор од заграда.
- Составува линеарни изрази.
- Заменува позитивни и негативни цели броеви во формули, линеарни изрази и изрази со мали степенови показатели, на пример, $3x^2 + 4$ или $2x^3$, вклучувајќи примери кои водат до равенка за решавање.

б) второ полугодие

Слика 4: Наставни цели за Алгебра и решавање проблеми во 8мо одд.

Согласно содржината и наставните цели на програмата, како и разгледувањето на учебникот се забележува следното:

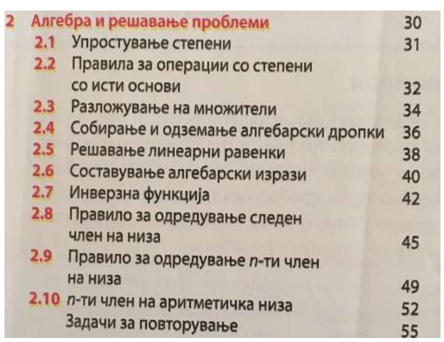
- Во првото полугодие, воведувањето е преку изведување на формули со користење на реални ситуации и геометриски фигури во рамнина и простор;
- Се посветува внимание на ослободување од загради преку групирање на слични членови и множење со позитивен и негативен број;
- Именувањето е со Алгебарски изрази, каде се наведени некои правила за нивно упростување, се спомнуваат примери на линеарни изрази и се објаснува множење членови во алгебарски изрази;
- Се посветува цела наставна единица за замена на конкретна вредност за непознатите во израз;
- Алгебарските изрази ги користи за воведување на равенки, нивно решавање со помош на функционална машина и решавање на равенки со една непозната, како и за аритметичка бројна низа и воведување на поимот функција;
- Во второто полугодие, кога имаме повторно навраќање на истото подрачје, прво се воведуваат равенки со примена во

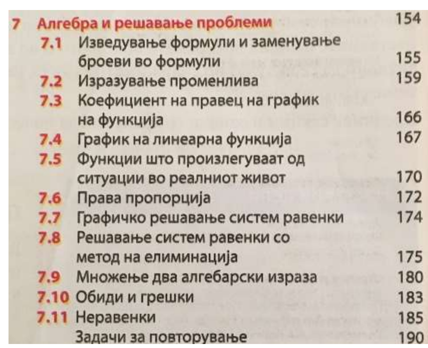
геометрија и елементарни реални ситуации, низи и линеарна функција, за најпосле со повторување на она од прво полугодие да се разгледаат накратко само со еден час повторно алгебарските изрази.

Недостатоците кои ги воочивме се:

- Именувањето е со Алгебарски изрази, но никаде не се прецизира поимот Алгебарски рационални изрази, иако цело време се работи со нив;
- Не постои прецизирање дека Алгебарските рационални изрази можат да бидат цели рационални и дробно-рационални изрази – алгебарски дропки;
- Не постои воведување на поим моном и полином, кои се цели рационални изрази, а цело време се работи со нив;
- Нема дефинирање на степен на моном и полином;
- Ослободувањето од загради е само со множење со позитивен и негативен број;
- Изостанува дефинирање на слични мономи, иако се спомнува слични членови, без никакви објаснувања.

Наставната содржина од 9то одделение за подрачјето Алгебра и решавање проблеми за прво и второ полугодие е дадена на слика 5, а) и слика 5, б) соодветно.

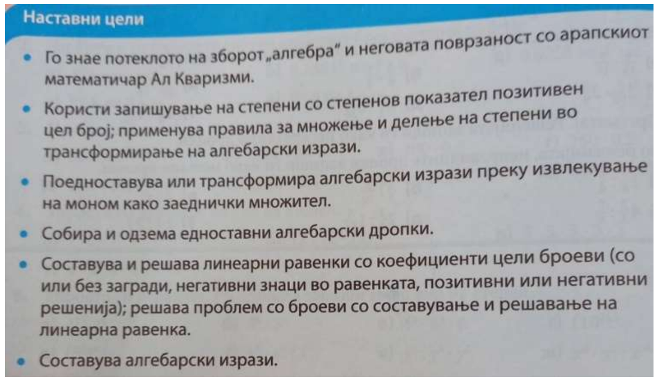




а) прво полугодие **б)** второ полугодие

Слика 5: Содржина за Алгебра и решавање проблеми во 9то одд.

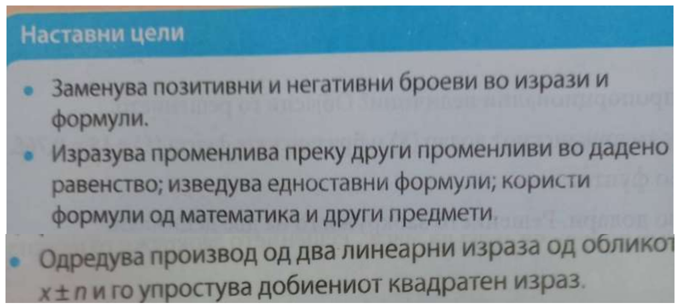
Наставните цели од 9то одделение за подрачјето Алгебра и решавање проблеми за прво и второ полугодие се дадени на слика 6, а) и слика 6, б) соодветно.



Наставни цели

- Го знае потеклото на зборот „алгебра“ и неговата поврзаност со арапскиот математичар Ал Кваризми.
- Користи запишување на степени со степенов показател позитивен цел број; применува правила за множење и делење на степени во трансформирање на алгебарски изрази.
- Поедноставува или трансформира алгебарски изрази преку извлекување на моном како заеднички множител.
- Собира и одзема едноставни алгебарски дропки.
- Составува и решава линеарни равенки со коефициенти цели броеви (со или без загради, негативни знаци во равенката, позитивни или негативни решенија); решава проблем со броеви со составување и решавање на линеарна равенка.
- Составува алгебарски изрази.

а) прво полугодие



Наставни цели

- Заменува позитивни и негативни броеви во изрази и формули.
- Изразува променлива преку други променливи во дадено равенство; изведува едноставни формули; користи формули од математика и други предмети.
- Одредува производ од два линеарни израза од обликот $x \pm n$ и го упростува добиениот квадратен израз.

б) второ полугодие

Слика 6: Наставни цели за Алгебра и решавање проблеми во 9то одд.

Согласно содржината и наставните цели на програмата, како и разгледувањето на учебникот се забележува следното:

- Во првото полугодие, се работи со оперирање на степени со иста основа, за потоа да се воведе разложување на множители кај алгебарски изрази, но само со извлекување на множител пред заграда;
- Потоа се собираат и одземаат алгебарски дропки;
- Се продолжува со решавање на линеарни равенки кои се воведуваат како поим за прв пат, иако учениците вакви равенки решаваат и во 8мо одделение;
- Составувањето на алгебарските изрази се врши преку користење на геометрија и елементарни реални ситуации;
- Во второто полугодие, кога имаме повторно навраќање на истото подрачје на алгебарските изрази се посветува многу малку внимание на множење на два изрази т.е без прецизирање дека се работи за множење на два полиноми.

Недостатоците кои ги воочивме се:

- Се спомнува поимот моном само во наставните цели, но во наставните единици ниту е дефиниран, ниту пак воопшто се спомнува;
- Се воведува собирање и одземање на алгебарски дропки без да се дефинираат истите, а потоа се продолжува со составување на алгебарски изрази (се собираат, одземаат и множат полиноми);
- Во второ полугодие меѓу другото се навраќа повторно на множење на два алгебарски изрази т.е множење на два бинома, без да се прецизира истото.

Во однос на алгебарските рационални изрази во споменативе одделенија може да ги дадеме следните добри аспекти, како и генералниот заклучок.

Единствен добар аспект во Кембрич програма во однос на алгебарските рационални изрази е нивната поширока примена во геометријата и елементарните реални ситуации, сепак генерално може да се заклучи дека:

- Алгебарските рационални изрази се дадени во склоп на Алгебарските изрази без нивно прецизирање;
- Алгебарските рационални изрази се дадени премногу елементарно за таа возраст на учениците со огромни недостатоци во дефинирање и разграничување на важни поими, кои се неопходни за нив. Ова делува демотивирачки за талентираните ученици и учениците што се заинтересирани за постигнување на високи резултати во математиката;
- Не се застапени формулите за скратено множење, кои се важни за оваа проблематика и се потребни за понатамошните степени на образование;
- Од претходно кажаното, материјалот е изложен многу конфузно и учениците во суштина не добиваат никакви прецизни знаења за алгебарските рационални изрази од она што треба да го усвојат како крајни применливи знаења.

3. АЛГЕБАРСКИ РАЦИОНАЛНИ ИЗРАЗИ ВО ПРВА ГОДИНА СРЕДНО ОБРАЗОВАНИЕ

Овде ќе ја разгледаме застапеноста и сложеноста во изучување на алгебарските рационални изрази во прва година средно образование преку наставните програми во гимназиско и средно стручно образование.

Наставната програма за гимназиско средно образование е изработена од Бирото за развој на образованието во 2001-та година и ја има следната содржина, слика 7:

3. АЛГЕБАРСКИ РАЦИОНАЛНИ ИЗРАЗИ **- Степен со показател природен број, операции со степени** **- Цели рационални изрази; мономи, полиноми; операции, разложување на множители, НЗС и НЗД** **- Дробно рационални изрази: поим и операции**	**18**	- Да повтори и да утврди за степени со показател природен број; - да ги повтори и да ги користи операциите множење и делење на степени со еднакви основи и степенување на степен, производ и количник; - да запишува броеви во обликот $a \cdot 10^k$; - да повтори за мономи, биноми..., полиноми и да ги продлабочи знаењата за операциите со нив; - да користи разложување на полиноми од типот am$\pm$bm, a^2-b^2, $a^2\pm 2ab+b^2$, $a^3\pm b^3$; - да одредува НЗС и НЗД за два и повеќе полинома; - да се запознае со алгебарска дропка и да одредува нејзин домен (област на определеност); - да проширува и скратува алгебарски дропки; - да се оспособи да ги извршува операциите со алгебарски дропки и да ги применува.	- организира дискусија; - дава инструкции; - демонстрира со објаснување; - организира работа во групи и во парови; - дава домашни задачи за индивидуална работа; - ги проверува и оценува задачите; - прави контролна задача/тест и ги оценува резултатите.	Математика: степени, цели рационални изрази (VII), дропки(V);

Слика 7: Наставни програма за I година, гимназиско образование.

За оваа наставна програма во употреба од 2002 – та година е учебникот [4].

Како што може да се види од наставната програма, но и од учебникот, алгебарските рационални изрази се изучуваат длабински со прецизно дефинирани поими. Прво се изучува оперирање на степени со показател природен број. Потоа, се работи со алгебарски цели рационални изрази, кои се дадени како веќе изучени, за потоа да се премине на алгебарски дропки и операции со нив.

Средното стручно образование е поделено на тригодишно и четиригодишно, а четиригодишното во зависност од насоките, математиката е застапена со 2 или со 3 наставни часа. Независно со колку часови е застапена, сепак оваа модуларна единица ги опфаќа истите наставни единици. Наставните програми се изработени од Бирото за развој на образованието 2013 година односно 2019 година и нивната содржина е дадени на слика 8.

Тема 2: ПОЛИНОМИ (18 часа)			
Цели	**Содржини**	**Поими**	**Активности и методи**
Ученикот/ученичката: - да ја искажува дефиницијата за степен и дава примери на степен со основа рационален број и показател цел број; - да множи, дели степени со еднакви основи или степени со исти степенови показатели и степенува степени; - да ја искажува дефиницијата за моном, собира, одзема, множи, дели, степенува мономи и одредува степен на моном; - да ја искажува дефиницијата за бином, трином и полином; - да собира, одзема полиноми и одредува степен на полином;	- Степени со основа Рационален број и показател цел број и операции со нив - Мономи и операции - Полиноми и операции - Разложување на полиноми	- Степен - Степенов показател - Основа на степен - Моном - Степен на моном - Бином - Трином - Полином - Степен на полином - Формули за скратено множење - Разложување на	Да се користат примери со повеќе операции со степени, со цел согледување на најекономичниот начин за пресметување на некој броен израз. Да се решаваат задачи за совладување на идеите и постапките за идентични трансформации на полиноми. *Да се користат разновидни примери при изучувањето на оваа тема (на пример, коефициентите на полиномите да бидат дропки, конечни децимални броеви, па и ирационални броеви, а не само цели броеви).*
- да множи и дели полином со моном и да множи полином со полином; - да ги искажува формулите за скратено множење-разлика од квадрати и бином на квадрат и да ги користи во задачи; - да разложува полиноми на множители со извлекување на заеднички множител пред заграда, со групирање и со примена на формулите за разлика од квадрати и бином на квадрат.	на множители	полином - Множител	*Да се решаваат задачи во кои се користат различни постапки за разложување на полиноми.* *Вежби и активности за оспособување на учениците во примена на формулите за скратено множење во двете насоки.* *При реализацијата на темата, наставникот треба да комбинира различни методи на активна настава.*

а) Наставен план за I година тригодишно средно стручно образование

	Модуларна единица 3: АЛГЕБАРСКИ РАЦИОНАЛНИ ИЗРАЗИ (22 часа)			
Ред. број	**Резултати од учење**	**Содржини и поими**	**Активности и методи**	**Критериуми на оценување***
1	**Ученикот/ученичката ќе биде способен/а да:** - множи и дели степени со исти основи или исти степенови показатели, да знае да степенува степен.	• Поим за степен со основа реален број и степенов показател цел број. • Множење степени со исти основи или исти степенови показатели • Делење степени со исти основи или исти степенови показатели • Степенување степен • Природен број во обликот $\overline{a_n a_{n-1} \ldots a_1 a_0} = 10^n a_n + 10^{n-1} a_{n-1} + \ldots + 10 a_1 + a_0$	**Активности:** • Наставникот го воведува поимот степен со показател цел број и операции со степени. Од учениците бара да ги извршуваат операциите со степени (истите ги проверува индивидуално). • Низ групна работа учениците вршат трансформација на изрази кои содржат операции со степени, а потоа вршат споредување на направените трансформации Методи: дискусија, дијалог, демонстрација, учење преку откривање, решавање проблеми.	**Ученикот/ученичката може да:** 1.1: Дефинира степен со основа природен број. Претвора степени со основа и показател природен број, во производ. 1.2: Ги извршува сите операции со степени со основа и показател цел број. 1.3: Ги извршува сите операции со степени. Го користи обликот на природен број $\overline{a_n a_{n-1} \ldots a_1 a_0} = 10^n a_n + 10^{n-1} a_{n-1} + \ldots + 10 a_1 + a_0$ 1.4: Решава посложени задачи со степени.
2	- дефинира и препознава моном, да определува коефициент и главна вредност во моном, да знае да дефинира и препознава слични мономи, да врши операции со мономи (собирање, одземање, множење, делење и степенување);	• Поим за моном • Слични мономи, собирање и одземање • Множење и делење мономи • Степенување мономи **Поими:** Моном, слични мономи.	**Активности:** • Наставникот го воведува поимот за моном и операциите со мономи. Од учениците бара да ги извршуваат операциите со мономи (истите ги проверува индивидуално). • Со помош на техниките за активна настава(вртелешка, ЗСНУ и сл.) учениците решаваат посложени задачи од операции со мономи, и се оспособуваат за самопроверување на стекнатите знаења. **Методи:** дискусија, дијалог, демонстрација, учење преку откривање, решавање проблеми	2.1: Дефинира и препознава моном, главна вредност и степен на моном. 2.2: Ги извршува операциите собирање и одземање на мономи. 2.3: Ги извршува сите операции со мономи. 2.4: Решава посложени задачи од мономи.
3	- дефинира полином, да знае да собира и одзема полиноми, да знае да множи полином со моном и полином со полином, да ги применува формулите за скратено множење (бином на квадрат, бином на куб, разлика од квадрати, разлика и збир од кубови), да знае да дели полином со моном и полином со полином;	• Поим за полином • Собирање и одземање полиноми • Множење полином со моном, множење полином со полином и формули за скратено множење (бином на квадрат, бином на куб, разлика од квадрати, разлика и збир од кубови) • Делење полином со моном, делење полином со полином и формули за скратено делење (бином на квадрат, бином на куб, разлика од квадрати, разлика и збир од кубови) **Поими:** Бином, трином, полином, Формули за скратено множење, делење на полином со полином, формули за скратено делење.	• Наставникот ги дефинира операциите со полиноми и бара од учениците да ги извршуваат (проверува индивидуално). • Преку групна работа, учениците ги изведуваат формулите за скратено множење. • Наставникот демонстрира примена на формулите за скратено делење, а потоа слична активност бара и од учениците (кои работат во групи). **Методи:** дискусија, дијалог, демонстрација, учење преку откривање, решавање проблеми	3.1: Дефинира полином. Го одредува степенот на полином. Ги искажува формулите за скратено множење. 3.2: Собира и одзема полиноми. Множи полином со моном и полином со полином. Ги применува формулите за скратено множење (бином на квадрат и разлика од квадрати) во конкретни задачи. 3.3: Ги извршува сите операции со полиноми. Ги користи формулите за скратено множење. 3.4: Ги користи формулите за скратено множење во посложени задачи.
4	- разложува полином на множители со извлекување на заеднички множител пред заграда, со помош на групирање или со примена на формули за скратено множење;	• Разложување полином на множители со извлекување на заеднички множител пред заграда • Разложување полином на множители со помош на групирање • Разложување полином со примена на формулите за скратено множење и делење • НЗД и НЗС на полиноми Поими: НЗД и НЗС на полиноми	• Наставникот ја објаснува постапката за разложување на полиноми со извлекување заеднички множител пред заграда преку одредување НЗД на полиноми. • Учениците разложуваат полиноми на множители со групирање (работат во групи) • Наставникот демонстрира разложување на множители на посложени полиноми на различни начини, а потоа бара од учениците да изведуваат слични активност. Преку	4.1: Разложува полиноми со извлекување на заеднички моном пред заграда. 4.2: Разложува полиноми со извлекување на заеднички бином пред заграда 4.3: Разложува полиноми со групирање и со формулите за скратено множење и одредува НЗД и НЗС на полиноми 4.4: Разложува на множители посложени полиноми.

			техниките на активна настава учениците разменуваат идеи и соработуваат. **Методи:** дискусија, дијалог, демонстрација, учење преку откривање, решавање проблеми	
5	- дефинира алгебарска дропка, да собира, одзема, множи и дели алгебарски дропки.	• Поим за алгебарска дропка и дефинициона област • Собирање, одземање, множење и делење на алгебарски дропки **Поими:** Алгебарска дропка.	• Наставникот го дефинира поимот за алгебарска дропка и ги воведува операциите со алгебарски дропки. Учениците ги извршуваат операциите со алгебарски дропки, а наставниот ги проверува индивидуално. • Низ групна работа учениците развиваат дискусија за областа на дефинираност на алгебарските дропки. • Наставникот демонстрира вршење на операции со алгебарски дропки, при претходно одредување на областа на дефинираност и НЗС на именителите, а потоа слична активност им задава на учениицте. Низ групна работа	5.1: Дефинира алгебраска дропка. 5.2: Ги извршува операциите со алгебарски дропки со исти именители. 5.3: Ги извршува сите операции со алгебарски дропки. 5.4: Решава пословени задачи од операции со алгебраски дропки и одредува област на дефинираност на алгебарските дропки
			учениците се оспособуваат за самопроверување и самооценување на знаењата. Методи: дискусија, дијалог, демонстрација, учење преку откривање, решавање проблеми	

б) Наставен план за I година четиригодишно средно стручно образование

Слика 8. Наставни планови за I година средно образование

Стариот учебник, кој се користеше во средно стручно образование е [5].

Според наставната содржина ја имаме истата констатација како и за гимназиското образование, дадена погоре.

4. Заклучок

Доволно е да се направи една споредба со наставниот материјал изучен во 7мо одделение по старата програма ([6], [7], [8]), каде се забележува дека алгебарските цели рационални изрази темелно и прецизно се изучени. И преминот во средно образование е многу поедноставен, бидејќи учениците во I година повторуваат за алгебарски цели рационални изрази и се надоврзуваат на изучување на алгебарски дропки.

Денес, она што се изучува по Кембриџ програмата е многу елементарно, непрецизно и конфузно. Оваа програма има многу недостатоци конкретно за изучување на алгебарски рационални изрази во однос на она што е потребно да се добие како знаење за средно образование. Во средно образование, денес професорот нема основа на која би ги вовел алгебарските рационални изрази, како што беше по старата програма во основното образование. За нив, ученикот во средно почнува од самата основа, без никакви предзнаења. Ова за ученикот значи тешкотии во совладувањето на материјалот и демотивација за предметот математика.

Затоа, би било одлично доколку програмата во основно образование претрпи одредени промени во однос на изучување на оваа проблематика. Добро би било во некоја идна програма за алгебарските рационални изрази да се задржи концептот од старата програма со нејзино збогатување со примена, која пак е добро обработена во Кембриџ програмата.

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STEM APPROACH IN TEACHING MATHEMATICS

Marija Miteva, Limonka Koceva Lazarova, Biljana Zlatanovska, Natasa Stojkovik

Abstract. Mathematics appears as essential subject in many academic studies. But students usually find math as difficult, and moreover consider its content as useless, thus avoid such studies. This is a reason for changing traditional approach in teaching mathematics and develop new one, which will emphasize problem-based learning. We will consider in this paper STEM approach in teaching mathematics and results of its implementation.

1. INTRODUCTION

The concept of STEM education is a concept which integrates Science, Technology, Engineering and Mathematics in the process of everyday teaching and learning. Education and teaching process should not only provide students with pure knowledge, but it should answer the question why do students need that knowledge and how to apply it in the future. During the process of learning, especially in the classes, the attention of the students at each age is usually at the highest level when they are considering a real life problem and are trying to solve it. This approach, i.e. including problem-based situations, characterizes STEM education. Thus, STEM approach means introducing certain concepts and their relations in the process of teaching, as something necessary for solving different problems that student will face up in the future, in their lives or carriers.

Students usually find mathematics as difficult subject in their education and are afraid of it, which is a reason for avoiding studying engineering, technology and anything else where mathematics appears as essential. This situation can be changed, as well as students' attitude toward mathematics, if STEM approach is implemented and real life problems are introduced in the classes. STEM approach can seriously contribute in improving the perception about mathematics among students. Moreover, project-based and problem-based learning and collaboration while solving certain problems can increase communication skills, creativity and critical thinking of students. It is important a STEM approach to be implemented in the earlier education, but is very important to become everyday practice in the higher education. About importance of STEM education, one can read [1-4].

In order to implement STEM approach in the process of education, teachers need an appropriate, well-developed curriculum for their lessons. Developing STEM curriculum progressively became research interest to many teachers nowadays. Some examples for real life problems can be found in [5]. In the frame of the Erasmus+ project *Mathematics of the Future: Understanding and Application of Mathematics with the help of Technology, FutureMath* [6], teachers who participate have developed STEM curriculum for different math topics, together with variety of teaching materials and examples how to use different digital tools, in order to easier implement STEM approach on math classes. In this new approach teachers usually start lecture with real life problems. We will present some results of implementing such approach in the classes.

2. RESULTS OF APPLYING STEM APPROACH ON MATH LESSONS

Implementing STEM approach on math lessons, the students had an opportunity to face up with the new trends of teaching and almost all of the advantages of the STEM approach were achieved. By STEM as an educational approach, the students has the best introduction to each lecture via using a real problem. It is more interesting for students to consider and try to solve real life situations, than listening math lectures. The real problem motivates them to think about similar real problems, which are already known to them, without having in mind their connection with mathematics. Considering real life problems which need math knowledge to be solved, students realize that they have to achieve appropriate math knowledge first, in order to successfully solve the problem. Thus, learning math formulas and expressions become necessary, and students are not wondering anymore why they have to learn it. Presenting the new material and using computer applications and mathematical software, made the lesson more interesting and fun for the students then the previous methods of lecturing. Implementing all of this encourages the students to collaborate and discuss one with another, but also with the teacher via creative questions. These questions are related to requests for clarification of introduced new terms and curiosity to learn more, which are the basis for deeper knowledge. The lessons generally passed quickly, creatively, with fun, and with the mutual satisfaction of both the students and the teacher. The students passed the pilot lecture as the quality time spent because the new material was already introduced. The biggest result is satisfied students. The smiling and satisfied students, for the teacher mean successful organized lesson.

Gaining practical knowledge during the studies is also very important in the educational process. The students move through the educational process and they acquire knowledge from many different (but connected) areas. Later, this knowledge they should apply in real situations in life. The initial places where students can apply theoretical knowledge in order to solve practical problems close to the real situations of their lives are the educational institutions. Educational institutions should teach the students how to do it. Only in this way, the students will be ready and successful for real life. Therefore, schools and universities must look for new ways and methodologies for practicing knowledge as a part of the process of teaching. Practical knowledge in teaching refers to students' knowledge of classroom situations and solving the practical problems they face in carrying out actions. The connection between Science, Technology, Engineering, and Mathematics guarantees the acquisition of practical knowledge in teaching. That is the STEM methodology. Having in mind that the new teaching methodology is based on solving practical problems, there is no doubt that students will gain practical knowledge simultaneously with the theoretical one. The new teaching methodology for practical knowledge per STEM means: connection of the real situations with the need for introducing new mathematical terms, which are provided in educational teaching programs; students should feel the need from introducing new mathematical knowledge; obtaining new knowledge with applications for solving a practical problem that is close to real problems of life; obtaining new knowledge which can be implemented in real situations, etc. The biggest benefit of gained practical knowledge is that the knowledge can be implemented in certain real-life situations. Applying different digital tools and software while solving certain problem-based situation students also gain practical knowledge.

We have organized several math lessons, with small groups of students (15-30), during which STEM approach was introduced. In order to receive feedback of the new methodology implemented, we have prepared short tests for checking the students' knowledge achieved on the lesson, and also a questionnaire as a survey about students' viewpoint for the new approach. It is interesting that almost all the students have answered the questions in the survey, although the questionnaire was not obligatory. In order to achieve objective answers, e-mail addresses of the students who answered the questions were not saved. Separate questionnaire was prepared for each piloting lesson, and the results are visible for each lesson separately. We have 107 answers to the 6 questionnaires with the same questions, but for the different lecture. There were 10 questions with the next offered answers*: I agree, I agree partially, I don't agree, I cannot say* and the students can choose only one of the offered answers. We will present in the next figures each question together with the results of the answers.

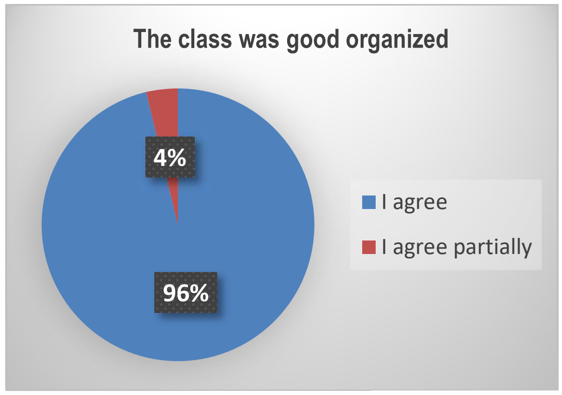

Figure 1: Answers to the first question

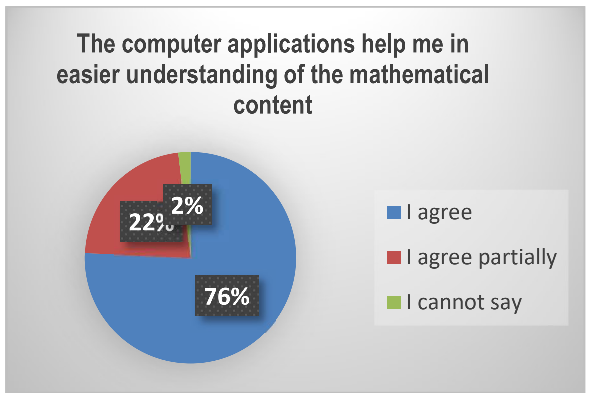

Figure 2: Answers to the second question

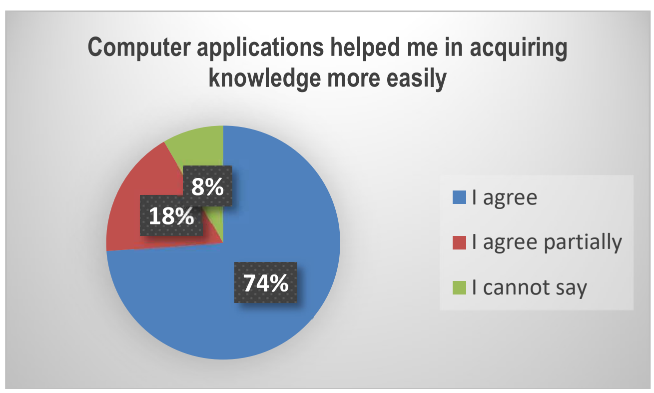

Figure 3: Answers to the third question

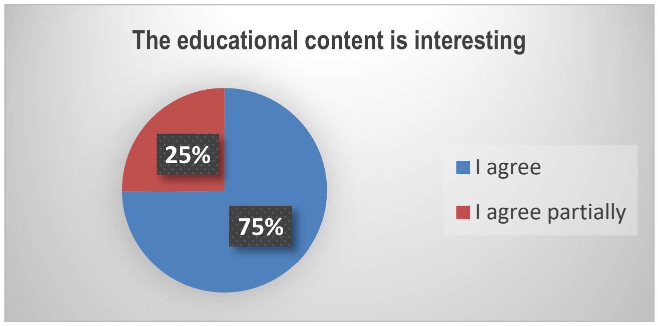

Figure 4: Answers to the fourth question

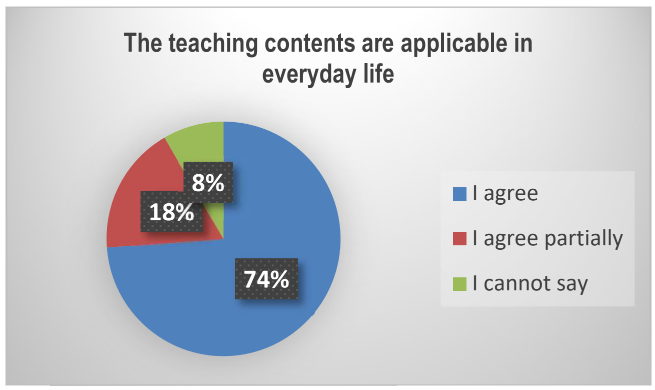

Figure 5: Answers to the fifth question

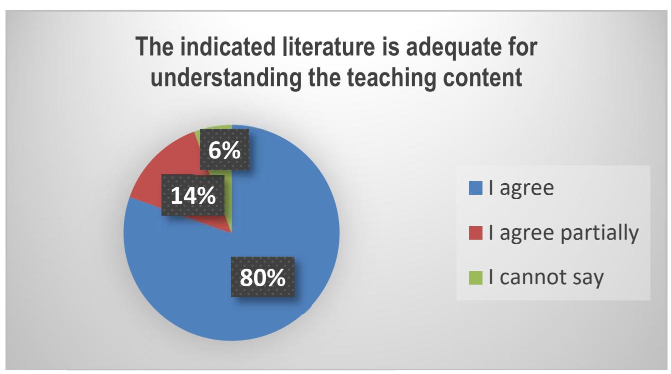

Figure 6: Answers to the sixth question

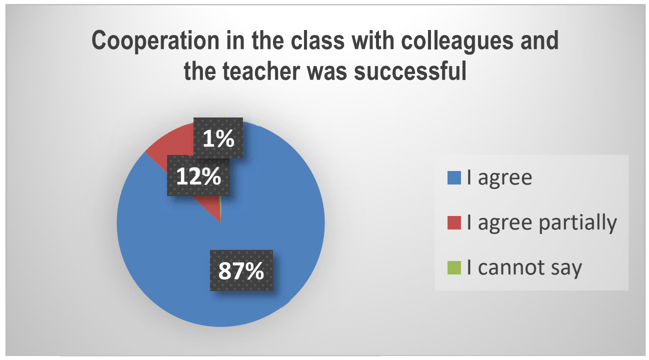

Figure 7: Answers to the seventh question

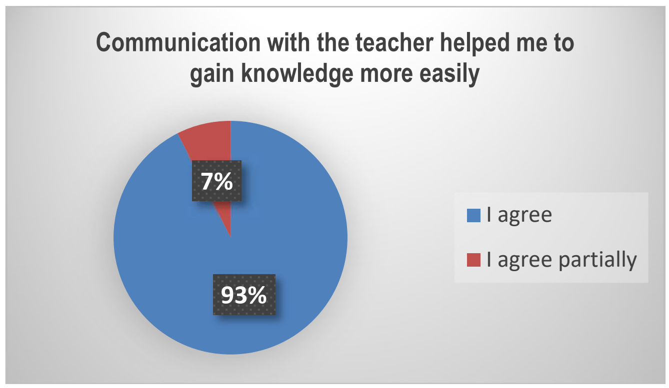

Figure 8: Answers to the eighth question

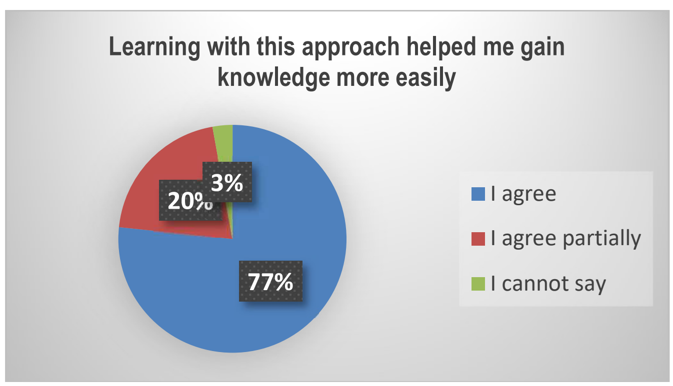

Figure 9: Answers to the ninth question

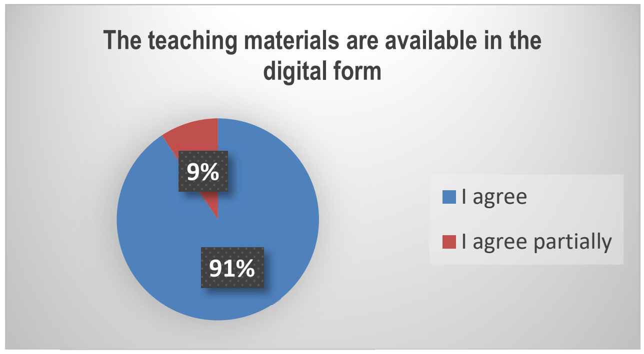

Figure 10: Answers to the tenth question

We believe that the answers are objective, because as we have said above, no email address was necessary to access the questionnaire, just the link of it was enough.

We have also prepared short test for checking students' knowledge immediately after the lesson. Unfortunately, we cannot boast with the results of the tests in this first phase of the implementation of the STEM approach, because we have not noticed big difference between these results and the results on the testing students' had before. But we are very pleased with the other results that we have notices with the implementation of the new method, because we managed to keep students' attention during the whole lesson and to burn the curiosity among them. They have very carefully joined the activities in the class; they were not boring and curiously have expected the solution of the problem set at the beginning of the lesson.

3. BENEFITS AND WEAK SIDES OF THE APPROACH

Both students and teachers have realized benefits from the implementation of the new methodology on the piloting lessons. Mathematics is usually understood by students as a science for itself, without any connection with reality. Using old methods during the math lessons, where math concepts were introduced with classical lectures, only with black board and chalk, full with mathematical theory, formulas and expressions, students are usually passive listeners. They are usually not involved in any activity on the classes, so their attention is decreasing continuously till the end of the lessons and they are getting boring. Thus, other method which will involve students in different activities during the lecture, as problem-solving situation, collaboration with others, etc. seems to be necessary to practice on everyday lessons. By connecting the 4 components of Science, Technology, Engineering and Mathematics, STEM has interesting access for presenting and introducing new material for educational classes on each educational level. Implementing STEM approach on lessons and using different digital tools in order to easily achieve knowledge is big step towards to make lectures interesting, to increase students' attention on the lessons and contribute in reaching positive attitude among students toward mathematics. Students will become active problem solvers, they will develop creative thinking, and the most important of all they will change the perception for the process of math education, realizing that math knowledge is important for their carrier and everyday life. Therefore, STEM finds wide application in improving the educational process and general satisfaction of students and teachers.

Changing the students' perception about mathematics, as a result of the new approach in teaching, will on a long time make students to not be afraid anymore of mathematics and make students not to avoid study programs where mathematics appears as essential. Furthermore, implementing new teaching methodology in the education can attract students to study STEM fields, which will be big achievement for the universities.

There are many other findings emerging from the implementation of the new teaching methodology. By STEM methodology, the students think more broadly and more deeply than usual. The STEM methodology determines the way for the students to research new and creative ways to solve real-world problems and connect themselves to the fields that interest them. Using STEM methodology contributes in producing students who think critically with the integration of knowledge and skills from multiple areas. The students get creative ideas to apply the acquired knowledge in solving real problems. Later, these students will get up innovators, leaders, and educators of society. They will be creative people who will lead society forward in development and progress.

The newly developed teaching methodology based on the STEM approach can essentially change the way of teaching science, technology, engineering and mathematics, and more important of it, can essentially improve students' results of studying and their perception about education, generally. Thus, this new approach in teaching mathematics has to be preferred for the teachers to use.

According to all above described that characterizes the newly developed teaching methodology, based on the STEM approach, no doubly there are many strengths of it. This new teaching methodology differs a lot from the classical methods of teaching. One of its major strengths is that the students are in the centre of the attention. They are no more passive listeners, but they are actively involved in the activities during the lessons. This new methodology and approach based on problem solving is strengthening students' creativity and problem-solving skills, as well as skills for collaboration and team work. The process of teaching with this new methodology has been changed from its roots. Students have new challenges on each new lesson, instead of being afraid from the material following. Teachers have also challenges to organize interesting lessons.

Although too many advantages, there are even some weaknesses of this new approach in teaching mathematics. Not each mathematical educational class can be organized via practical problems from real life's situations. In mathematics there exist terms with abstract nature which cannot be connected to practical problems. The good organization of a class with this methodology requires full dedication to the teacher and a lot of spent time.

Also, very often students do not have available appropriate digital tools and software which can help them in solving certain problems. Even the software and digital tools are available, very often they do not have an experience in working with it. This can be considered as a weakness, but as an advantage at the same time, because students simultaneously with the new lecture can practice digital tools and similar resources.

4. CONCLUSIONS

According to all above, the concept of STEM education and STEM principles can easily be implemented into STEM studies if teacher gives to the students a problem situation at the beginning of each lecture, in order students to realize that certain theoretical knowledge is necessary for solving such problem. In the continuation of the lesson, teacher can introduce theory of the subject, but students will realize it as something that they have to achieve in order to solve problem, not something useless and boring which is part of the subject's curriculum and that they have to memorize. In order to reach time on the lessons, teacher can give as homework to the students a problem situation related to the material which will be introduced on the next lesson. However, starting the lecture with a problem to be solved is the essential thing that makes this new approach, and changes the perception about mathematics and education at all.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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APPLICATION OF SEMIVARIOGRAMS AND KRIGING IN GEOTECHNICAL MODELLING

Marija Maneva, Igor Peshevski, Milorad Jovanovski, Zoran Misajleski, Daniel Velinov

Abstract. Semivariograms and kriging are very important tools for geotechnical modelling. The application of the recent results on these techniques, aided with computer software, yields new insights in different areas of science and engineering. In this paper, we use spherical semivariograms and ordinary kriging, for modeling of the specific geotechnical parameters in the coal deposit Brod-Gneotino. The obtained values for distribution of these parameters are quite convenient and effective for realistic slope analysis, exploration and long term planning of surface mines development. Although, in this paper we have case study, to proposed technique can be applied in general, whether it is an active open pit mine or design for a new one.

1. INTRODUCTION

The coal as an energetic mineral resource is from a fundamental importance in Macedonia, since it is the dominant source for electricity production. In Macedonia, coal deposits can be found in several so called sediment basins, characterized with tertiary geological age. Nowadays, only the deposits located in Pelagonical sediment basin are used for coal exploitation. More specifically, the coal is exploited at two surface mines in the Pelagonia basin: The so called open pits of "Podinska serija" in REK Bitola and the relatively recently activated open pit Brod-Gneotino.

Have in mind its importance, the coal deposit Brod-Gneotino has been subject of many geological, hydrogeological and geomechanical investigations and explorations, so there is a plenty data for detailed definition of the geology, hydrogeology and geomechanics of this deposit.

The application of the modern mathematical trends and computer software facilitates the process of interpretation of obtained data from the performed investigations, by creating of different types of geological and geotechnical datasets and models. Using the tools of spatial analysis, developed in the second half of the last century, aided by computer software, fast data processing can be performed. In comparison, the manual processing is quite complicated and time consuming.

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Key words and phrases. Spatial analysis, geostatistics, spherical semivariogram, kriging, volume weight, direct shear, cohesion, coal deposit.

The computer software Leapfrog^{TM} (more precisely its module Leapfrog Geo) is used for this study devoted on geotechnical modeling of the coal deposit Brod-Gneotino. With this geological modeling, 3D representation of the spatial structures of the deposit is performed. Likewise, with the geotechnical modeling, the measured data of geotechnical parameters is presented in 3D, and related to the defined geological structures.

3D modeling and spatial analysis of the collected data is being used in the mining for a quite long time. Before using modern computer into mathematical modeling, generating the 3D models, was made by using two dimensional specialized charts, cross-sections and diagrams. In the last three decades, the number of studies devoted on 3D modeling has rapidly increased, as a consequence of using specialized computer software. In this software, the modern 3D representations (models) are created with high resolution, and by using interpolating algorithms, [1].

The expansion and advancement of the 3D modeling in geology include integration of large amount of geological data, as well as additional available lithological, structural, geochemical, geophysical, geotechnical and other type of data. The constructed 3D representations can be used as interactive tools for exploring of mineral deposits, [6]. Three dimensional geological modeling (3DGM) is helpful for the geologists in quantitative analysis of three dimensional spatial structures, defining the spatial relations between geological objects. 3DGM technology gives us a technical support for drawing information and conclusions for the geology, 3D modeling and quantitative calculation of mineral resources in the deposits, [12].

Geotechnical modeling is fundamental in designing the mines with open pit or underground excavation. Fully defined and representative geotechnical model will provide information about the engineering and geological characteristics of the rock structure, defining its behavior in the domains of the mining. The model is composed of individual rock structures, showing similar geotechnical properties. The definition of this individual domains and its comprehensive approach is a keynote for the process of exploitation and related hazards, through facilitating the optimal solutions for the projects of the mines, [2], [8].

The development of the software improves the modeling, allowing the obtained models to be built in realistic 3D environment, by using implicit and semi implicit ways of modeling. The ways of modeling depend on the choice of data, application of certain trends and defining the geology, and additionally the obtained surfaces can be manually modified by the software's user. These options in the software are used for creating more complicated models, with renewing ability, by adding new data or new geological rules, and presenting more interpretations for large amount of data. The used mathematical models and software allow better understanding of the complexity of the deposits. We must be careful about the following things: the used data for modeling should be appropriate, representative and their quality should be assessed before we start with the mathematical modeling aided by computer software, [8].

The geotechnical model contributing to design of the mines should be based on comprehensive approach in geology, structures, alterations and erosions, and how they influence at engineering and geological characteristics of rock structures. The geotechnical engineers must have excellent understanding for the hazards and constraints of each separate model, as well as their influence of creating geotechnical domains, [8].

The mathematical modeling used here will be based on the theory from spatial analysis, or more precisely on semivariograms and ordinary kriging. Once a theoretical semivariogram is fixed, we are ready for spatial prediction. For spatial prediction, the geostatistics uses kriging. The term kriging is given in honor of the South African mining engineer, Daniel Gerhardus Krige. The question of expressing in a function the structure of spatial dependence or correlation, known in the geostatistics literature as structural analysis, is a key issue in the subsequent process of optimal prediction (kriging), as the success of the kriging methods is based on the functions yielding information about the spatial dependence detected. The functions referred to above are covariance functions (also called covariograms) and semivariograms, but they must meet a series of requisites, for example stationary and intrinsic hypothesis. As we only have the observed realization, in practice, the semivariograms derived from it may not satisfy such requisites. For this reason, one of the theoretical models (also called the valid models) that do comply must be fitted to it. Kriging aims to predict the value of a regularized function, $Z(\mathbf{s})$, at one or more non-observed points or blocks from a collection of data observed at n points (or blocks in the case of block prediction) of a domain D, and provides the best linear unbiased predictor (BLUP) of the regionalized variable under study at such non-observed points or blocks. Thus, the predictor support can be a point or a block, [9].

The paper is organized as follows. After the introduction, in Section 2, the theoretical foundations of this study is given, i.e. details on spherical semivariograms and ordinary kriging is presented. Section 3 is central part in this paper and is devoted on application of the methods of spatial analysis in geostatistical modeling of parameters important for slope stability and designing of surface mines. Conclusions of the paper are based on the previously obtained results of the case study Brod-Gneotino. The most important, the presented methods, approach and conclusions can be applied in general case of planning and designing of open pit and underground mining operations.

2. SEMIVARIOGRAMS AND KRIGING

In spatial analysis, the semivariogram is given by the following formula

$$\gamma(s_i - s_j) = \frac{1}{2} V(Z(s_i) - Z(s_j)),$$

for all $s_i, s_j \in D$, where D is continuous domain under study, and by $V(\cdot)$ denotes the variance of $\cdot$.

Under the second-order stationary hypothesis:

The random function $\{Z(s) : s \in D\}$ is said to be second-order stationary, if it has finite second-order moments and following hold:

a) The mathematical expectation exists and is constant, so it does not depend on the location s, $E(Z(s)) = \mu(s) = \mu$,

b) The covariance exists for every pair $Z(s)$ and $Z(s+h)$ depends only on the vector h joining the locations s and $s+h$, but not specifically on them, i.e. it holds

$$C(Z(s), Z(s+h)) = C(h), \text{ for all } s \in D \text{ and vectors } h;$$

and the intrinsic hypothesis (with no drift):

The random function $\{Z(s) : s \in D\}$ is said to be intrinsic if, for any given vector h of translation, the first-order increments $Z(s+h) - Z(s)$ are second-order stationary, i.e. $E(Z(s+h) - Z(s)) = \mu(s)$, where $\mu(s)$, the drift, is linear in h, and

$$C((Z(s+h) - Z(s)), (Z(s+h+h') - Z(s+h'))) = C(h, h'),$$

which is equivalent to

$$\frac{1}{2} V(Z(s+h) - Z(s)) = \gamma(h),$$

which is only a function of h,

it can be written as:

$$\gamma(h) = \frac{1}{2} V(Z(s+h) - Z(s)) = \frac{1}{2} E\left((Z(s+h) - Z(s))^2\right),$$

showing how the dissimilarity between $Z(s+h)$ and $Z(s)$ develops with distance h.

The semivariogram is the instrument used par excellence to describe the spatial dependence in the regionalized variable. The reason is that it covers a broader spectrum of regionalized variables than the covariance function, which is confined to second-order stationary random functions. This spectrum includes intrinsically stationary random functions, in which the covariance cannot be defined. In the second-order stationary framework, both the semivariogram and the covariogram are theoretically equivalent, [9]. Indeed,

$$C(h) = C(0) - \gamma(h).$$

In practice, the mean is unknown and it must be estimated from the data, which introduces a bias. The semivariogram of an intrinsically stationary random function depends on the vector h that connecting the locations (both on the distance between s and $s+h$, and also on the direction, but not on the locations themselves). Hence, in general terms, it is anisotropic. In the case when semivariogram depends only on distance, it is called isotropic.

The semivariogram is a non-decreasing monotone function, so that the variability of the first increments of the random function increases with distance. The semivariograms that correspond to second-order stationary random functions have a typical behavior at intermediate and large distances: They rise from the origin and increase monotonically with distance until approaching its limiting value, the a priori variance of the random function, $C(0)$, either exactly or asymptotically, [9].

The slope of the semivariogram indicates the change in the dissimilarity of the values of the regionalized variable with distance. The above-mentioned limiting value of the semivariogram is called the variance sill, or simply the sill (m), and the distance at which the sill is reached is termed the range, which defines the threshold of spatial dependence, i.e. the zone of influence of the random function. This means that, the range is the distance beyond which the values of the regionalized variable have no spatial dependence.

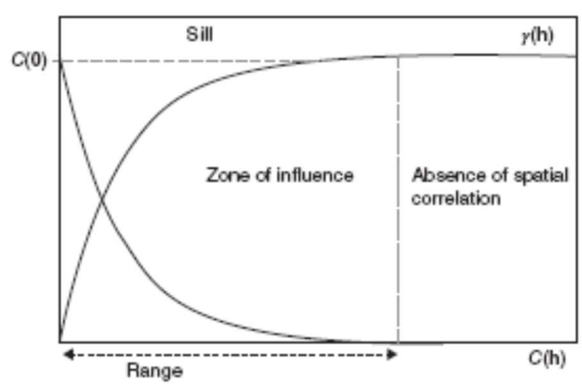

Figure 1: Example of Bounded semivariogram and its covariogram counterpart

Figure 1 illustrates that the larger the range, the larger is the zone of influence of the random function. In the case when the sill is reached asymptotically there is not a well-marked range, but a practical range (the distance at which the semivariogram takes the value $0,95m$). This practical range is closely related to the scale parameter, a, of the semivariogram (if the sill is reached exactly, i.e. in the case of a flat sill, a coincides with the range). Semivariograms that do not reach a sill are quite prevalent, and in particular this fact can occur when dealing with non-stationary random functions, in example when we have existence of a drift, intrinsically stationary random functions, or even second-order stationary random functions, in the case when the range exceeds the largest distance for which the semivariogram can be estimated.

Here we consider semivariogram that is associated with the second-order stationary hypothesis. Thus, it has a covariogram counterpart. This type of semivariograms also received the name of transition models because the distance at which the sill is reached represents the transition from the state of existence of spatial correlation to the state of absence of such spatial correlation, [9].

2.1. The spherical model

This model is valid on $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$. It is defined as

$$\gamma(|h|)=\begin{cases} m\left(1,5\dfrac{|h|}{a}-0,5\left(\dfrac{|h|}{a}\right)^3\right), & |h|\le a \\ m, & |h|>a \end{cases},$$

where $m=C(0)$ is the value of the semivariogram when it reaches the sill, and a is the range. The spherical semivariogram have a linear behavior near the origin, which indicates continuity, but a certain degree of irregularity in the random function. Also, it can be easily checked,

$$\frac{\partial\gamma(|h|)}{\partial h}=m\left(\frac{1,5}{a}-\frac{|h|^2}{a^3}\right),$$

hence the slope of the semivariogram at the origin is $1,5\dfrac{m}{a}$. The tangent at the origin intersects the sill at $|h|=\dfrac{2}{3}a$. Considering behavior at large distances, we have that it reaches the sill at $|h|=a$. This well-defined range, along with its simple polynomial expression and its validity on $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$, are some of the reasons for the wide use of the spherical semivariogram in practical applications. But the main reason is that an almost linear behavior up to a certain distance (the range) and then stabilization matches a large variety of observed regionalizations, [9].

Next, we will give the kriging equations, which give us a prediction

of the value of the random function $Z(s)$ at a non-observed point (or block) as a linear combination of the values of the random functions at the sampled points (or blocks) or at a set of them that are close to the prediction point. Clearly, we will also give the expression of the prediction error variance, or simply the kriging variance, which indicates how accurate the kriging prediction is. These equations are obtained by imposing on the predictor the classical conditions of unbiasedness and minimum variance, i.e. by imposing on the prediction error zero expectation and minimum variance (that is, minimizing the mean squared prediction error). It is nevertheless necessary to make the following clarification: the minimization of the mean-squared prediction error stems from the assumption that the semivariogram is known. Usually, this is not the case in many practical situations, since the kriging prediction is based on the empirical semivariogram (to which a theoretical semivariogram is fitted to). Moreover, it is difficult to measure the consequences of not using the true semivariogram. More precisely, in the case of point observation support, the point kriging predictor $Z^*(s_0)$ at the non-observed point s_0 is given by

$$Z^*(s_0) = \sum_{i=1}^{n} \lambda_i Z(s_i),$$

where $Z(s_i)$ are the values observed at the n points in the neighborhood of s_0, the prediction point, and λ_i are the kriging weights obtained by imposing on the prediction error the classical conditions above referred. In many occasions we are interested in block prediction. That is, our aim is to predict the average value of the random function being studied in a block V, given by $Z_V(s)$, which is assigned to the point $s \in V$

$$Z_V(s) = \frac{1}{|V|} \int_V Z(s')\, ds',$$

where $|V|$ is the area or the volume of V, s' sweeps throughout V and s is a point in V to which the average value of the block is assigned.

In such cases, when the observations are based on points, the predictor of the average value of the random function in V is given by

$$Z^*(V) = \sum_{i=1}^{n} \lambda_i Z(s_i).$$

In the case of a block observation support, that is, the data available are the average values in blocks v_i, the block predictor in V is

$$Z^*_V = \sum_{i=1}^{n} \lambda_i Z_{v_i}(s_i),$$

where $Z_{v_i}(s_i)$ to be the average value in the block v_i, which is assigned to the point $s_i \in v_i$

$$Z_{v_i}(s_i) = \frac{1}{|v_i|}\int_{v_i} Z(s')\,ds',$$

as s' sweeps all over v_i, [9].

Clearly, the quality of kriging predictions is built on the size of the sample and the quality of the data, but it also confides in:

- the location of the realizations (if they are uniformly distributed in the domain under study, there will be better coverage and more information about what happens in that domain than if realizations are grouped);
- the distance between the points (or blocks) observed and the point or block to be predicted (more trust should be placed in nearby realizations than in distant realizations);
- the spatial continuity of the random function being studied (it is easier to predict the value of a regular random function at a point or over a block than of a random function that fluctuates markedly).

At the end of this section, let us make a remark that the main advantage of kriging over other spatial interpolation techniques (inverse distance method, splines, polynomial regression, among others) is that not only does it take into account geometric characteristics and the number and organization of locations, but also considers the structure of the spatial correlation that is deduced from the information available through semivariogram structures, hence yielding more reliable predictions. Therefore, the weights that kriging is using are not calculated on the basis of an arbitrary rule that can be used in some cases but not others, but rather on the behavior of the function that represents the structure of spatial correlation. In this sense, it is a more flexible method than those mentioned above, [9]. Additionally,

- Kriging makes it possible to measure how accurate are the predictions using the prediction error variance (the kriging variance) and can yield a map of the standard deviation of the prediction errors;
- kriging variance does not depend on the actual realization of the random function, which acts as a probability shelter, meaning we can calculate it before learning the values of the variables at those points, providing we know the structure of the spatial dependence of that random function. This is extremely useful when designing networks of optimum observations, that is, for selecting locations to be observed that provide the least kriging variance;

- kriging is an exact interpolator, which means that at the points that make up part of the sample, the kriging prediction corresponds with the value observed, the kriging variance therefore being zero. In short, the observation support can be points or blocks. In the first case the prediction support can be also points or blocks; in the second case, it can be only blocks, [9].

3. SEMIVARIOGRAMS AND KRIGING IN MODELING

In this section, we are going to use spherical semivariograms, in order to obtain realistic 3D representation of the certain geotechnical parameters on sand and coal in the coal deposit Brod-Gneotino. With these mathematical tools and the module Leapfrog Geo from the software Leapfrog^{TM}, the collected data [3]-[5] (see also [7], [10]-[11]), is analyzed and the following quasi-homogeneous domains in the Brod-Gneotino deposit are obtained: quasi-homogeneous domains of sand's volume weight, quasi-homogeneous domains of internal friction of sand determined by the direct shear, quasi-homogeneous domains of cohesion of sand determined by direct shear, quasi-homogeneous domains of the angle of internal friction of coal determined with triaxial tests and quasi-homogeneous domains of the angle of internal friction of coal determined with triaxial tests.

In the sequel, we are going to give detailed survey of the upper mentioned quasi-homogeneous domains.

3.1. Quasi-homogeneous domains of sand's volume weight

The data used for mathematical modeling of quasi-homogeneous domains belong to geological age of Quaternary, Pliocene and Miocene age, hence the created quasi-homogeneous domains will correspond to the part of geological model with Quaternary, Pliocene and Miocene age, without the part belonging on trepel and coal layers.

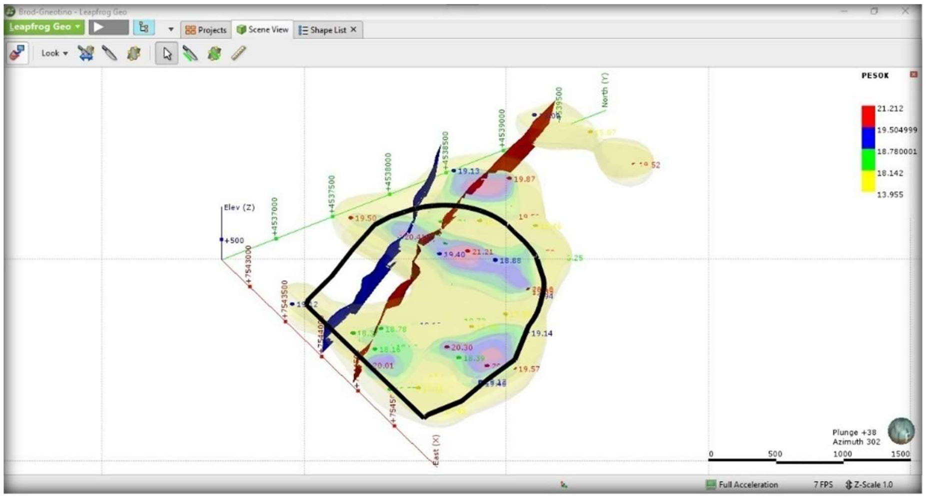

Figure 3: 3D representation of quasi-homogeneous domains of sand's volume weight

The obtained values for volume weight of the sand in the surface layer in Brod-Gneotino are divided on two parts:

- Gravelly sand (SP,SW) with ***γ*=20,10 [kN/m³]**
- Silty sand (SFc, SFs)) with ***γ*=18,74 [kN/m³].**

By the 3D representation of the quasi-homogeneous domains of the volume weight of the sand, we have the following:

- ✓ Data used for mathematical modeling of these domains are in the range from 13,95 to 21,21 [kN/m^3], with median **18,78 [kN/m^3]**,
- ✓ Domain with the highest values (19,50 – 21,21 [kN/m^3]) is located in the north part and in the part of east final slope of surface layer,
- ✓ The domains with lower values are more represented i.e. 13,95 – 18,14 [kN/m^3] and 18,14 – 18,78 [kN/m^3]),
- ✓ Since the obtained values belong in the intervals with lower values, we can consider the option, when analyzing the slope stability, to use lower values for volume weight of sand, compared to the obtained values.

3.2. Quasi-homogeneous domains of internal friction of sand determined by direct shear

Here, for mathematical modeling of quasi-homogeneous domains of the angle of internal friction of the sand, determined with direct shear, we use data belonging to geological layers with Pliocene and Miocene age, hence the created quasi-homogeneous domains will correspond to the part of geological model with Pliocene and Miocene age, without the part belonging on trepel and coal layers.

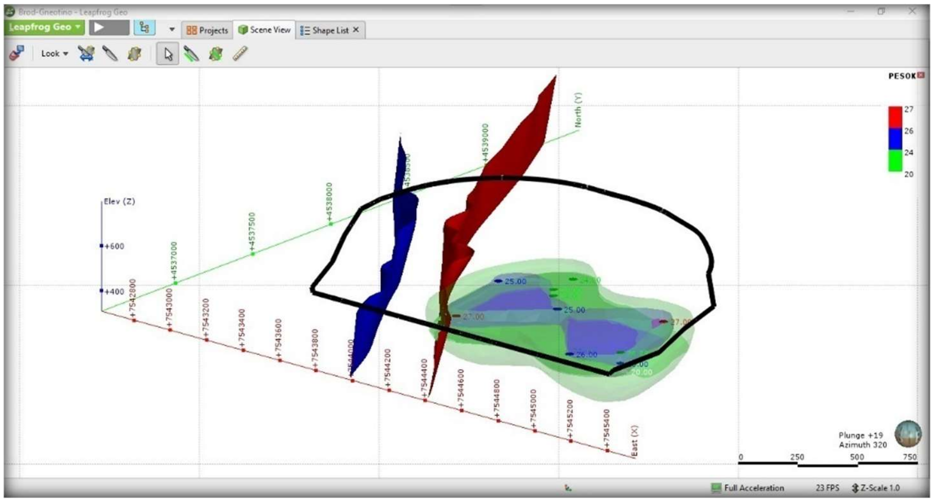

Figure 4: 3D representation of quasi-homogeneous domains of the angle of internal friction of the sand determined with direct shear

The obtained values for the angle of internal friction obtained with the direct shear of the sand in the surface layer in Brod-Gneotino are divided on two parts:

- Gravelly sand (SP,SW) with $\varphi = 28°$ and
- Silty sand (SFc, SFs)) with $\varphi = 17,65°$.

By the 3D representation of the quasi-homogeneous domains of the angle of internal friction obtained with direct shear of the sand, we have the following:

- ✓ Data used for mathematical modeling of these domains are in the range from $\varphi = 20°$ to $\varphi = 27°$, with median $\varphi = 24°$, located in the non-fault domain,
- ✓ Domain with the highest values ($26°$-$27°$) is almost in punctate structure,
- ✓ The domains with lower values are more represented i.e. $20° - 24°$ and $24° - 26°$, and they are located in south-east part of the border of the surface mine,
- ✓ It can be noticed that the obtained values for this geotechnical parameter are located outside of the domains of separated quasi-homogeneous

domains. Since the domains with the values $20° - 24°$, the mentioned values can be used for analysis of slope stability of surface mine.

3.3. Quasi-homogeneous domains of cohesion of sand determined by direct shear

The data used for the mathematical modeling of quasi-homogeneous domains of the cohesion of sand determined by direct shear, belonging to geological layers with Pliocene and Miocene age, hence the created quasi-homogeneous domains will correspond to the part of geological model with Pliocene and Miocene age, without the part belonging on trepel and coal layers.

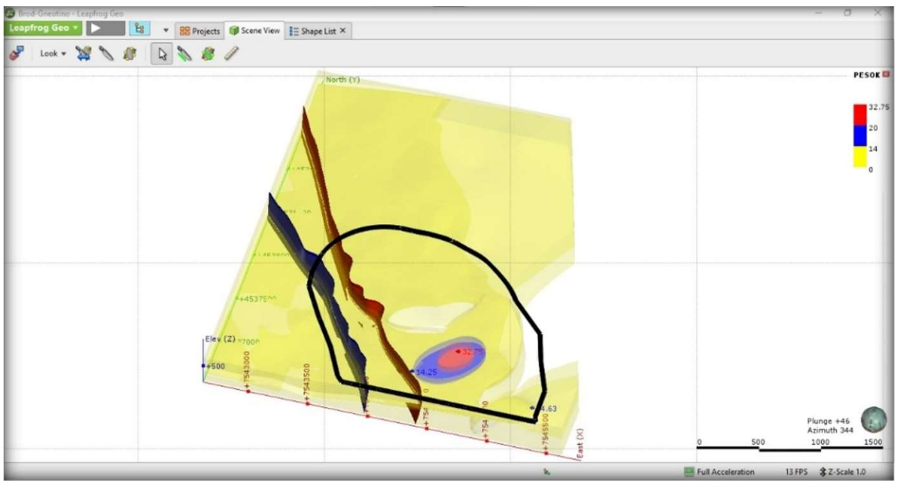

Figure 5: 3D representation of quasi-homogeneous domains of the cohesion of the sand determined with the direct shear

The obtained values for cohesion by direct shear of the sand in the surface mine in Brod-Gneotino are divided on two parts:

- Gravelly sand (SP,SW) with $c = 0,00[kN/m^3]$ and
- Silty sand (SFc, SFs)) with $c = 8,00[kN/m^3]$.

By the 3D representation of the quasi-homogeneous domains of the cohesion obtained with the direct shear of the sand, we derive the following conclusions:

✓ Data used for mathematical modeling of these domains are in the range from $c = 0,00[kN/m^3]$ to $c = 32,75[kN/m^3]$, with median $c = 0,00[kN/m^3]$, located in the non-fault domain,

- ✓ These quasi-homogeneous domains are formed by 13 numerical data, from whose only three of them are not zeros, giving the reason why modeling of the domains is not made like in the other geotechnical parts,
- ✓ The most representative domain with median value $c = 0,00[kN/m^3]$ is with correlation with the regular value of the cohesion of the sands,
- ✓ Non-zero values of the cohesion are $14,25$; $14,63$ and $32,75[kN/m^3]$. They are located in the medium part of the surface mine and they belong of the group of silty sands.
- ✓ The final slopes in the frame of the whole border of the surface mine are characterized with cohesion value $c = 0,00[kN/m^3]$.

3.4. Quasi-homogeneous domains of the angle of internal friction of coal determined with triaxial tests

The data used for modeling of the angle of the internal friction of the coal, determined by the triaxial test belong in the geological sections of the five coal layers with Miocene age, hence the created quasi-homogeneous domains will refer to these parts from the geological model.

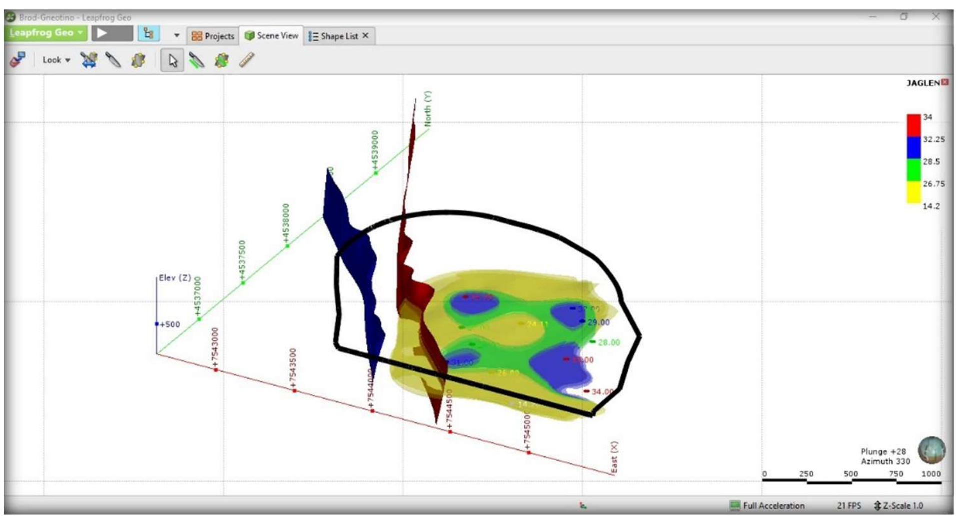

Figure 6: 3D representation of quasi-homogeneous domains of the angle of internal friction of the coal determined with the triaxial tests

The obtained value for the angle of the internal friction of the coal in the surface mine Brod-Gneotino is $\varphi = 24°$. From the 3D representation of the quasi-homogeneous domains of the angle of the internal friction of the coal, determined with triaxial tests, we can derive the following conclusions:

- ✓ Data used for creating of these domains are in the range from $\varphi = 14,20°$ to $\varphi = 34,00°$, with median $\varphi = 28,50°$ and are located in the non-fault domain,
- ✓ The domains with higher values ($28,50° - 32,25°$ and $32,25° - 34,00°$) are the least spatially represented and they are located mainly in the east final slopes, as well as in the line with the first fault structure in the surface mine,
- ✓ The domains with lower values ($14,20° - 26,75°$ and $26,75° - 28,50°$) are more spatially represented covering whole space in non-fault domain form the surface mine, especially the domain with the lowest values,
- ✓ It can be noticed that the obtained value for this geotechnical parameter belongs in the quasi-homogeneous domain with the lowest values which is spatially the most represented, but in stability analysis of the east final slopes, as well as in the line with the first fault structure, can be used higher values from the obtained ones.

3.5. Quasi-homogeneous domains of the angle of internal friction of coal determined with triaxial tests

The data used for modeling of the cohesion of the coal, determined by the triaxial tests belong in the geological parts of the five coal layers originated from Miocene period, hence the created quasi-homogeneous domains will refer to these parts from the geological model.

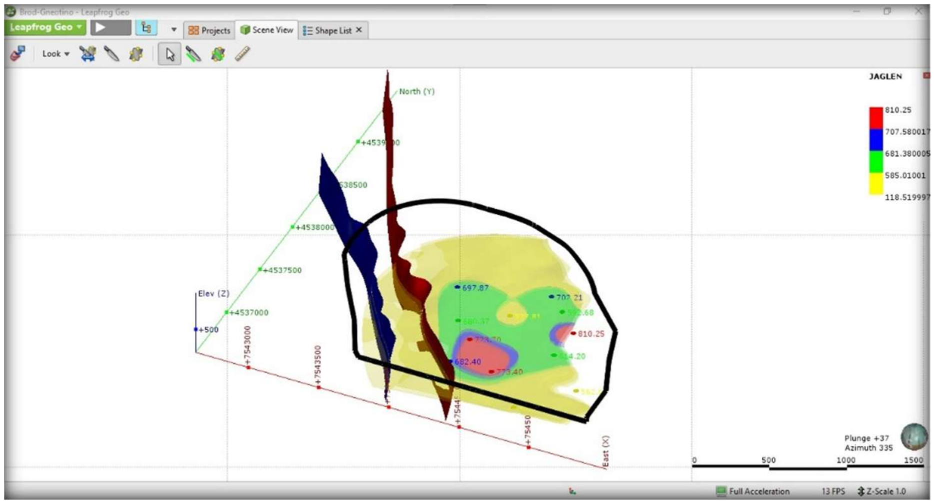

Figure 6: 3D representation of quasi-homogeneous domains of the cohesion of the coal determined with the triaxial tests

The obtained value for the cohesion of the coal in the surface mine Brod-Gneotino is $c = 50,00\ [kN / m^3]$. From the 3D representation of the quasi-homogeneous domains of the cohesion of the coal, determined with triaxial tests, we can derive the following conclusions:

- ✓ Data used for creating of these domains are in the range from $c = 118,52\ [kN / m^3]$ to $c = 810,25\ [kN / m^3]$, with median $c = 681,38\ [kN / m^3]$ and are located in the non-fault domain,
- ✓ The domains with higher values ($681,38 - 707,5\ [kN / m^3]$ and $707,58 - 810,25\ [kN / m^3]$) are the least spatially represented and they are located in the small part in the east final slopes, as well as in the middle part in the surface mine,
- ✓ The domains with lower values ($118,52 - 585,01\ [kN / m^3]$ and $585,01 - 681,38\ [kN / m^3]$) are more spatially represented covering whole space in non-fault domain form the surface mine.

CONCLUSIONS

The used mathematical models, aided by computer techniques, allow combination of geological model and numerical models for geotechnical parameters by geotechnical sections/domains, parts in combined models. These models display the distribution of values of the geotechnical parameters in different geological layers/mediums.

In the analysis of the stability of slopes in the excavation blocks at the surface mine Brod-Gneotino, the obtained values for geotechnical parameters are of primary importance. Considered quasi-homogeneous domains and their spatial analysis in the different parts of the surface mine, can make a significant contribution into adopting certain values for the geotechnical parameters, which on the other side are used for slope analysis. The scope of this paper, alongside with the theoretical approach of the theory of spatial analysis, especially semivariograms and kriging, is the selection of the geotechnical values needed for stability analysis of the final slopes of the surface mine. The studied domains show that in certain parts of the deposit, the obtained laboratory values are appropriate, while in other parts it can be used lower or higher values in order to be obtained more realistic analysis of the slope stability, as process, highly important for exploitation and long term planning and development of the open pit mine. Same approach can be suggested also for the case of underground mining.

All of this emphasize the importance of the spatial analysis aided with computer programs in the geotechnical modeling. The application of these methods is important for spatial and statistical processing of geotechnical data. Using corresponding interpolations results in selection of realistic values of the geotechnical parameters in all levels of project design is thus highly recommended.

This mathematical approach is quite general, easy for application and is of great benefit for geotechnical characterization of different types of mineral deposits.

Finally, due to the nature of the geological materials, other mathematical approaches can be suggested and are being used for geotechnical characterization in geotechnics such as in kinematic stability analyses for hard rock masses, calculations of bearing capacity, settlement, and others. All of these analysis find benefit from the geostatistical tools. These aspects overcome the scope of this paper.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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A GEOGRAPHICALLY WEIGHTED REGRESSION APPROACH IN REGIONAL MODEL FOR REAL ESTATE MASS VALUATION

Natasha Malijanska, Sanja Atanasova, Gjorgji Gjorgjiev, Igor Peshevski, Daniel Velinov

Abstract. Real estate mass valuation models of a market value have a tendency to generate real estate property values as close as to the real market values. Property valuation theory, as one of the primary factors influencing property value, considers location. The main statistical tool used for modelling in this investigation is geographically weighted regression. More precisely, the paper is striving to establish a mass valuation real estate property model considering the implementation of spatial data as a significant factor in determining the market value of condominiums in Skopje.

1. INTRODUCTION

The great importance of real estate, both in economic as well as in social life creates a need for trustworthy data about its own value, which will be helpful in making decisions during its management and usage.
The value, in the publication Uniform standards of professional appraisal practice by the Appraisal Foundation, is defined as "*the monetary relationship between properties and those who buy, sell, or use those properties*". Value expresses an economic concept. As such, it is never a fact, that is, it is always an opinion about the value of the property at a given time in accordance with a certain definition of value. Real estate appraisal or property valuation is *"the act or process of developing an opinion of value of the property"*, [1].

It is important to distinguish the term market value from the term market price, which is the amount for which real estate is sold on a certain date. In addition to the market, investment, liquidation value, value according to the principle of continuity and many other types of real estate value can be also estimated. The market value by the International Valuation Standards Council in their publication International Valuation Standards is defined as *"the estimated amount for which an asset or liability should exchange on the valuation date between a willing buyer and a willing seller in an arm's length transaction, after proper marketing and where the parties had each acted knowledgeably, prudently and without compulsion"*, [8].

Regarding the method of valuation, i.e., the number of real estates that are apprised, there is an individual and mass valuation.

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Key words and phrases. Geographically weighted regression approach, multivariate regression, sensitive analysis, real estate mass valuation.

The individual valuation is an estimate of the value specifically intended for individual real estate, taking into account its specific characteristics and referring to a specific date. Unlike individual valuation, mass valuation is a process of valuing a group of real estate, on a given date, using common data, applying standardized methods and conducting statistical tests to ensure unity and equality in valuation. When assessing a large number of real estates, it is difficult to emphasize each of their qualities, so special attention is paid to defining what is common to all real estate that is valued, i.e., significant factors for their value. The mass valuation, by the Appraisal Foundation in their publication Uniform Standards of Professional Appraisal Practices, is defined as a *"process of valuing a universe of properties as of a given date using standard methodology, employing common data and allowing for statistical testing"*, [1].

Mass valuation is based on the same basic principles as individual valuation. However, mass valuation includes many real estates for a certain date, which is why mass valuation techniques include equations, tables, and plans, collectively called models.

Mass valuation models attempt to represent the market for a certain type of real estate in a particular area. The structure of such models can be seen as a two-step process:

- Model specification and
- Model calibration.

The model specification provides a framework for simulating supply forces and real estate market demand. This step involves selecting the variables of supply and demand, that need to be considered and defining their correlation towards the value as well as their own correlation. Model calibration is the process of adjusting the mathematical model for mass valuation, the tables, and the estimates for the current market. The structure of the model can be valid for several years, but it is usually calibrated or updated each year. For longer periods, a complete market analysis is required, [3]. The purpose of the mass valuation is to reflect the current conditions in the local market.

When specifying the mass valuation model, firstly the variables are identified (supply and demand) that can impact the value of the real estate and then they are defined as mathematical conversions such as logarithms, which are often used to transform nonlinear data. At the same time, the mathematical form of the model is defined. It can be used in linear (additive) and nonlinear (including multiplier) forms. Next, the model is calibrated, i.e., the data are analysed so we can determine the adjustments or the coefficients that represent the contribution to the value of the real estate of the selected variables.

The construction of the models requires a good theoretical foundation, data analysis, and research methods. The best valuation models are expected to be accurate, rational, and explainable. Regression analysis is one of the most used methods in statistics, it is used for understanding, modelling, predicting, and explaining complex phenomena. In regression analysis, the predicted variable is called a dependent variable, and the variables used for prediction are called independent variables. Regression analysis allows the creation of a model for predicting the values of a dependent variable, based on the values of other independent variables or only one independent variable.

Building a regression model is an iterative process that involves finding effective independent variables to explain the dependent variable we are trying to model or understand. By repeating the regression procedure, we determine which variables are effective predictors, and then we constantly subtract and/or add variables until we find the best possible regression model. The process of building a model is a research process. It is necessary to identify explanatory variables in consultation with theory, experts in the field, and based on common sense. We need to be able to state and justify the expected relationship between each explanatory variable and the dependent variable before the analysis, and we need to question the models where these relationships do not match.

The first law of geography, given by Waldo Tobler, is that "*everything is related to everything else, but near things are more related than distant things*", [12]. Foundations of many spatial statistical methods are based on this law. Geographically weighted regression (GWR) is a method used in spatial statistical analysis, discovering geographical variations in the relationship between a response variable and a set of covariates. GWR has been applied in a variety of disciplines and studies, aided by the increased availability of geo-referenced data at finer scales, and by an appreciation that global regression models can mask substantively important departures from average trends at local levels.

Based on the established infrastructure related to the mass valuation of real estate, the aim of this research paper is defined as the first attempt to establish a model for mass valuation of real estate for parts of the city of Skopje, while explicitly incorporating the spatial factor. The research also focuses on the application of data that the Agency of Cadastre, registers as real estate transactions that take place within the state, and which are the only official and relevant data source. Considering that in the value of the real estate, and consequently in the assessment of the value, the location has a great impact, the intention is to base the research on GeoInformation systems with which the spatial factor will be easily implemented in the model as well as the control of this component will be more extensive.

The rest of the paper is arranged as follows. In Section 2, the basic concerning geographically weighted regression are provided. Central part in this paper is Section 3. In this section are given two different models of real estate mass valuation, both using geographically weighted regression. The impacts of the age of the building and garage area are separately depicted and analysed. At the end, certain comparison of mass valuation models performance is given.

2. GEOGRAPHICALLY WEIGHTED REGRESSION

The main objective of spatial analysis is to identify the nature of the relationships that variables exhibit, [5-7]. Usually, this is made by calculating statistics or estimating parameters with observations taken from different spatial units across a study area, [9-12]. The obtained statistics or estimates of the parameter are assumed to be constant across space although this might be a very questionable assumption to make in many circumstances. In general, it is reasonable to assume that there might be intrinsic differences in relationships over space or that there might be some problem with the specification of the model from which the relationships are being measured and which manifests itself in terms of spatially varying parameter estimates. In either case it would be useful to have a means of describing and mapping such spatial variations as an exploratory tool for developing a better understanding of the relationships being studied, [2].

The most used model in geographical analysis is the model of simple linear regression. Using this technique, a particular variable (the dependent variable), is modeled as a linear function of a set of independent or predictor variables. The model states as follows:

$$y_i = a_0 + \sum_{k=1}^{m} a_k x_{ik} + \varepsilon_i \tag{1}$$

where y_i is the ith observation of the dependent variable, x_{ik} is the ith observation of the kth independent variable, ε_i are independent normally distributed error terms with zero means, and each a_k are determined from the observations. The number of observations is n. Using the least squares method, a_κ, $k = 1, 2, ..., m$ are estimated. In context of matrices, the upper equation can be written as

$$\hat{a} = (x^t x)^{-1} x^t y \tag{2}$$

where the independent observations are the columns of x and the dependent observations are the single column vector y. The column vector $\hat{a}$ contains the coefficient estimates. Each of these estimates can be looked of as a “rate of change” between one of the independent variables and the dependent variable.
For example, if y were agreed condominiums prices, and x contained several variables related to the attributes of the condominiums and its surrounding environment, coefficients could be used to estimate the change in

condominiums price for an extra square meter of garage, an extra bedroom, or the condominiums being located one kilometer closer to the nearest school.

Note that these rates of change are assumed to be universal. Wherever an apartment is located, for example, the marginal price increase associated with an additional bedroom is fixed. It might be more reasonable to assume that rates of change are determined by local culture or local knowledge, rather than a global utility assumed for each commodity. Returning to the example, the value added for an additional bedroom might be greater in a neighborhood populated by families with children where extra space is likely to be viewed highly beneficial in a neighborhood populated by singles or elderly couples, in which case extra space might be viewed as a negative feature. These variations in relationships over space, such as those described above, are referred to as spatial nonstationarity, [2].

Geographically weighted regression (GWR) addresses problems like the one in the previous paragraph. It is a relatively simple technique, extending the traditional regression framework of equation (1). Local variations in rates of change are allowed, so that the coefficients in the model are specific to a location i, rather than being global estimates. The regression equation in this case is given by

$$y_i = a_{i0} + \sum_{k=1}^{m} a_{ik} x_{ik} + \varepsilon_i \tag{3}$$

where a_{ik} is the value of the k th parameter at location i. Note that (1) is a special case of (3), by putting all of the functions are constants across space. As will be shown below, the point i at which estimates of the parameters are obtained is completely generalizable and need not only refer to points at which data are collected. Using GWR, it is quite easy to compute parameter estimates. For instance, for locations lying between data points, which makes it possible to produce detailed maps of spatial variations in relationships. Although the model in equation (3) appears to be a simple extension of (1), a problem with calibrating (3) is that the unknown quantities are in fact functions mapping geographical space onto the real line, rather than simple scalars as in (1). In a typical data set, samples of the dependent and independent variables are taken at a set of sample points and it is from these that the parameters must be estimated. In the traditional model, these estimates are constant for all i but in equation (3) this is clearly not the case. For model (3), it seems intuitively appealing to base estimates of a_{ik} on observations taken at sample points close to i. If some degree of smoothness of the a_{ik}, $k = 1, 2, ..., m$ is assumed, then reasonable approximations may be made by considering the relationship between the observed variables in a region geographically close to i.

By the use of a weighted least squares approach to calibrating regression models, different emphases can be placed on different observations in generating the estimated parameters. In ordinary least squares, the sum of the squared differences of predicted and actual y_i, is minimized by the coefficient estimates. In weighted least squares a weighting factor w_i is applied to each squared difference before minimizing, so that the inaccuracy of some predictions carries more of a penalty than others. If w is the diagonal matrix consisting of all w_i, then the estimated coefficients satisfy

$$\tilde{a} = (x^t wx)^{-1} x^t wy \,. \qquad (4)$$

In Geographically weighted regression, weighting an observation in accordance with its proximity to i would allow an estimation of a_{ik} to be made that meets the criterion of "closeness of calibration points" set out above. Note that usually in weighted regression models the values of w_i are constant, so that only one calibration has to be carried out to obtain a set of coefficient estimates. In this case w varies with i, a different calibration exists for every point in the study area. In this case, the parameter estimation formula could be written more generally as

$$a(i) = (x^t w(i)x)^{-1} x^t w(i) y \,. \qquad (5)$$

Comparing this method and that of kernel regression and kernel density estimation, we can say the following: In kernel regression, y is modeled as a nonlinear function of x by weighted regression, with weights for the i th observation depending on the proximity of x and x_i, for each i with the estimator being

$$\tilde{a}(x) = (x^t w(x)x)^{-1} x^t w(x) y \,. \qquad (6)$$

The main difference between the two methods is that in (6), kernel regression, the weighting system depends on the location in "attribute space" of the independent variables, whereas in geographically weighted regression (see (5)) it depends on location in geographical space. The output in (5) is typically a set of localized parameter estimates in x space so that highly nonlinear and nonmonotonic relationships between y and x can be modeled. The typical output in (6) is a set of parameter estimates that can be mapped in geographic space to represent nonstationarity or parameter "drift", [2].

From the above, w_i is a weighting scheme established on the proximity of i to the sampling locations around i, without an explicit relationship being stated. The choice of such a relationship will be considered in continuation. Firstly, consider the implicit weighting scheme of (2). Here

$$w_{ij} = 1, \tag{7}$$

for all i and j. Here j represents a specific point in space at which data are observed and i represents any point in space for which parameters are estimated. This means that, in the global model each observation has a weight one. An initial step toward weighting based on locality might be to exclude from the model calibration observations that are further than some distance d from the locality. This is equivalent to putting their weights to be zero, giving a weighting function by

$$\begin{cases} w_{ij} = 1, & d_{ij} < d \\ w_{ij} = 0, & otherwise \end{cases}. \tag{8}$$

The use of (8) allows efficient computation, since for every point for which coefficients are to be computed; only a subset (often quite small) of the sample points need to be included in the regression model. Hence, the spatial weighting function in (8) suffers the problem of discontinuity. As i vanes around the study area, the regression coefficients could vary drastically as one sample point moves into or out of the circular buffer around i and which defines the data to be included in the calibration for location i. Although instant changes in the parameters over space might genuinely occur, in this case changes in their estimates would be artifacts of the arrangement of sample points, rather than any underlying process in the phenomena under investigation. One way to address this problem is to make w_{ij} a continuous function of d_{ij}, where d_{ij} is the distance between i and j. In this case, it can be seen from (5) that the coefficient estimates would then vary continuously with i. A straightforward choice for the weight function w_{ij} might be

$$w_{ij} = \mathrm{e}^{-\beta d_{ij}^{2}}, \tag{9}$$

so that if i is a point in space at which data are observed, the weighting of that data will be unity and the weighting of other data will decrease according to a Gaussian curve as the distance between i and j increases. In the latter case the inclusion of data in the calibration procedure becomes “fractional.” For example, in the calibration of a model for point i, if $w_{ij} = 0,5$, then data at point j contribute only half the weight in the calibration procedure as data at point i itself. For data far away from i the weighting will be asymptotically zero, effectively excluding these observations from the estimation of parameters for location i.

Adjustments of (8) and (9) may be made, having the computationally desirable property of excluding all data points greater than some distance from

i and also the desirable property of continuity. An example is the bisquare function given by

$$w_{ij} = \begin{cases} (1 - d_{ij}^{\ 2} / d^2)^2, & d_{ij} < d \\ 0, & otherwise \end{cases}. \tag{10}$$

This excludes points outside radius d, but tapers the weighting of points inside the radius, so that w_{ij} is a continuous and once differentiable function for all points less than d units from i.

Whatever the specific weighting function employed, the essential idea of Geographically weighted regression is that for each point i there is a "bump of influence" around i corresponding to the weighting function in a way that sampled observations close to i have more influence in the estimation of i's parameters than do sampled observations farther away.

The following problem occurs when use GWR: The estimated parameters are, in part, functions of the weighting function or kernel selected in the method. In (8)**,** for example, as d becomes larger, the closer will be the model solution to that of OLS and when d is equal to the maximum distance between points in the system, the two models will be equal. Equivalently, in (9) as β tends to zero, the weights tend to one for all pairs of points so that the estimated parameters become uniform and GWR becomes equivalent to OLS. Conversely, as the distance- decay becomes greater, the parameter estimates will increasingly depend on observations in close proximity to i and hence will have increased variance. The problem is therefore how to select an appropriate decay function in GWR. Consider the selection of β in (9), one possible solution is β to be chosen on a least squares criteria. If the error terms in (3) are assumed to be Gaussian, then this also fulfills a maximum likelihood criterion. Hence, the way to proceed would be to minimize the quantity

$$\sum_{i=1}^{n} (y_i - y_i^*(\beta))^2 , \tag{11}$$

where $y_i^*(\beta)$ is the fitted value of y_i using a distance-decay of β. For the sake of finding the fitted value of y_i, it is necessary to estimate the a_{ik}'s at each of the sample points and then combine these with the z-values at these points. However, when minimizing the sum of squared errors suggested above, a problem is encountered. Let β was made very large so that the weighting of all points except for i itself become negligible. Hence the fitted values at the sampled points will tend to the actual values, so that the value of (11) tends to zero. This means that under such an optimizing criterion, the value of β tends to infinity, but clearly this degenerate case is not useful. First, the parameters of such a model, are not defined in this limiting case. Second, the estimates will fluctuate wildly throughout space in order to give locally good fitted values at each i.

The *cross-validation* (CV) approach suggested for local regression by Cleveland (1979) and for kernel density estimation by Bowman (1984), is a solution to this problem. Here, a score of the form

$$\sum_{i=1}^{n}(y_i - y_{\neq i}^{*}(\beta))^2$$

is used where $y_{\neq i}^{*}(\beta)$ is the fitted value of y_i with the observations for point i omitted from the calibration process. This approach has the desirable property of countering the wrap-around effect, since when becomes very large, the model is calibrated only on samples near to ***i*** and not at i itself. Plotting the CV score against the required parameter of whatever weighting function is selected will therefore provide guidance on selecting an appropriate value of that parameter. If it is desired to automate this process, then the CV score could be maximized using an optimization technique such as a Golden Section search, [2].

3. MODELLING WITH GEOGRAPHICALLY WEIGHTED REGRESSION

According to the theoretical settings, experience, available research and data made available from the *Register of Leases and Real Estate Prices*, a set of proposed explanatory variables has been identified that are considered to determine the market value of the real estate. Despite the good reasons for including any available real estate data as variables in the model, it was found that some of the explanatory variables were statistically significant and some were statistically insignificant. For this reason, statistical tests have been conducted to make a number of possible combinations of proposed input explanatory variables, requiring models that best explain the dependent variable and thus perform the model specification. The analysis of the proposed explanatory variables gave the results shown in the table below. Also, through the statistical analysis, multicollinearity is calculated between the explanatory variables, i.e., VIF value. In which the value taken as a limit is the value 7.5, i.e., if the VIF value is less than 7.5 there is no multicollinearity between the explanatory variables.

The following table shows the result for significance and multicollinearity based on the analysis of the explanatory variables.

Table 1: Result of the analysis of variables

Summary of variable significance				Multicollinearity
Variable	**Significant**	**Negative**	**Positive**	**VIF**
Area	100	0	100	1.69
Garage (area)	100	0	100	1.19
Distance to closet mall	100	100	0	4.17
Age	98.07	100	0	1.67
Elevator	87.67	0	100	1.64
Distance to closest university	85.09	99.14	0.86	2.97
Distance to school	79.93	0	100	1.33
Balcon area	74.53	0.02	99.98	1.19
Floor number	73.79	0	100	1.21
High quality interior	69.43	0	100	1.03
Distance to closest park	65.00	62.99	37.01	3.71
Rooms	60.68	18.84	81.16	1.47
Own heating system	60.21	8.77	91.23	1.93
Distance to closest hospital	54.98	16.58	83.42	1.90
Distance to closest kinder garden	44.19	42.45	57.55	1.38
Distance to city centre	43.51	48.20	51.80	2.56
Basement area	39.99	77.79	22.21	1.30
Communal heating system	33.04	24.30	75.70	2.32
Distance to closest bus station	22.33	72.18	27.82	1.33

The results obtained from the analysis of the explanatory variables show that there is a high significance of certain structural, but also spatial characteristics for the real estate that is subject to transaction. It can also be noted that we do not have a redundant explanatory variable, i.e., there is no multicollinearity between the explanatory variables. In the process of defining an appropriate model, it is necessary to experiment with different variables to explain the value of the real estate. It is important to be aware that the coefficients of the explanatory variables (and their statistical importance) may change radically depending on the combination of variables we include in the model.

For the purposes of the research, two models were created with GWR, while for assessing the quality of the created models, the statistical parameters R^2, adjusted $R^2$and Akanke's Information Criterion (AICc) were used. R^2 and adjusted R^2 are statistically derived from the regression equation to quantify model performance. The value of R^2 ranges from 0 to 1. If the model explains the dependent variable perfectly R^2 is 1.0. As an example, if you get a value of R^2 of 0.49, it can be interpreted with the words: "the model explains 49 percent of the variations in the dependent variable". Adjusted R^2 is always slightly lower than the value for R^2, as it reflects the complexity of the model (number of variables). Consequently, the adjusted R^2 is a more accurate measure of model performance. The Akaike information criterion (AIC) is an estimator of prediction error and thereby the relative quality of statistical models for a given set of data. AIC estimates the relative amount of information lost by a given model: the less information a model loses, the higher the quality of that model.

Model 1

Model 1 created with GWR is specified only with structural features of residential property. As explanatory variables for which statistical tests showed the greatest signification are as follows: Area, Garage (area), Balcony area and Age. Using these explanatory variables, the first model for which the following statistical indicators are obtained is formed in the table below, through which we can see the success of the model.

Table 2: Results of the analysis – GWR for Model 1

OID	VARNAME	VARIABLE	DEFINITION
0	Bandwidth	2137.860192	
1	ResidualSquares	54169319433	
2	EffectiveNumber	21.807198	
3	Sigma	8063.135068	
4	AICc	17821.981138	
5	R2	0.794736	
6	R2Adjusted	0.78961	
7	Dependent Field	0	PRICE_EU
8	Explanatory Field	1	AREA
9	Explanatory Field	2	AREA_BALCO
10	Explanatory Field	3	AREA_GARAG
11	Explanatory Field	4	AGE

Table 3: Correlation coefficient between the projected prices by Model 1 and the actual purchase prices for the control group points

Correlations

		price_eu	Predicted
price_eu	Pearson Correlation	1	,899**
	Sig. (2-tailed)		,000
	N	95	95
Predicted	Pearson Correlation	,899**	1
	Sig. (2-tailed)	,000	
	N	95	95

**. Correlation is significant at the 0.01 level (2-tailed).

The correlation analysis between the appraised market value of the residential property that has been sold, obtained with model 1 and the actual purchase price performed in transactions for the control group of transactions, calculated with Pearson the correlation coefficient in the SPSS software for this model is 0.899, i.e., 89.9 %.

Model 2

Model 2 created with GWR uses the same structural and explanatory variables as Model 1 and supplemented by three spatial explanatory variables that the analysis showed were statistically significant: Distance to the closest mall, Distance to the closest hospital and Distance to the closest university. By applying all these explanatory variables, the following statistical indicators shown in the following table are obtained:

Table 4: Results of the analysis –GWR for Model 2

OID	VARNAME	VARIABLE	DEFINITION
0	Bandwidth	2137.860192	
1	ResidualSquares	47401929637.900002	
2	EffectiveNumber	31.111344	
3	Sigma	7585.142586	
4	AICc	17724.392795	
5	R2	0.82038	
6	R2Adjusted	0.813815	
7	Dependent Field	0	PRICE_EU
8	Explanatory Field	1	AREA
9	Explanatory Field	2	AREA_BALCO
10	Explanatory Field	3	AREA_GARAG
11	Explanatory Field	4	AGE
12	Explanatory Field	5	DIST_MAL
13	Explanatory Field	6	DIST_HOS
14	Explanatory Field	7	DIST_UNI

The correlation analysis between the appraised market value of the residential property that has been sold, obtained with model 2 and the actual purchase price performed in the transactions for the control group points, calculated with Pearson correlation coefficient in the SPSS software for this model is 0.906, i.e., 90.6%.

Table 5: Correlation coefficient between the projected prices by Model 2 and the actual purchase prices for the control group points

Correlations

		price_eu	Predicted
price_eu	Pearson Correlation	1	,908**
	Sig. (2-tailed)		,000
	N	95	95
Predicted	Pearson Correlation	,908**	1
	Sig. (2-tailed)	,000	
	N	95	95

**. Correlation is significant at the 0.01 level (2-tailed).

When calibrating mass valuation models where spatial regression models are used, they have a variable value that varies depending on the location. In order to register this variation, spatial data in raster data format is used. Hence, a significant advantage in using the GWR model and applying GeoIS is the ability to create a series of raster layers of variable coefficients. This allows the identification of spatial variations within the research area, which can help in effective decision making. Such records can provide an excellent insight into the key parameters that affect the value of the property in a particular area. For example, the age of the property can have a significant negative impact on the value of the property in newly developed areas where most of the properties are completely new, and on the other hand it can have a positive impact in an old part of the city where older buildings have architectural features and historical significance. In order to emphasize the importance of these models, the results of the age factor of the building will be presented. As expected, the age of the building is inversely proportional to the value of the property, i.e., the older construction reduces the value of the property due to obsolescence, deterioration and depreciation. The analysis of the raster data model of the coefficient for the age of the building showed that the impact of this factor varies through the field of research and less impact (lower coefficients) this factor is observed in the central area of the city, while the impact of the age of the building increases as we move away from the central urban area, to the settlements of Karposh, Aerodrom, where new buildings are being built and the demand for new buildings is higher.

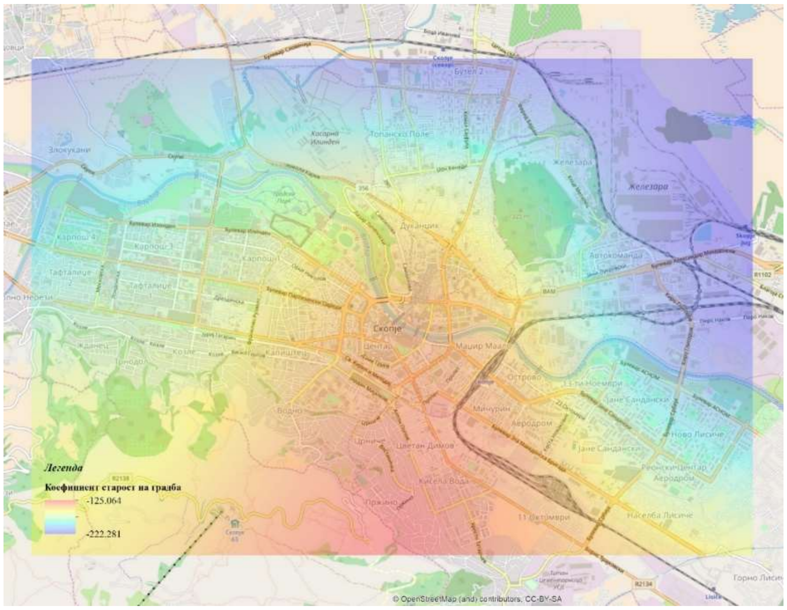

Figure 1: Value of the coefficient before the variable age

The analysis of the raster data model of the coefficient for the impact of the garage surface showed that this impact is greater in the municipalities of Centar and Karposh, while in the municipalities of Aerodrom and Chair, that impact is less, as expected, due to the existence of more and larger parking spaces.

Figure 2: Values of the coefficient in front of the variable area of the garage

4. CONCLUSIONS

Based on the results obtained from quality control of the established models for mass valuation we can conclude that both models meet the statistical checks and have a satisfactory accuracy of market value prediction. However, although they have satisfactory accuracy, it is necessary to emphasize the difference between the number and type of explanatory variables that these models incorporate and how they affect the end result.

Table 6: Comparison of mass valuation models performance

	Model 1 - GWR	*Model 2 - GWR*
Coefficient of determination – R^2	79.5%	82.0%
Akaike Information Criterion – AICc	17822	17724
Pearson correl.	89.9%	90.8%
Input data	Non-spatial	Spatial
Number of explanatory variables	4	7

Model 2 has higher R^2 coefficient, which means that the created model fits much better in the data. A higher percentage shows that the dependent variable (the value of the residential property) is better explained by the selected independent variables, while this percentage is lower in Model 1. Also, the AICc value of the first model is lower than the one of Model 2.

As for the accuracy of the prediction, which is calculated as the correlation coefficient between the projected prices of the control transactions that were omitted from the creation of the models and the actual prices of their purchase, Model 2 has a higher Pearson correlation factor than Model 1.

The results show that the use of Geographically weighted regression (GWR) in predicting market real estate values is a great basis for developing mass valuation models. In doing so, the incorporation of spatial explanatory variables can have a positive impact on real estate mass valuation models.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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