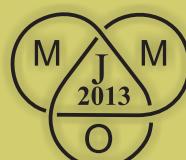


20-та Македонска
математичка олимпијада



17-та JMMO



ARMAGANKA-Library Olympiads

Mathematical Olympiads

Macedonian Mathematical Olympiad 2016
Balkan Mathematical Olympiad 2016
European Girl's Mathematical Olympiad 2016
European Mathematical Cup 2015
Junior Macedonian Mathematical Olympiad 2016
Junior Balkan Mathematical Olympiad 2016
Mediterranean Mathematical Olympiad 2016

Aleksa Malcheski, Ph.D.
Slagjana Brsakoska, Ph.D.
Risto Malcheski, Ph.D.
Bojan Prangoski, Ph.D.
Daniel Velinov, Ph.D.
Sanja Atanasova, Ph. D
Pavel Dimovski, Ph.D.
Tomi Dimovski, M.Sc.
Vesna Andova, Ph.D
Samoil Malcheski, Ph. D
Methodi Glavche
Dimitar Treneski

President: Alekса Malcheski
Publisher: Union of mathematicians of Macedonia-Armaganka
Address: **ul. 2 br. 107A**
Vizbegovo, Butel, Skopje, Republic of Macedonia

CIP - Каталогизација во публикација

Национална и универзитетска библиотека "Св. Климент Охридски", Скопје

51(079.1)

MATHEMATICAL olympiads : Macedonian mathematical olympiads 2001:

Balkan mathematical olympiads : European girl's mathematical olympiad

: European mathematical cup / Alekса Malchevski ... [and others.]. -

Skopje : Macedonian mathematical society-Armaganka, 2001. - 35 стр. : ilustration.

; 24 см. - (#Библиотека #Олимпијади)

Authors: Alekса Malcheski, Slagjana Brsakoska, Risto Malcheski, Bojan Prangoski, Daniel Velinov, Sanja Kostadinova, Pavel Dimovski, Tomi Dimovski, Vesna Andova, Samoil Malcheski, Dimitar Treneski, Methodi Glavche

ISBN 978-9989-646-74-4 (кн.)

ISBN 978-9989-646-18-8 (бидл.)

1. Malchevski, Alekса [автор] [уредник]

а) Математика - Задачи од натпревари

COBISS.MK-ID 101387274

Foreword





**XXIV Regional competition of mathematics for secondary school
03.03.2001 year**

I class

1. AB Prove that the product of four following natural numbers :

- a) Is divisible by 24; b) is not a square of any natural number

Solution. (a) Let $n, n+1, n+2, n+3$ be four following natural numbers.

Hence, their product $n(n+1)(n+2)(n+3)$ has at least two even numbers, one number divisible by 4 and at least one number divisible by 3. Hence, that number is divisible by $2 \cdot 3 \cdot 4 = 24$.

$$\begin{aligned} \text{b) From } n(n+1)(n+2)(n+3) &= (n^2 + 3n)(n^2 + 3n + 2) = \\ &= (n^2 + 3n)^2 + 2(n^2 + 3n) + 1 - 1 = (n^2 + 3n + 1)^2 - 1 \end{aligned}$$

it follows that the product is not a square of any natural number n .

2.A If $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$ and $ac > 0$, then $\frac{a+b}{2a-b} + \frac{c+b}{2c-b} \geq 4$. Prove it!

Solution. $\frac{1}{a} + \frac{1}{c} = \frac{2}{b} \Rightarrow \frac{a+c}{ac} = \frac{2}{b} \Rightarrow b = \frac{2ac}{a+c}$. Hence,

$$\begin{aligned} \frac{a+b}{2a-b} + \frac{c+b}{2c-b} &= \frac{a + \frac{2ac}{a+c}}{2a - \frac{2ac}{a+c}} + \frac{c + \frac{2ac}{a+c}}{2c - \frac{2ac}{a+c}} = \frac{a^2 + 3ac}{2a^2} + \frac{c^2 + 3ac}{2c^2} = \\ &= \frac{1}{2} + \frac{3c}{2a} + \frac{1}{2} + \frac{3a}{2c} = 1 + \frac{3}{2} \left(\frac{c}{a} + \frac{a}{c} \right) \geq 1 + \frac{3}{2} \cdot 2 = 1 + 3 = 4. \end{aligned}$$

The equality holds for $a = c$.

2.B Mile and Zlatko were writing their homework. Ana came to their house and asked them about the subjects of their homework. They gave her these answers:

Mile: "If I am doing math then Zlatko is doing Macedonian language and literature".

Zlatko: "I am doing Macedonian language and literature or Mile is not doing math".

Then Ana thought for a while and said: "Either both of you are telling the truth or both of you are not telling the truth".

Is Ana telling the truth?

Solution. Let us denote with p and q the following statements:

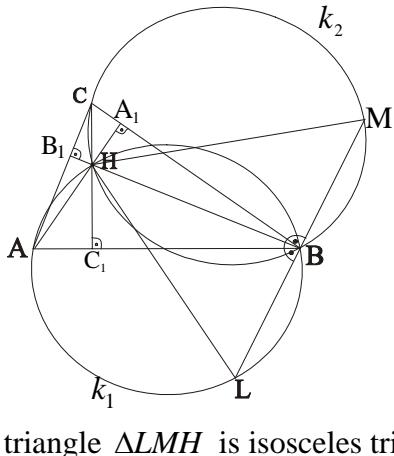
p : "Mile is doing math".

q : "Zlatko is doing Macedonian language and literature".

The formula $(p \Rightarrow q) \Leftrightarrow (q \vee \neg p)$ is tautology. Hence, Mile and Zlatko either both are telling the truth or both are not telling the truth. So, Ana is telling the truth.

3.A Let H be the orthocenter of the triangle ABC . Prove that the circumcircles of the triangles ABH, BCH and CAH have equal radii.

Solution. Let k_1 and k_2 be the circumcircles of the triangles ABH and BCH respectively and let HL and HM be the diameters of k_1 and k_2 respectively



(see the picture). From $\angle HBL = 90^\circ = \angle HBM$, it follows that L, B, M are collinear.

(1) $\angle HLB = \angle HAB$, as the peripheral angles over the arc HB of the circle k_1

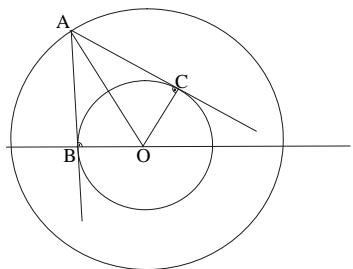
(2) $\angle HMB = \angle HCB$, as the peripheral angles over the arc HB of the circle k_2 .

The right triangles HAC_1 and HCA_1 are similar, so

(3) $\angle HAC_1 = \angle HCA_1$, i.e. $\angle HAB = \angle HCB$.

From (1), (2) and (3) it follows that $\angle HLB = \angle HMB$, i.e. $\angle HLM = \angle HML$. Hence, the triangle $\triangle LMH$ is isosceles triangle, so, $\overline{HL} = \overline{HM}$.

3.B Let r and $2r$ are the radii of two concentric circles. Prove that the angle between the tangents in any point of the circle with radius $2r$ to the other circle is 60° .



Solution. Let AB and AC be the tangents in the point A . From the right triangle $\triangle OAB$ we have $\overline{OA} = 2\overline{OB}$, so, $\angle OAB = 30^\circ$. Analogously, we have $\angle OAC = 30^\circ$, so, $\angle BAC = 60^\circ$.

4.AB Determine the biggest natural number that is less than the value of the expression $\underbrace{\sqrt{6 + \sqrt{6 + \dots + \sqrt{6}}}}_{2001 \text{ roots}} + \underbrace{\sqrt[3]{6 + \sqrt[3]{6 + \dots + \sqrt[3]{6}}}}_{2001 \text{ roots}}$, knowing that $\sqrt[3]{6} > 1,8$.

Solution. From $\sqrt{6} > 2,4$ and $\sqrt[3]{6} > 1,8$ it follows that

$$4,2 < \sqrt{6 + \sqrt[3]{6}} < \sqrt{6 + \sqrt{6 + \dots + \sqrt{6}}} + \sqrt[3]{6 + \sqrt[3]{6 + \dots + \sqrt[3]{6}}}$$

On the other hand we have that:

$$\begin{aligned} \sqrt{6 + \sqrt{6 + \dots + \sqrt{6}}} &< \sqrt{6 + \sqrt{6 + \dots + \sqrt{6+3}}} = 3 \\ \sqrt[3]{6 + \sqrt[3]{6 + \dots + \sqrt[3]{6}}} &< \sqrt[3]{6 + \sqrt[3]{6 + \dots + \sqrt[3]{6+2}}} = 2 \\ \sqrt{6 + \sqrt{6 + \dots + \sqrt{6}}} + \sqrt[3]{6 + \sqrt[3]{6 + \dots + \sqrt[3]{6}}} &< 5. \end{aligned}$$

So, the number we are looking for is 4.

II class

1.A Let a and b be the solutions of the equation $x^2 - 3cx - 8d = 0$, and c and d be the solutions of the equation $x^2 - 3ax - 8b = 0$. Determine the sum $a+b+c+d$, if a, b, c, d are different real numbers.

Solution. From the Viet formula it follows:

$$a+b=3c, c+d=3a, ab=-8d, cd=-8b.$$

If we add the first two equations we have $a+b+c+d=3(a+c)$. From the same two equations we have: $b=3c-a$, $d=3a-c$ and if we replace them into the other two equations we have:

$$a(3c-a)=-8(3a-c), c(3a-c)=-8(3c-a).$$

If we subtract these two equations we have: $c^2-a^2=32(c-a)$.

From $a \neq c$, it follows that $a+c=32$. Hence, $a+b+c+d=3(a+c)=3 \cdot 32=96$.

1.B The formula $y=(k+1)x^2-2kx+k-1$, $k \in \mathbb{R} \setminus \{-1\}$ determines the set of parabolas

a) Determine the geometric set of the apexes of the parabolas;

b) Solve the equation: $(k+1)x^2-2kx+k-1=0$.

Solution. a) Let $T(x, y)$ is the apex of the parabola $y=(k+1)x^2-2kx+k-1$.

Then

$$x=\frac{k}{k+1}=1-\frac{1}{k+1}, y=-\frac{1}{k+1}.$$

If we eliminate k from the equations above then the equation of the geometric set of points is: $y=x-1$. So, the geometric set of the apexes of the parabolas is a line.

b) From $D=4k^2-4(k^2-1)=4$, it follows that $x_1=1$, $x_2=\frac{k-1}{k+1}$.

2.A Determine all triplexes (x,y,z) of real numbers that satisfy the system of equations: $x+\sqrt{y}=3$, $y+\sqrt{z}=3$, $z+\sqrt{x}=3$.

Solution. $x-y=\sqrt{z}-\sqrt{y}$, $y-z=\sqrt{x}-\sqrt{z}$, $z-x=\sqrt{y}-\sqrt{x}$. Let $x \leq y$.

Then $\sqrt{z}-\sqrt{y} \leq 0$ i.e. $z \leq y$ and $\sqrt{x}-\sqrt{z} \geq 0$, i.e. $x \geq z$. It follows that $y \leq x$. So, $x=y$. Analogously we have $x=z$. So, $x=y=z$.

Now, it remains to solve the equation: $x+\sqrt{x}=3$, i.e. $\sqrt{x}=3-x$, where $x \in [0,3]$.

Its solution is: $\frac{7-\sqrt{13}}{2}$, so, the solution of the system is:
 $\left(\frac{7-\sqrt{13}}{2}, \frac{7-\sqrt{13}}{2}, \frac{7-\sqrt{13}}{2}\right)$.

2.B Determine all triplexes (x,y,z) of real numbers which satisfy the system of equations: $x+y=z$, $x^2+y^2=z$, $x^3+y^3=z$.

Solution. With squaring and raising to the third power the first equation we have:

$$x^2+y^2+2xy=z^2, z+2xy=z^2, xy=\frac{1}{2}(z^2-z)$$

$$x^3+y^3+3xy(x+y)=z^3, z+\frac{3}{2}(z^2-z)z=z^3, z(z-1)(z-2)=0.$$

If $z=0$, then $x+y=0$, $xy=0$, so, the solution is $(0,0,0)$.

If $z=1$, then $x+y=1$, $xy=0$, so, the solution is $(0,1,1)$ and $(1,0,1)$.

If $z=2$, then $x+y=2$, $xy=1$, so, the solution is $(1,1,2)$.

Hence, the solutions of the system are: $(0,0,0)$, $(0,1,1)$, $(1,0,1)$ and $(1,1,2)$.

3.AB Let u be a complex number. Determine all complex numbers z , such that $\frac{u - \bar{uz}}{1 - z}$ is a real number.

Solution. The complex number a is a real number if and only if $a = \bar{a}$. We have that

$$\frac{u - \bar{uz}}{1 - z} = \overline{\left(\frac{u - \bar{uz}}{1 - z} \right)}, \quad \frac{u - \bar{uz}}{1 - z} = \frac{\bar{u} - u\bar{z}}{1 - \bar{z}}$$

$$u - \bar{uz} - u\bar{z} + u\bar{z}\bar{z} = \bar{u} - u\bar{z} - \bar{u}z + uz\bar{z}, \quad (\bar{u} - u)(1 - z\bar{z}) = 0.$$

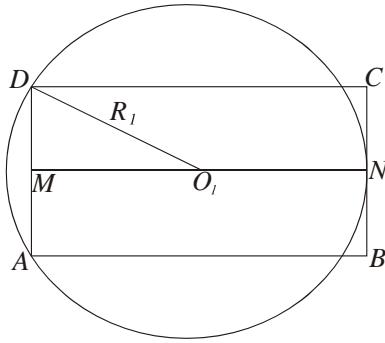
If u is a real number then $u = \bar{u}$, so the solution is every $z \in \mathbb{C} \setminus \{1\}$.

If u is a complex number then the solution is every $z \in \mathbb{C} \setminus \{1\}$ such that $|z| = 1$.

4.AB Let $ABCD$ be a rectangle. Let BC be the tangent in the point N of the circle $k_1(O_1, R_1)$ which passes through the points A and D and let CD be the tangent of the circle $k_2(O_2, R_2)$ which passes through the points A and B . If $a = \overline{AB}$, $b = \overline{AD}$ then prove that $R_1 + R_2 \geq \frac{5}{8}(a+b)$.

Solution. Obviously N is the middle point of the side BC . Let M be the middle point of the side AD . From the right triangle MO_1D it follows

that: $R_1^2 = \overline{MD}^2 + \overline{MO_1}^2 = \left(\frac{b}{2}\right)^2 + (a - R_1)^2$.



Now, $R_1 = \frac{b^2 + 4a^2}{8a}$. Analogously, $R_2 = \frac{a^2 + 4b^2}{8b}$.

$$R_1 + R_2 = \frac{b^3 + 4a^2b + a^3 + 4ab^2}{8ab} = \frac{(a+b)(a^2 + b^2 + 3ab)}{8ab} \geq \frac{(a+b)(2ab + 3ab)}{8ab} = \frac{5(a+b)}{8}$$

III class

1.AB Solve the inequality: $\lg(5^x + x - 20) > x - x\lg 2$.

Solution. From $x - x\lg 2 = x(\lg 10 - \lg 2) = x\lg 5 = \lg 5^x$ it follows that $\lg(5^x + x - 20) > \lg 5^x$. Then $5^x + x - 20 > 0$, $5^x + x - 20 > 5^x$. So the solution is every $x > 20$.

2.A Prove that for arbitrary α, β and γ at least one of the numbers: $\sin\alpha\cos\beta$, $\sin\beta\cos\gamma$, $\sin\gamma\cos\alpha$ is not bigger than $\frac{1}{2}$.

Solution. Let us assume the contrary i.e. let each of these numbers be bigger than $\frac{1}{2}$.

Then

$$\sin \alpha \cos \beta \cdot \sin \beta \cos \gamma \cdot \sin \gamma \cos \alpha > \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{1}{8},$$

$2\sin \alpha \cos \beta \cdot 2\sin \beta \cos \gamma \cdot 2\sin \gamma \cos \alpha > \frac{1}{8} \cdot 8$, $\sin 2\alpha \sin 2\beta \sin 2\gamma > 1$, which is not possible because $\sin 2\alpha \cdot \sin 2\beta \cdot \sin 2\gamma \leq 1$.

2.B Solve the equation: $\sin 2x = \sqrt{2} \sin^2 x + (2 - \sqrt{2}) \cos^2 x$.

Solution. If $\cos x = 0$, then $\sin^2 x = 1$, $\sin 2x = 2\sin x \cos x = 0$, so the equation has no solution.

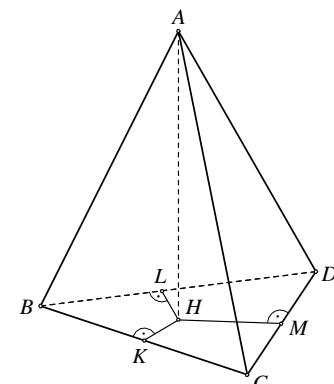
So, $\cos x \neq 0$. If we divide the given equation by $\cos^2 x$ then
 $\sqrt{2}\operatorname{tg}^2 x - 2\operatorname{tg} x + 2 - \sqrt{2} = 0$, $(\operatorname{tg} x - 1)(\sqrt{2}\operatorname{tg} x + \sqrt{2} - 2) = 0$. So, $\operatorname{tg} x = 1$ or $\operatorname{tg} x = \sqrt{2} - 1$.

Solutions of these equations are: $x_1 = \frac{\pi}{4} + k\pi$, $x_2 = \frac{\pi}{8} + m\pi$ ($k, m \in \mathbb{A}$).

3. A The edges AB, AC and AD of the tetrahedron are diameters of the balls T_{AB}, T_{AC} and T_{AD} . Prove that these three balls cover the tetrahedron.

Solution. Let AH be the perpendicular line to the face BCD of the tetrahedron. Let HL, HM and HK be the perpendicular lines to the sides BD, DC and CB of the triangle BCD . We have to prove that each of the pyramids $ABKL, ACKM$ and $ADML$ is covered with the balls T_{AB} , T_{AC} and T_{AD} , respectively.

From the construction of the pyramid $ABKL$ it follows that $\angle AKB = \angle ALB = \angle AHB = 90^\circ$, and that's why the points K, H, L belong on the sphere S_{AB} . So, the segments BL, BH, BK, AL, AH and AK are in the ball and that's why the triangles AKB, AHK and ALB and the quadrilateral $BLHK$ are in the ball T_{AB} . Hence, the pyramid $ABHKL$ is covered by the ball T_{AB} .



3.B The radius R of the circumcircle and the radius r of the incircle in the right triangle ABC are related as $5:2$, respectively. Determine the acute angles of the triangle.

Solution. We use the notation as on the picture. From $\overline{AB} = 2R$ and $\overline{AB} = \overline{AD} + \overline{DB} = \overline{OD} \operatorname{ctg} \frac{\alpha}{2} + \overline{OD} \operatorname{ctg} \frac{\beta}{2} = r \left(\operatorname{ctg} \frac{\alpha}{2} + \operatorname{ctg} \frac{\beta}{2} \right)$ it follows that

$$2R = r \left(\operatorname{ctg} \frac{\alpha}{2} + \operatorname{ctg} \frac{\beta}{2} \right). \text{ Hence, } \operatorname{ctg} \frac{\alpha}{2} + \operatorname{ctg} \frac{\beta}{2} = 2 \frac{R}{r} = 2 \cdot \frac{5}{2} = 5.$$

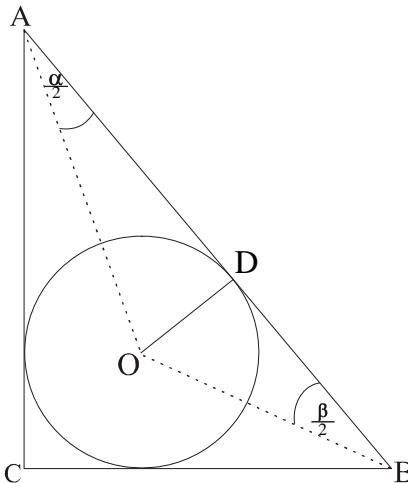
From the right triangle ABC it follows that $\alpha + \beta = 90^\circ$ i.e. $\frac{\alpha}{2} + \frac{\beta}{2} = 45^\circ$. From the equation

$$\operatorname{ctg}\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \frac{\operatorname{ctg}\frac{\alpha}{2} \cdot \operatorname{ctg}\frac{\beta}{2} - 1}{\operatorname{ctg}\frac{\alpha}{2} + \operatorname{ctg}\frac{\beta}{2}} = 1, \text{ it}$$

follows that $\operatorname{ctg}\frac{\alpha}{2} \operatorname{ctg}\frac{\beta}{2} = 6$. Let $\alpha < \beta$. The solutions of the system of equations:

$$\operatorname{ctg}\frac{\alpha}{2} + \operatorname{ctg}\frac{\beta}{2} = 5, \quad \operatorname{ctg}\frac{\alpha}{2} \operatorname{ctg}\frac{\beta}{2} = 6 \quad \text{are } \operatorname{ctg}\frac{\alpha}{2} = 3 \text{ and } \operatorname{ctg}\frac{\beta}{2} = 2.$$

Hence $\alpha = 2\arccos 3 = 36^\circ 52' 12''$, $\beta = 2\arccos 2 = 53^\circ 07' 48''$.



4.AB Let $\overline{AB} = a, \overline{BC} = b, \overline{CD} = c, \overline{AD} = d$ be the sides of a convex quadrilateral, s be the half perimeter, and 2φ be the sum of its opposite angles.

Prove that its area is

$$P = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \varphi}.$$

Solution. Let $\beta + \delta = 2\varphi$ (see the picture). Then

$$P = P_{ABC} + P_{ACD} = \frac{ab}{2} \sin \beta + \frac{cd}{2} \sin \delta \text{ i.e.}$$

$2P = ab \sin \beta + cd \sin \delta$. By raising to the second power the last equation we have:

$$4P^2 = a^2 b^2 \sin^2 \beta + c^2 d^2 \sin^2 \delta + 2abcd \sin \beta \sin \delta$$

If we put $\sin^2 \beta = 1 - \cos^2 \beta$, $\sin^2 \delta = 1 - \cos^2 \delta$ in the last equation we have:

$$4P^2 = (ab + cd)^2 - (ab \cos \beta - cd \cos \delta)^2 - 2abcd(1 + \cos \beta \cos \delta - \sin \beta \sin \delta)$$

.

If we put $1 + \cos 2\varphi = 2 \cos^2 \varphi$, in the last equation we have

:

$$4P^2 = (ab + cd)^2 - (ab \cos \beta - cd \cos \delta)^2 - 4abcd \cos^2 \varphi.$$

(1)

Let $\overline{AC} = e$. From the cosine theorem for the triangles ABC and ACD it follows that:

$$e^2 = a^2 + b^2 - 2ab \cos \beta, \quad e^2 = c^2 + d^2 - 2cd \cos \delta.$$

If we subtract these two equations we have: $ab \cos \beta - cd \cos \delta = \frac{1}{2}(c^2 + d^2 - a^2 - b^2)$.

If we put this equation in (1) we have

$$4P^2 = (ab + cd)^2 - \frac{1}{4}(c^2 + d^2 - a^2 - b^2)^2 - 4abcd \cos^2 \varphi, \text{ i.e.}$$

$$4P^2 = \frac{1}{4}(-a + b + c + d)(a - b + c + d)(a + b - c + d)(a + b + c - d) - 4abcd \cos^2 \varphi$$

$$P^2 = \frac{-a + b + c + d}{2} \cdot \frac{a - b + c + d}{2} \cdot \frac{a + b - c + d}{2} \cdot \frac{a + b + c - d}{2} - abcd \cos^2 \varphi$$

From $\frac{a + b + c + d}{2} = s$, it follows that :

$$P = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \varphi}.$$

IV class

1.A In the infinite arithmetic series of integer numbers, one of the terms is a square of natural number. Prove that this series has infinite terms which are squares of natural numbers.

Solution. Let (a_n) be an infinite arithmetic series of integer numbers with difference d . If $m^2, m \in \mathbb{N}$ is a term of (a_n) then $(m+kd)^2 = m^2 + d(2mk + k^2d)$ is also a term of (a_n) .

1.B The binomial coefficients on the second, third and fourth member in binomial expansion of the bynom $(1+x)^n$ are terms of arithmetic series. Determine n .

Solution. From $\binom{n}{2} - \binom{n}{1} = \binom{n}{3} - \binom{n}{2}$ it follows that $n^2 - 9n + 14 = 0$. Solutions of this equation are $n_1 = 7$ and $n_2 = 2$, but the only solution of the problem is $n = 7$.

2.A Solve the inequality $(\sqrt{2}+1)^{\frac{|x|-1}{x+1}} \leq (\sqrt{2}-1)^{-\frac{x}{6}}$.

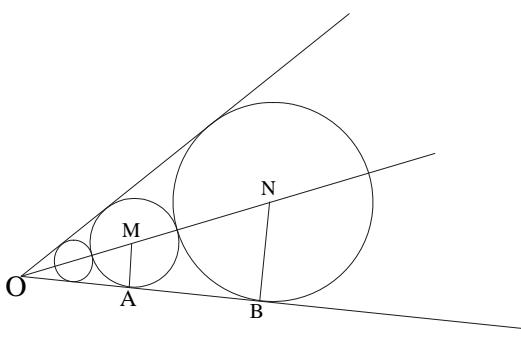
Solution. From $(\sqrt{2}-1)^{-1} = \frac{1}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \sqrt{2}+1$, it follows that

$$(\sqrt{2}+1)^{\frac{|x|-1}{x+1}} \leq (\sqrt{2}+1)^{\frac{x}{6}}.$$

Now $\frac{|x|-1}{x+1} \leq \frac{x}{6}$, i.e. $\frac{6|x|-6-x^2-x}{6(x+1)} \leq 0$.

If $x \geq 0$, then $\frac{5x-6-x^2}{6(x+1)} \leq 0$, i.e. $\frac{(x-2)(x-3)}{6(x+1)} \geq 0$. Hence $x \in [0,2] \cup [3, \infty)$.

If $x < 0$, then $\frac{-7x-6-x^2}{6(x+1)} \leq 0$, i.e. $\frac{(x+1)(x+6)}{6(x+1)} \geq 0$. Hence $x \in [-6,-1) \cup (-1,0)$. Finally $x \in [-6,-1) \cup (-1,2] \cup [3, \infty)$.



2.B Sides of the angle are tangents of circles as it is shown on the picture. Determine the sum of:

a) perimeters b) areas of all circles.

Solution. Let $\overline{AM} = r_1$, $\overline{BN} = r$, $\overline{OM} = a$, $\overline{MN} = b$. From the similarity of the triangles OBN and OAM it follows that $\frac{r}{r_1} = \frac{a+b}{a}$. On the other hand $\sin 30^\circ = \frac{r_1}{a}$,

so $a = 2r_1$. Hence $r = r_1 \left(1 + \frac{b}{a}\right) = r_1 \left(1 + \frac{b}{2r_1}\right) = r_1 \left(1 + \frac{r+r_1}{2r_1}\right)$ i.e. $r_1 = \frac{r}{3}$.

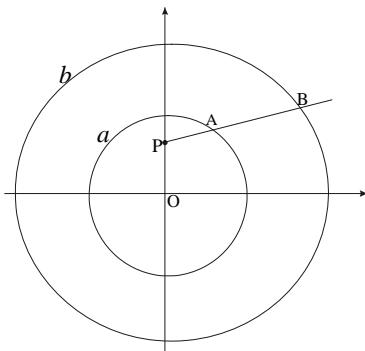
Analogously $r_2 = \frac{r_1}{3} = \frac{r}{9}$, etc.

a) The sum of the perimeters is $2\pi(r + r_1 + r_2 + \dots) = 2\pi\left(r + \frac{r}{3} + \frac{r}{9} + \dots\right) = 2\pi r \frac{3}{2} = 3\pi r$.

b) The sum of the areas is

$$\pi(r^2 + r_1^2 + r_2^2 + \dots) = \pi\left(r^2 + \frac{r^2}{3^2} + \frac{r^4}{3^4} + \dots\right) = \pi r^2 \frac{9}{8} = \frac{9}{8}\pi r^2.$$

3.AB Let $k_1(O, r)$ and $k_2(O, R)$ be two concentric circles, $r < R$, and let $P \neq O$ be an interior point of k_1 . Half-straight line l with the initial point P intersect the circles k_1 and k_2 in the points A and B respectively. Prove that the length of the segment \overline{AB} has the biggest value if half-straight line l is perpendicular to OP .



Solution: Let us put the circles k_1 and k_2 in the coordinate system with origin of coordinates in O . WLOG if we assume that P is on the y -axis. If $d = \overline{OP}$, then the equation of the half-straight line l is $y = kx + d$, $x \geq 0$ (WLOG if we assume that $x \geq 0$). If $A(x_A, y_A)$ and $B(x_B, y_B)$ then $x_A^2 + y_A^2 = r^2$ and $y_A = kx_A + d$, $x_A \geq 0$. From these two equations it follows that $(1+k^2)x_A^2 + 2kdx_A + d^2 - r^2 = 0$. From $r > d$ it follows $(1+k^2)r^2 - d^2 > 0$, so the equation has

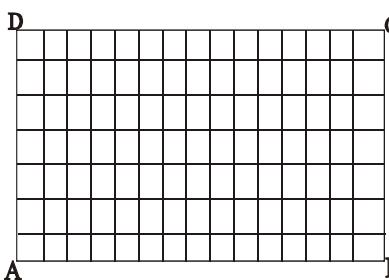
two real solutions. From $r > d$ it follows that $(kd)^2 < r^2 - d^2 + (kr)^2$, i.e.

$(kd)^2 < (1+k^2)r^2 - d^2$. Because $x_A \geq 0$ we consider the

$$\text{solution: } x_A = \frac{-kd + \sqrt{(1+k^2)r^2 - d^2}}{1+k^2}. \text{ Analogously } x_B = \frac{-kd + \sqrt{(1+k^2)R^2 - d^2}}{1+k^2}.$$

Hence $y_A = kx_A + d$, $y_B = kx_B + d$, so the distance between the points $A(x_A, y_A)$ and $B(x_B, y_B)$ is: $D(k) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} = \sqrt{R^2 - r^2}$

$$\sqrt{R^2 - \frac{d^2}{1+k^2}} + \sqrt{r^2 - \frac{d^2}{1+k^2}}$$



This expression reaches its maximal value if the denominator is minimal i.e. if $k = 0$. Hence, l is perpendicular to OP .

4.AB Rectangle $ABCD$ is intersected with m -lines that are parallel to the side AB and with m -lines parallel to the side BC . Determine the number of the rectangles construct in this way.

Solution: There are $m+2$ horizontal lines, including the lines AB and CD , and the same number of the vertical lines. Every two horizontal lines are determining one rectangle. The number of such rectangles is

$$C_{m+2}^2 = \binom{m+2}{2}. \text{ The number of the rectangles determine from the vertical lines is}$$

$$\text{also } C_{m+2}^2 = \binom{m+2}{2}. \text{ Hence, the number of all rectangles is}$$

$$\binom{m+2}{2}^2 = \frac{(m+2)^2(m+1)^2}{4}.$$

**XLIV Republic competition of mathematics for secondary school
Strumica, 31 III-2000 year**
I class

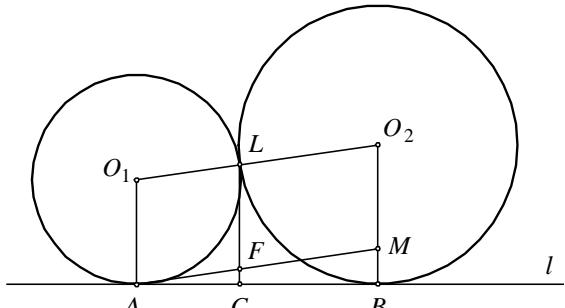
- 1.** Is it possible for the natural numbers from 1 to 1991 to be written in series such that each number can be written twice, so that on the second time writing number k , $k \in \{1, 2, \dots, 1991\}$ will be exactly k places after the first writing?

Solution. Let us assume that this is possible. Then, number $k \in \{1, 2, \dots, 1991\}$ is on the m_k -th and on the $(m_k + k)$ -th position. Then

$$\begin{aligned} 1 + 2 + \dots + 3982 &= (m_1 + (m_1 + 1)) + (m_2 + (m_2 + 2)) + \dots + (m_{1991} + (m_{1991} + 1991)) \\ 7930153 &= 2(m_1 + m_2 + \dots + m_{1991}) + 1983036 \\ 5947117 &= 2(m_1 + m_2 + \dots + m_{1991}) \end{aligned}$$

which is a contradiction. So, that kind of writing is impossible.

- 2.** The circles k_1 and k_2 have an outside common point L (see the picture). The distance from L to the common tangent t of k_1 and k_2 is equal to 1. If r_1 and r_2 are the radii of the k_1 and k_2 respectively then prove that $\frac{1}{r_1} + \frac{1}{r_2} = 2$.



Solution. WLOG let us assume that $r_1 \leq r_2$. Let A and B be the common points of t with the circles k_1 and k_2 respectively. Let $M \in BO_2$ be the point such that $AM \parallel O_1O_2$ and let F be the intersect point of the segments LC and AM . Triangles ΔAFC and ΔAMB are similar so $\overline{FC} : \overline{MB} = \overline{AF} : \overline{AM}$ i.e.

$$\frac{1-r_1}{r_2-r_1} = \frac{r_1}{r_1+r_2}$$

Hence $r_1 + r_2 - r_1^2 - r_1 r_2 = r_1 r_2 - r_1^2$ i.e. $r_1 + r_2 = 2r_1 r_2$.

- 3.** Let a, b, p, q, r, s be the natural numbers such that $qr - ps = 1$ and $\frac{p}{q} < \frac{a}{b} < \frac{r}{s}$.

Prove the inequality: $b \geq q + s$.

Solution. From the inequality $\frac{p}{q} < \frac{a}{b}$ it follows that $aq - bp > 0$. Because $aq - bp \in \mathbb{A}$ it follows that $aq - bp \geq 1$. (1)

From the inequality $\frac{a}{b} < \frac{r}{s}$ it follows that $br - as > 0$. Because $br - as \in \mathbb{A}$ it follows that $br - as \geq 1$. (2)
Then $b = b(qr - ps) = bqr - bps = (bqr - qas) + (qas - bps) = q(br - as) + s(aq - bp)$. From (1) and (2) it follows that $b \geq q + s$.

4. Let P be a point on the bisector of the right angle at the point C . Let l be a line through P and let A and B be the intersect points of the line l and the sides of the right angle. If $a = \overline{AC}$ and $b = \overline{BC}$ then prove that the value of the expression $\frac{1}{a} + \frac{1}{b}$ doesn't depend of the position of the line l .

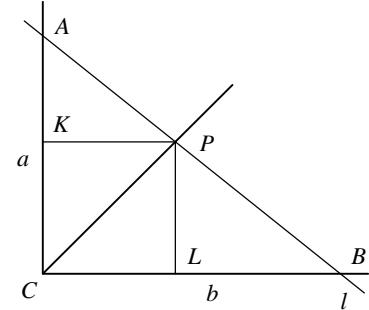
Solution. Let PK and PL be the perpendicular lines through the point P on the sides AC and BC respectively. Then the triangles ΔAKP and ΔPLB are similar, so

$$\frac{\overline{AK}}{\overline{KP}} = \frac{\overline{PL}}{\overline{LB}}$$

i.e. $\frac{a-h}{h} = \frac{h}{b-h}$, where $h = \overline{PK} = \overline{LP}$. From $\frac{a-h}{h} = \frac{h}{b-h}$, it follows that

$$ab = ah + bh \text{ i.e. } \frac{1}{h} = \frac{1}{a} + \frac{1}{b}$$

The fact that the number h doesn't depend of the position of the line l lead us to the conclusion.



II class

1. Determine all complex numbers which satisfy the system of equations: $z^{19}w^{25} = 1$, $z^5w^7 = 1$, $z^4 + w^4 = 2$.

Solution. From $z^5w^7 = 1$ it follows that $z^{15}w^{21} = 1$, so $\frac{z^{19}w^{25}}{z^{15}w^{21}} = 1$ i.e. $z^4w^4 = 1$. From the last

and the third equation of the system we have that z^4 and w^4 are the solutions of the equation $t^2 - 2t + 1 = 0$, i.e. $z^4 = w^4 = 1$. Now $1 = z^5w^7 = zw^3$ i.e. $z = w$. The solutions of the system are $(1,1)$, $(-1,-1)$, (i,i) and $(-i,-i)$.

2. If $f(x) = ax^2 + bx + c$ is a function such that $|f(x)| \leq 1$ for $|x| \leq 1$, then prove that $|a| \leq 2$.

Solution. We have:

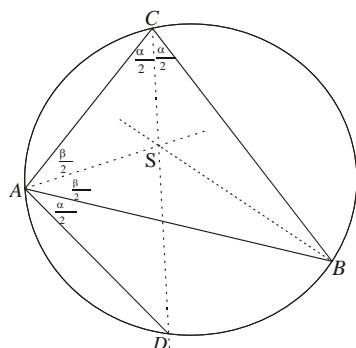
$$f(0) = c \Rightarrow |c| \leq 1,$$

$$|f(1)| = |a + b + c| \leq 1 \text{ and}$$

$$|f(-1)| = |a - b + c| \leq 1.$$

$$|2a| = |(a + b + c) + (a - b + c) - 2c| \leq |a + b + c| + |a - b + c| + 2|c| \leq 1 + 1 + 2 = 4 \Rightarrow |a| \leq 2.$$

3. If S is the incentre of the triangle ABC , and D is the intersect point of the line AS and the circumcircle of the triangle ABC , then prove that $\overline{DB} = \overline{DC} = \overline{DS}$.



Solution. D is the middle point of the arc BC , so $\overline{DB} = \overline{DC}$.

We have: $\angle DBC = \angle DAC = \frac{\alpha}{2}$,

$$\angle DSB = 180^\circ - (\angle SBD + \angle BDS) = 180^\circ - (\angle SBC + \angle CBD + \angle BCA) =$$

$$= 180^\circ - \left(\frac{\alpha}{2} + \frac{\beta}{2} + \gamma \right) = \frac{\alpha + \beta}{2} = \angle SBD. \text{ So, the triangle } \Delta SBD \text{ is isosceles and } \overline{DB} = \overline{DS}.$$

4. Let A be the set of all 7-digit numbers with different digits 1,2,3,4,5,6 and 7. Prove that there are no two numbers in A such that one of them is divisible by the other.

Solution. Let us assume the contrary i.e. let $a, b \in A$, are such that $a = bk$, $k \geq 2$.

From $k \leq \frac{7654321}{1234567} < 7$, it follows that $k \in \{2, 3, 4, 5, 6\}$. From $1+2+3+4+5+6+7=28$, it

follows that each number can be written as $3l+1$, i.e. each number is not divisible by 3, and that's why they are not divisible by 6. So $k \neq 3$ and $k \neq 6$.

I case: $k=2$. If $a=2b$ then $a+b=3b$. But $a+b$ is not divisible by 3, so $a=3s+1$ and $b=3p+1$. So $k \neq 2$.

II case: $k=5$. If $a=5b$, then $a+b=6b$. But $a+b$ is not divisible by 3, so $k \neq 5$.

III case: $k=4$. If $a=4b$, then $a-b=3b$. But $a-b$ is divisible by 9, so $a-b=3m$. Now it follows that $3m=b$ which is impossible. So $k \neq 4$.

III class

1. If $a > 1$, $b > 1$, $c > 1$ then prove the inequality $\log_{abc}^3 a \cdot \log_a b \cdot \log_a c \leq \frac{1}{27}$.

Solution. From $\log_a a = 1$, $\log_a b > 0$, $\log_a c > 0$ it follows that

$$\frac{\log_a a + \log_a b + \log_a c}{3} \geq \sqrt[3]{\log_a a \cdot \log_a b \cdot \log_a c}, \text{ i.e. } \log_a^3 abc \geq 27 \log_a b \cdot \log_a c.$$

2. Let M be an inpoint of the triangle ABC and let $AM \cap BC = \{A_1\}$, $BM \cap CA = \{B_1\}$ and

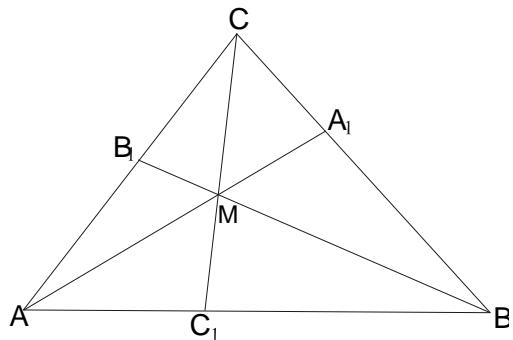
$$CM \cap AB = \{C_1\}. \text{ Prove the inequality } \frac{\overline{AM}}{\overline{A_1M}} + \frac{\overline{BM}}{\overline{B_1M}} + \frac{\overline{CM}}{\overline{C_1M}} \geq 6.$$

Solution. Let S be the area of the triangles ΔABC and S_1, S_2, S_3 be the areas of the

triangles $\Delta MBC, \Delta MCA, \Delta MAB$ respectively. From the fact that the triangles ABC and MBC have the same base it follows that $\overline{AA_1} : \overline{MA_1} = S : S_1 = (S_1 + S_2 + S_3) : S_1$ i.e.

$$\frac{\overline{MA}}{\overline{MA_1}} = \frac{S_2 + S_3}{S_1} = \frac{S_2}{S_1} + \frac{S_3}{S_1}. \text{ We can conclude by the analogy that}$$

$$\frac{\overline{MB}}{\overline{MB_1}} = \frac{S_3 + S_1}{S_2} = \frac{S_3}{S_2} + \frac{S_1}{S_2} \text{ and } \frac{\overline{MC}}{\overline{MC_1}} = \frac{S_1 + S_2}{S_3} = \frac{S_1}{S_3} + \frac{S_2}{S_3}. \text{ So,}$$



$$\frac{\overline{AM}}{\overline{A_1M}} + \frac{\overline{BM}}{\overline{B_1M}} + \frac{\overline{CM}}{\overline{C_1M}} = \left(\frac{S_1}{S_2} + \frac{S_2}{S_1} \right) + \left(\frac{S_2}{S_3} + \frac{S_3}{S_2} \right) + \left(\frac{S_3}{S_1} + \frac{S_1}{S_3} \right) \geq 2\sqrt{\frac{S_1}{S_2} \frac{S_2}{S_1}} + 2\sqrt{\frac{S_2}{S_3} \frac{S_3}{S_2}} + 2\sqrt{\frac{S_3}{S_1} \frac{S_1}{S_3}} = 6$$

3. Prove that if the inequality $a \cos x + b \cos 3x > 1$ has no solution then $|b| \leq 1$.

Solution. If $f(x) = a \cos x + b \cos 3x$ then $f(x) \leq 1$ for every x .

$$f(\pi) = -(a+b) \leq 1, f(0) = a+b \leq 1 \text{ so } |a+b| \leq 1. \text{ On the other hand } f\left(\frac{\pi}{3}\right) = \frac{a}{2} - b \leq 1 \text{ and}$$

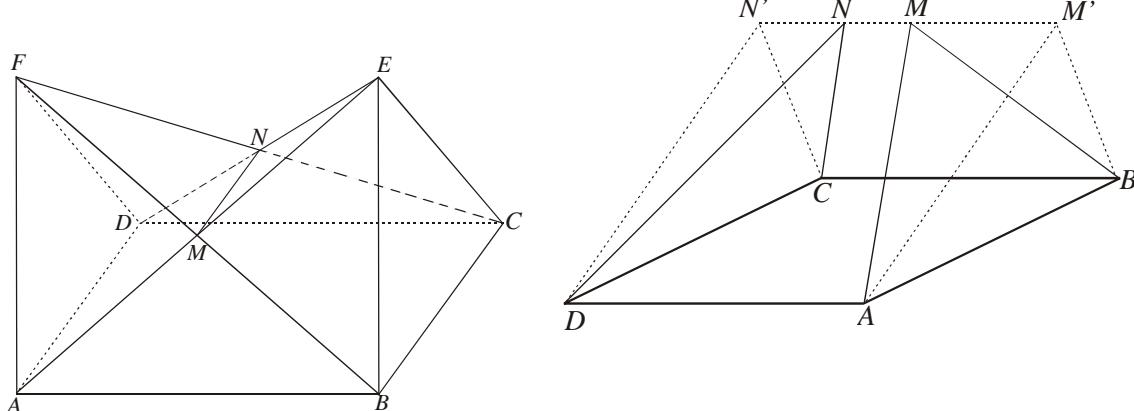
$$f\left(\frac{2\pi}{3}\right) = b - \frac{a}{2} \leq 1, \text{ so } \left| \frac{a}{2} - b \right| \leq 1, \text{ i.e. } |2b-a| \leq 2. \text{ Now}$$

$$|b| = \frac{1}{3}|3b| = \frac{1}{3}|a+b+2b-a| \leq \frac{1}{3}(|a+b| + |2b-a|) \leq \frac{1}{3}(1+2) = 1.$$

4. Two pyramids have a square for a common base. The pyramids are on the same side of the square. Their heights pass through the middle points of the two opposite sides of the square and their length is b . Determine the volume of the figure that is common for the both pyramids.

Solution. Let $ABCDE$ and $ABC'D'E'$ are the given pyramids. From the rectangle $ABEF$ it follows that M is the middle point of the segment AE . Similarly, N is the middle point of the segment CF . So, $\overline{AB} = a$ and the length of the heights in the triangle $\Delta ABM'$ from M' is $\frac{b}{2}$.

On the other hand MN is the median of the triangle ΔADE , so $\overline{MN} = \frac{a}{2}$. Now it follows that



$$\overline{MM'} = \overline{MN} = \frac{a}{4} \text{ and}$$

$$V_{ABCDMN} = V_{ABCDM'N'} - 2V_{CDNN'} = \frac{1}{2}a \cdot \frac{b}{2}a = 2 \cdot \frac{1}{3} \cdot \frac{1}{2}a \cdot \frac{b}{2}a = \frac{a^2b}{4} - \frac{a^2b}{24} = \frac{5}{24}a^2b.$$

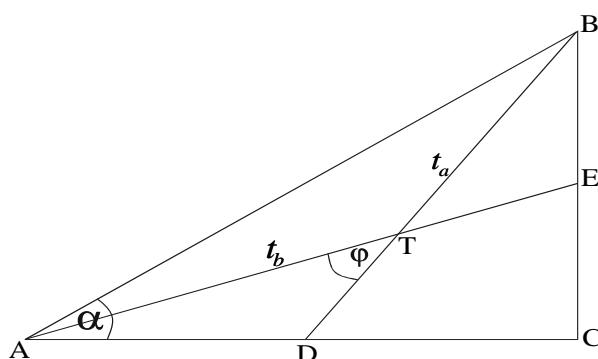
IV class

1. Determine the minimum value of the expression: $\frac{x^2 + y^2 + z^2}{xy + yz}$, if $x > 0, y > 0, z > 0$.

Solution. From $x^2 + \frac{1}{2}y^2 \geq 2\sqrt{\frac{1}{2}x^2y^2} = \sqrt{2}xy$ and $\frac{1}{2}y^2 + z^2 \geq 2\sqrt{\frac{1}{2}y^2z^2} = \sqrt{2}yz$ it follows that

$$\frac{x^2 + y^2 + z^2}{xy + yz} = \frac{\left(x^2 + \frac{1}{2}y^2\right) + \left(\frac{1}{2}y^2 + z^2\right)}{xy + yz} \geq \frac{\sqrt{2}xy + \sqrt{2}yz}{xy + yz} = \sqrt{2}. \text{ For } x=1, y=\sqrt{2}, z=1 \text{ we have}$$

$$\frac{1^2 + \sqrt{2}^2 + 1^2}{1 \cdot \sqrt{2} + \sqrt{2} \cdot 1} = \sqrt{2}. \text{ Now, the minimum value of the expression } \frac{x^2 + y^2 + z^2}{xy + yz} \text{ is } \sqrt{2}.$$



2. Determine a right triangle for which the angle between the median lines to the legs has a maximal value.

Solution. Let $\overline{AC} = b, \overline{BC} = a$. From the cosine theorem for the triangle ΔADT it follows that $2\overline{AT} \cdot \overline{DT} \cdot \cos \varphi = \overline{AT}^2 + \overline{DT}^2 - \overline{AD}^2$, i.e.

$$\frac{4}{9}\overline{AE} \cdot \overline{BD} \cdot \cos \varphi = \left(\frac{2}{3}\overline{AE}\right)^2 + \left(\frac{1}{3}\overline{BD}\right)^2 - \frac{b^2}{4}. \quad (1)$$

If we put $\overline{AE}^2 = b^2 + \frac{a^2}{4}$ and $\overline{BD}^2 = a^2 + \frac{b^2}{4}$ in (1) we have

$$\frac{1}{9}\sqrt{(4b^2 + a^2)(4a^2 + b^2)} \cdot \cos \varphi = \frac{2}{9}(a^2 + b^2), \text{ i.e.}$$

$$\cos \varphi = \frac{2(a^2 + b^2)}{\sqrt{(4b^2 + a^2)(4a^2 + b^2)}} \quad (2)$$

Now,

$$\sin \varphi = \sqrt{1 - \cos^2 \varphi} = \frac{3ab}{\sqrt{(a^2 + 4b^2)(b^2 + 4a^2)}} \quad (3)$$

From (2) and (3) it follows that $\tan \varphi = \frac{3ab}{2c^2}$. If we put $a = c \sin \alpha, b = c \cos \alpha$ in the last equation we have $\tan \varphi = \frac{3}{4} \sin 2\alpha$. Maximum value of $\tan \varphi$ can be reached for $\sin 2\alpha = 1$, i.e. for $\alpha = \frac{\pi}{4}$. So, the triangle we are looking for is isosceles right triangle.

3. Determine all positive values of parameter a , for which all negative solutions x of the equation $\cos((8a-3)x) = \cos((14a+5)x)$ form an increasing arithmetic series.

Solution. Let us write down the equation (1) as $\cos((8a-3)x) - \cos((14a+5)x) = 0$, i.e. $-2 \sin \frac{(8a-3)x + (14a+5)x}{2} \cdot \sin \frac{(8a-3)x - (14a+5)x}{2} = 0$. The given equation is equivalent with the system of equations $\begin{cases} (8a-3)x + (14a+5)x = 2k\pi \\ (8a-3)x - (14a+5)x = 2n\pi \end{cases}$, i.e. with system $\begin{cases} (11a+1)x = k\pi \\ (3a+4)x = n\pi \end{cases}$. From $a > 0$ it follows that $11a+1 > 0$ and $3a+4 > 0$. Hence, $x = \frac{k\pi}{11a+1}$ or $x = \frac{n\pi}{3a+4}$. Let us denote by $x_k = \frac{k\pi}{11a+1}$ and by $x_n = \frac{n\pi}{3a+4}$. From $x \geq 0$ it follows that $n, k = 0, 1, 2, \dots$, so (x_k) and (x_n) are two increasing arithmetic series with first term 0 and differences $d_1 = \frac{\pi}{11a+1}$ and $d_2 = \frac{\pi}{3a+4}$ respectively.

Now we'll prove the following proposition: *Numbers x_k and x_n form an increasing arithmetic series if and only if for $d_1 \leq d_2$ there is $m \in \mathbb{N}$ such that $d_2 = md_1$, or for $d_2 \leq d_1$ there is $m \in \mathbb{N}$ such that $d_1 = md_2$.*

Proof: Let $d_1 \leq d_2$. Then d_1 is the second term of the new arithmetic series (the first one is 0).

From $\frac{\pi}{11a+1} \leq \frac{\pi}{3a+4}$ it follows that the difference of the new arithmetic series is d_1 . But d_2 is the

term of that arithmetic series, so, there is $m \in \mathbb{N}$ such that $d_2 = md_1$.

Contrary, let $m \in \mathbb{N}$ is such that $d_2 = md_1$. Hence, $x_n = d_2 m(n-1)$ and every term of the series (x_n) is the term of the series (x_k) . So, the series we are looking for is (x_k) .

The proof is analogous for $d_2 \leq d_1$.

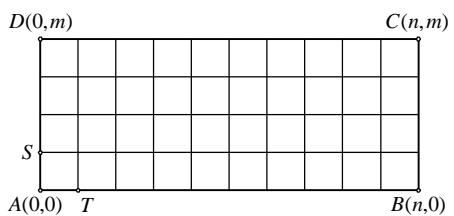
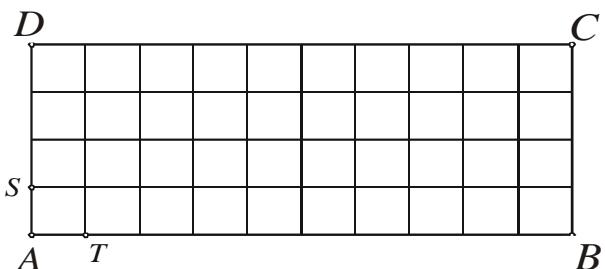
Now, let $d_1 \leq d_2$. There is $m \in \mathbb{N}$ such that $d_2 = md_1$, i.e. $\frac{\pi}{3a+4} = m \frac{\pi}{11a+1}$ from where it follows that $a = \frac{4m-1}{11-3m}$. From $m \in \mathbb{N} \Rightarrow 4m-1 > 0$, $a > 0 \Rightarrow 11-3m > 0$, so, $m < \frac{11}{3}$. Hence, $m \in \{1,2,3\}$

and the appropriate values for a are $\frac{3}{8}, \frac{7}{5}, \frac{11}{3}$.

Now, let $d_2 \leq d_1$. There is $m \in \mathbb{N}$ such that $d_1 = md_2$, i.e. $\frac{\pi}{11a+1} = m \frac{\pi}{3a+4}$, from where it follows that $a = \frac{4-m}{11m-3}$. From $m \in \mathbb{N}$ and $a > 0$ it follows that $11m-3 > 0$ and $4-m > 0$. Hence, $m \in \{1,2,3\}$, and the appropriate values for a are $\frac{3}{8}, \frac{2}{19}, \frac{1}{30}$.

So, $a \in \left\{ \frac{3}{8}, \frac{2}{19}, \frac{1}{30}, \frac{7}{5}, \frac{11}{3} \right\}$.

4. The rectangle $ABCD$ with dimensions m and n where m and n are natural numbers and $n = mk$, is divided by $m \cdot n$ unique squares (see the picture). Every path from the point A to the point C which goes through the divided segments only to the right and up, has length $m+n$. Determine how many times the number of paths from A to C which passes through the point T is bigger than the number of paths from A to C which passes through the point S .



Solution. Let us denote with series of 0-es (for the move to the right for the unique length) and with 1-es (for the move up for the unique length) every path from A to C . Hence, the number of the paths through the point T is $C_{m+n-1}^{n-1} = \binom{m+n-1}{n-1}$ the number of the paths through the point

S is $C_{m+n-1}^{m-1} = \binom{m+n-1}{m-1}$. From $\frac{\binom{m+n-1}{m-1}}{\binom{m+n-1}{n-1}} = k$, it follows that the number of the paths through the point T is k -times bigger than the number of paths through the point S .

8-th Macedonian Mathematical Olympiad

1. Prove that if $m \cdot s = 2000^{2001}$ where $m, s \in \mathbb{A}$ then the equation $mx^2 - sy^2 = 3$ has no solution in \mathbb{A} .

Solution. $mx^2 - sy^2 = 3$, $m \cdot s = 2000^{2001}$. If we multiply the first equation by m then $m^2x^2 = 3m + msy^2$.

Now $m^2x^2 = 3m + msy^2 \equiv 2000^{2001}y^2 \equiv (-1)^{2001}y^2 \equiv -y^2 \pmod{3}$. But

$m^2x^2 \equiv 0 \pmod{3}$ or $m^2x^2 \equiv -1 \pmod{3}$ and $-y^2 \equiv 0 \pmod{3}$ or

$-y^2 \equiv -1 \pmod{3}$, so, we have $m^2x^2 \equiv -y^2 \equiv 0 \pmod{3}$

$\Rightarrow 3 \mid y$, $3 \mid mx \Rightarrow 9 \mid y^2$, $9 \mid (mx)^2$. According to this

$3m = m^2x^2 - msy^2 \Rightarrow 3m \equiv 0 \pmod{9} \Rightarrow 3 \mid m$. But

$m \cdot s = 2000^{2001} \Rightarrow 3 \nmid m$, so the equation has no solution.

2. Does there exist a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $n \geq 2$ holds $f(f(n-1)) = f(n+1) - f(n)$?

Solution. Let $n \in \mathbb{N}$ and $n \geq 2$. For every $i = 2, \dots, n$ holds

$$f(f(i-1)) = f(i+1) - f(i).$$

If we sum all these equations we have

$$\sum_{i=2}^n f(f(i-1)) = \sum_{i=2}^n (f(i+1) - f(i)) = f(n+1) - f(2)$$

i.e. $f(n+1) = f(2) + \sum_{i=2}^n f(f(i-1)) \geq n$ (f is a function on \mathbb{N}).

Now $f(i-1) \geq i-2$, $f(f(i-1)) \geq f(i-2) \geq i-3$, so

$$\begin{aligned} f(n+1) &= f(2) + f(f(1)) + f(f(2)) + \sum_{i=4}^n f(f(i-1)) \geq \\ &\geq 3 + \sum_{i=4}^n (i-3) = 3 + \frac{(n-2)(n-3)}{2}. \end{aligned}$$

Now $f(n+1) \geq \frac{n^2 - 5n + 12}{2} > 2n + 1$ for $n \geq 10$. Let

$f(2n) = M > 4n + 3$. Then

$$f(M) = f(2) + \sum_{i=2}^{M-1} f(f(i-1)),$$

$$f(2n) = f(2) + \sum_{i=2}^{2n-1} f(f(i-1))$$

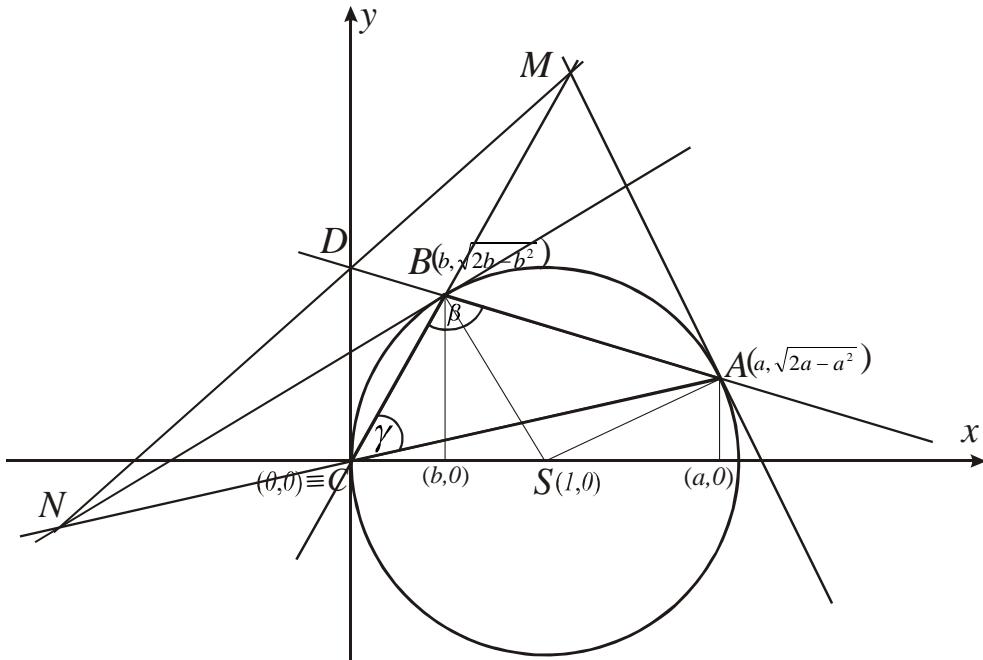
$$f(M) - f(2n) = f(f(2n)) + f(f(2n+1)) + \dots + f(f(M-2)).$$

But $f(f(2n)) = f(M)$ so we have

$$-f(2n) = f(f(2n)) + f(f(2n+1)) + \dots + f(f(M-2))$$

which is not possible. Therefore there is no such function.

3. Let ABC be a triangle with no two sides equal, and let k be an circumcircle around ABC . Let t_A, t_B, t_C are the tangents of the circle k in the points A, B, C , respectively. Prove that $AB \not\parallel t_C$, $AC \not\parallel t_B$, $BC \not\parallel t_A$ and that the points $AB \cap t_C$, $AC \cap t_B$, $BC \cap t_A$ are collinear.



Solution. Let us put the triangle ABC in the coordinate system with C as a origin of coordinates and t_C as y -axis. Then every point of the k satisfy the equation $(x-1)^2 + y^2 = 1$. Let $A(a, \sqrt{2a-a^2})$ and $B(b, \sqrt{2b-b^2})$. Let $A = 2a - a^2$ and $B = 2b - b^2$.

Then the equations of the lines t_A and OB are

$$y - \sqrt{A} = \frac{1-a}{\sqrt{A}}(x-a) \text{ and } y = \frac{\sqrt{B}}{b}x, \text{ respectively. The coordinates of the}$$

intersect point $M(x_M, y_M)$ of these two lines are $x_M = \frac{ab}{D_B}$, $y_M = \frac{a\sqrt{B}}{D_b}$ where $D_b = \sqrt{A}\sqrt{B} - b(1-a)$. Similarly, the coordinates of the intersect point $N(x_N, y_N)$ of the lines AC and t_B are $x_N = \frac{ab}{D_a}$ and $y_N = \frac{b\sqrt{A}}{D_a}$, where $D_a = \sqrt{A}\sqrt{B} - a(1-b)$. If $D(x_D, y_D)$ is the intersect point of the lines AB and t_C then $x_D = 0$ and $y_D = \frac{b\sqrt{A} - a\sqrt{B}}{b-a}$.

The equation of the line MN is $y - y_N = \frac{y_M - y_N}{x_M - x_N}(x - x_N)$ and for $x = 0$ we have

$$y = y_N - \frac{\frac{b\sqrt{A}}{D_a} + \frac{a\sqrt{B}}{D_b}}{\frac{ab}{D_b} - \frac{ab}{D_a}} x_n = \dots = \frac{b\sqrt{A} - a\sqrt{B}}{b-a}.$$

Now, it follows that the line MN intersects t_C in the point D , so, the points M, N and D are collinear.

If $t_A \parallel BC$ then $\angle MAB = \angle CBA$, so

$\angle MAB = 90^\circ - \angle BAS = 90^\circ - (90^\circ - \gamma) = \gamma$. But $\angle CBA = \beta$, so $\beta = \gamma$ i.e. ABC is not a triangle with no two sides equal. Contradiction!.

4. Let M be a finite set and let $\Omega \subseteq \mathcal{P}(M)$ such that:

- (1) If $|A \cap B| \geq 2$ for $A, B \in \Omega$ then $A = B$;
- (2) There are $A, B, C \in \Omega$ such that $A \neq B \neq C \neq A$ and $|A \cap B \cap C| = 1$;
- (3) For every $A \in \Omega$ and for every $a \in M \setminus A$ there is unique $B \in \Omega$ such that $a \in B$ and $A \cap B = \emptyset$.

Prove that there are numbers p and s such that:

- (a) For every $a \in M$ the number of the sets which include the point a is p .
- (b) For every $A \in \Omega$, $|A| = s$;
- (c) $s+1 \geq p$

Solution. (a) Let $x \in M$. Let $\langle x \rangle = \{X \in \Omega \mid x \in X\}$. We'll prove that $(\forall x, y \in M) |\langle x \rangle| = |\langle y \rangle|$. We define the mapping

$$\varphi : \langle x \rangle \rightarrow \langle y \rangle \text{ by } \varphi(T) = \begin{cases} T', & y \in T \\ T, & y \notin T \end{cases}, \text{ where } T' \text{ is the set for which } y \in T'$$

and $T \cap T' = \emptyset$. The set T' exists because of the condition (3) and at the same time φ is bijection. Hence $|\langle x \rangle| = |\langle y \rangle|$, so, if $a \in M$ and $p = |\langle a \rangle|$ it follows that the number of all sets from Ω which include the point a is p .

(b) Let $A, B \in \Omega$ and $A \neq B$.

1° $A \cap B = \emptyset$. Let $a \in A$. Then $a \notin B$. From $|\langle a \rangle| \geq 3$ (because of the condition (2)) it follows that there is $C \in \Omega$ such that $C \cap B \neq \emptyset$ and $C \cap B = \{a'\}$. We define the mapping $\psi : A \rightarrow B$ by $\psi(a) = a'$. There is $D \in \Omega$ such that $D \cap C = \emptyset, D \cap B = \{x'\}$ and $D \cap A = \{x\}$. Now we put $\psi(x) = x'$. ψ is bijection (ψ is onto mapping because of the condition (3)) and that's why $|A| = |B|$.

2° $A \neq B$, $0 < |A \cap B| \leq 1$. Let $A \cap B = \{a\}$. We define mapping $\psi : A \rightarrow B$ by $\psi(a) = a$. From the condition (2) it follows that there is $C \in \Omega$ such that $a \in C, C \neq A, C \neq B$. Let $x \in A$ be an arbitrary element. Because of the condition (3) there is $D, E \in \Omega$ such that $x \in D, D \cap B = \emptyset, D \cap C = \{x''\}, E \cap A = \emptyset$ and $E \cap B = \{x'\}$. We define $\psi(x) = x'$; ψ is bijection. Hence $|A| = |B|$.

Let $A \in \Omega$ be an arbitrary set and let $s = |A|$. Then from 1° and 2° it follows that $(\forall B \in \Omega) |B| = s$.

(c) Let $A \in \Omega$ be an arbitrary set. Because of the (b) we have $|A| = s$. Let $a \notin A$. From (a) it follows that $|\langle a \rangle| = p$. Because of the condition (3) there is $B \in \Omega$ such that $B \cap A = \emptyset$ and $a \in B$. For every $C \in \langle a \rangle, C \cap A \neq \emptyset$ (because of the condition (1)). Because $|\langle a \rangle| = p$ there are $p - 1$ sets C with such conditions. Because $|A| = s$ we have $p - 1 \leq s$ i.e. $p \leq s + 1$.

**БАЛКАНСКА МАТЕМАТИЧКА ОЛИМПИЈАДА
МОЛДАВИЈА, 2000**

- 1. (Албанија).** Да се најдат сите функции $f : \mathbf{R} \rightarrow \mathbf{R}$ такви, што

$$f(xf(x) + f(y)) = (f(x))^2 + y, \quad (1)$$

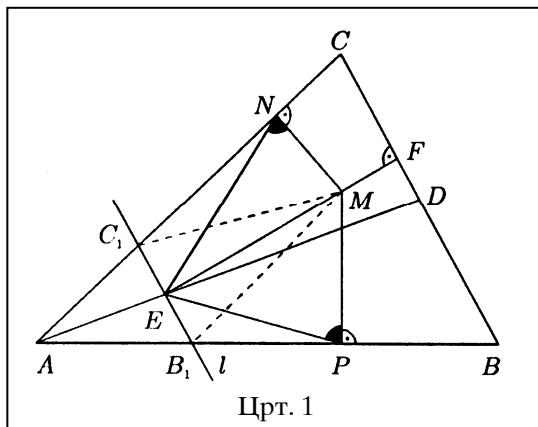
за секои $x, y \in \mathbf{R}$.

Решение. Нека функцијата $f(x)$ ги задоволува условите на задачата. Ако $f(0) = a$, тогаш ставајќи $x = 0$ во (1) добиваме $f(f(y)) = a^2 + y$, за секој $y \in \mathbf{R}$. Понатаму, во последното равенство ставаме $y = -a^2$ и добиваме $f(f(-a^2)) = 0$, т.е. за $b = f(-a^2)$ важи $f(b) = 0$. Сега, од равенството (1) при $x = b$ добиваме $f(f(y)) = y$, за секој $y \in \mathbf{R}$. Повторно од (1), ако x го замениме со $f(x)$ добиваме $f(xf(x) + f(y)) = x^2 + y$, па затоа $(f(x))^2 + y = x^2 + y$, т.е. $(f(x))^2 = x^2$, за секој $x \in \mathbf{R}$. Според тоа, $f(1) = \pm 1$.

- Ако $f(1) = 1$, тогаш $f(1+f(y)) = 1+y$ за секој $y \in \mathbf{R}$, па затоа $(f(1+f(y)))^2 = (1+y)^2$ односно $(1+f(y))^2 = (1+y)^2$. Конечно, $f(y) = y$, за секој $y \in \mathbf{R}$.
- Ако $f(1) = -1$, тогаш $f(-1+f(y)) = -1+y$ за секој $y \in \mathbf{R}$, па затоа $(f(-1+f(y)))^2 = (1+y)^2$ односно $(-1+f(y))^2 = (1+y)^2$. Конечно, $f(y) = -y$, за секој $y \in \mathbf{R}$.

Според тоа, добивме две решенија $f(x) = x$ и $f(x) = -x$. Непосредно се проверува дека овие функции ја задоволуваат равенката (1).

- 2. (Македонија).** Во разностран остроаголен триаголник ABC точката E е внатрешна точка за тежишната линија AD , ($D \in BC$). Точката F е ортогонална проекција на E врз правата BC . Нека M е внатрешна точка на за отсечката EF , а точките N и P се ортогоналните проекции на точката M врз правите AC и AB , соодветно. Докажи, дека симетралите на аглиите PMN и PEN се паралелни.



Решение. Ќе докажеме дека $\angle ENM = \angle EPM$, (црт. 1). Нека l е права низ E паралелна на BC . Со B_1 и C_1 да ги означиме пресечните точки на правата l со правите AB и AC , соодветно. Јасно, E е средина на

B_1C_1 . Освен тоа $EF \perp BC$, па затоа $EF \perp B_1C_1$, што значи дека ΔB_1C_1M е рамнокрак. Според тоа $\angle B_1C_1M = \angle MB_1C_1$. Но, $\angle MNC_1 = \angle MEC_1 = 90^\circ$ па затоа точките E, M, N и C_1 лежат на иста кружница, од што следува дека $\angle B_1C_1M = \angle ENM$. Аналогно, $\angle MPB_1 = \angle MEB_1 = 90^\circ$, т.е. точките E, M, P и B_1 лежат на иста кружница, од што следува $\angle EPM = \angle MB_1C_1$. Сега да го разгледаме читириаголникот $EPMN$. Аглите PMN и PEN ќе ги поистоветуваме со соодветните внатрешни агли на четириаголникот и за расудувањата, кои следуваат, не е важно дали овој четириаголник е конвексен или не е. Расудувањата остануваат во сила и кога четириаголникот се дегенерира во триаголник, т.е. кога E лежи на PN . За симетралата на $\angle PEN$ можни се два случаи: таа минува низ точката M или сече некоја од отсечките MP или MN . Ако симетралата минува низ M , тогаш $\Delta EMN \cong \Delta EMP$, од што следува дека $\overline{MN} = \overline{MP}$. Но, тогаш $\Delta MNC_1 \cong \Delta MPB_1$, па затоа $\angle MC_1N = \angle MB_1P$. Но, тоа значи дека $\angle AC_1E = \angle AB_1E$, т.е. $\angle ACB = \angle ACB$, што не е можно бидејќи ΔABC е разностран. Ако симетралата на $\angle PEN$ сече некоја од отсечките MP или MN , на пример MP во точка Q , тогаш од

$$\angle PMN = 360^\circ - 2\angle PEQ - 2\angle MPE$$

следува дека

$$\angle EQP = 180^\circ - (\angle PEQ + \angle MPE) = \frac{1}{2} \angle PMN$$

што значи дека симетралите на аглите PEN и PMN .

- 3. (Југославија).** Да се најде максималниот број на правоаголници со страни 1 и $10\sqrt{2}$ кои може да се исечат од правоаголник со страни 50 и 90 со сечења, паралелни на страните на почетниот правоаголник.

Решение. Правоаголникот $ABCD$ да го поставиме во координатен систем, така што неговите темиња да имаат координати $A(0,0), B(0,50), C(90,50), D(90,0)$. Прво ќе докажеме како од правоаголникот може да се исечат 315 правоаголници со страни 1 и $10\sqrt{2}$. Бидејќи $90 > 6 \cdot 10\sqrt{2}$. Можеме да го поделиме $ABCD$ на два правоаголника со димензии $50 \times 60\sqrt{2}$ и $(90 - 60\sqrt{2}) \times 50$. Од првиот четириаголник со хоризонтални прави на растојание 1 една од друга можеме да исечиме 50 ленти со димензија $1 \times 60\sqrt{2}$, после што од секоја лента можеме да исечиме по шест правоаголници со димензии $1 \times 10\sqrt{2}$. Според тоа, првиот правоаголник ќе го поделиме на $50 \cdot 6 = 300$ правоаголници со димензии $1 \times 10\sqrt{2}$. Бидејќи $90 - 60\sqrt{2} > 5$ и $50 > 3 \cdot 10\sqrt{2}$ од вториот правоаголник можеме да исечиме правоаголник со димензии $5 \times 30\sqrt{2}$ од кој потоа сечиме $5 \cdot 3 = 15$ правоаголници со димензии $1 \times 10\sqrt{2}$, што значи дека исековме 315 правоаголници со димензии $1 \times 10\sqrt{2}$.

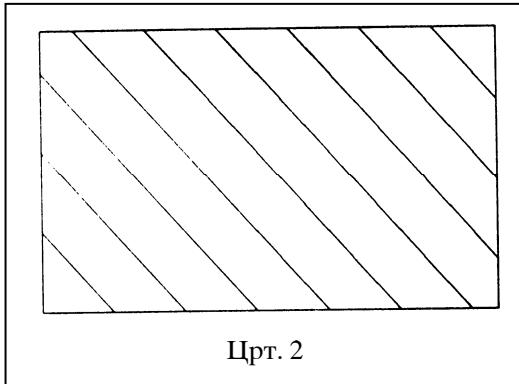
Ќе докажеме дека 315 е максималниот број правоаголници со димнзии $1 \times 10\sqrt{2}$ кои можеме да ги исечиме од дадениот правоаголник со димензии 50×90 . Да ги разгледаме пресеците на правите со равенки

$$L_1 : x + y = 10\sqrt{2}, L_2 : x + y = 20\sqrt{2}, \dots, L_9 : x + y = 90\sqrt{2}$$

со правоаголникот $ABCD$.

Да забележиме, дека од $100\sqrt{2} - 90 > 50$ следува дека правата чија равенка е $x + y = 100\sqrt{2}$ не го сече правоаголникот $ABCD$.

Бидејќи секоја од горните прави зафаќа агол од 45° со координатните оски, не е тешко да се докаже дека збирот на отсечките од овие прави, кои лежат во право-



Црт. 2

аголник $1 \times 10\sqrt{2}$ со страни паралелни на координатните оски, е $\sqrt{2}$. Да ја означиме должината на отсечката од правата L_i , која лежи во внатрешноста на правоаголникот $ABCD$ со l_i . Со елементарни пресметувања наоѓаме дека

$$l_1 = 20, l_2 = 40, l_3 = 60, l_4 = l_5 = l_6 = 50\sqrt{2},$$

$$l_7 = 140\sqrt{2} - 140, l_8 = 140\sqrt{2} - 160, l_9 = 140\sqrt{2} - 180$$

Со собирање наоѓаме, дека вкупната должина во внатрешноста на правоаголникот $ABCD$ е еднаква на $570\sqrt{2} - 360$. Бидејќи еден правоаголник со димензии $1 \times 10\sqrt{2}$ во себе содржи отсечка со вкупна должина $\sqrt{2}$, ако имаме t правоаголници, тогаш $t\sqrt{2} \leq 570\sqrt{2} - 360$, што значи дека $t \leq [\frac{570\sqrt{2} - 360}{\sqrt{2}}] = 315$, што и требаше да се докаже.

- 4. (Румунија).** Еден природен број r го нарекуваме “степен”, ако $r = t^s$, каде $t \geq 2, s \geq 2$ се природни броеви. Докажи, дека за секој природен број n постои множество A од природни броеви, кое ги задоволува следните услови:

- a) A има n елементи;
- b) секој елемент од A е “степен”; и
- c) за секои $r_1, r_2, \dots, r_k \in A, 2 \leq k \leq n$ бројот $\frac{r_1+r_2+\dots+r_k}{k}$ е “степен”.

Решение. Прво ќе ја докажеме следната

Лема. За секој $m \in \mathbf{N}$ постои таков $d \in \mathbf{N}$, што сите броеви од множеството $\{d, 2d, \dots, md\}$ се степени.

Доказ. Нека p_1, p_2, \dots, p_k се сите прости делители на $m!$, а q_1, q_2, \dots, q_m се првите m прости броеви. Ќе бараме d од облик $p_1^{x_1} p_2^{x_2} \dots p_k^{x_k}$ така,

што dl да е точно q_l –ти степен на $l = 1, 2, \dots, m$. Ако го запишеме l во обликот $p_1^{\alpha_1(l)} p_2^{\alpha_2(l)} \dots p_k^{\alpha_k(l)}$ добиваме, дека треба да е $x_i + \alpha_i(l) \equiv 0 \pmod{q_l}$, $i = 1, 2, \dots, k$ т.е. секој x_i треба да го задоволува условот

$$x_i + \alpha_i(l) \equiv 0 \pmod{q_l}, \text{ за } l = 1, 2, \dots, m.$$

Бидејќи броевите q_1, q_2, \dots, q_m се по парови заемно прости, од кинеската теорема за остатоци следува дека последниот систем има решение, што значи дека лемата е докажана. ♦

Сега решението на задачата следува ако во претходната лема ставиме $m = n! \frac{n(n+1)}{2}$ и земеме $r_j = jn!d$, за $j = 1, 2, \dots, n$. Очигледно аритметичката средина на било кои k од дадените броеви е од облик id , каде

$$i < n! \frac{n(n+1)}{2} = m$$

па затоа таа е степен.

18th BALKAN MATHEMATICAL OLYMPIAD
Belgrade. Yugoslavia, May 5, 2001

- Let n be a positive integer. Show that if a and b are integers greater than 1 such that $2^n - 1 = ab$, then the number $ab - (a - b) - 1$ is of the form $k \cdot 2^{2m}$, where k is odd and m is a positive integer.

Solution A. The number T can be written as $ab - a + b - 1 = (a+1)(b-1)$, and therefore it suffices to find the form of the numbers $a+1$ and $b-1$.

We observe the following cases:

Case 1: If $n = 2s$, $s \in \mathbb{N}$, we have: $2^n - 1 = 2^{2s} - 1 = (2^s - 1)(2^s + 1) = ab$ and therefore we can set $a = 2^s - 1$, $b = 2^s + 1$ and we will have $a+1 = 1 \cdot 2^s$, $b-1 = 1 \cdot 2^s$, therefore $T = (a+1)(b-1) = 1 \cdot 2^{2s}$, $m = s$ and $k = 1$.

Case 2: If $n = (2t+1)(2r+1)$; $t, r \in \mathbb{N}$ (odd mixed), we have

$$ab = (2^{2t+1})^{2r+1} - 1 = (2^{2t+1} - 1)((2^{2t+1})^{2r} + (2^{2t+1})^{2r-1} + \dots + (2^{2t+1}) + 1),$$

and therefore we have $a = 2^{2t+1} - 1$, so $a+1 = 2^{2t+1}$ and

$$b = (2^{2t+1})^{2r} + (2^{2t+1})^{2r-1} + \dots + (2^{2t+1}) + 1$$

implies $b-1 = 2^{2t+1}(2z+1)$, $z \in \mathbb{N}$, and so $T = (a+1)(b-1) = (2z+1)2^{2(2t+1)}$, with $k = 2z+1$ and $m = 2t+1$.

Case 3: If $n = p$ where p is an odd prime, because $2^n - 1$ is odd it implies that a, b are odd, and let $a = 2L - 1$ and $B = 2U + 1$. The numbers L and U cannot be of the form $L = 2^{d-1}$ and $U = 2^{f-1}$, since:

- (i) If $L = 2^{d-1}$ then $a = 2^d - 1$ and since $a | (2^n - 1)$ we must have $n = \text{multiple of } d$; contradiction, since n is prime;
- (ii) If $U = 2^{f-1}$ then $b = 2^f + 1$ and since $b | (2^n - 1)$ we must have $n = \text{even multiple of } f$; contradiction, since n is prime.

Therefore we have that the numbers L and U are of the form: $L = 2^{d-1}(2h+1)$ and $U = 2^{f-1}(2v+1)$ and therefore $a+1 = 2^d(2h+1)$ and $b-1 = 2^f(2v+1)$. Now it suffices to show $d = f$.

We have $2^n - 1 = ab = (2^d(2h+1) - 1)(2^f(2v+1) + 1)$ or

$$2^n = 2^{d+f}(2h+1)(2v+1) + 2^d(2h+1) - 2^f(2v+1).$$

If $d > f$, dividing by 2^d the last relation we have even number equal to odd number; contradiction. Similarly, if $d < f$, dividing by 2^f we arrive to a contradiction; therefore $d = f$ and so $T = (a+1)(b-1) = 2^{2d}(2h+1)(2v+1)$, $m = d$ and $k = (2h+1)(2v+1)$.

Solution B. Denote by $\deg_2 x$ the greatest $k \in \mathbb{N}$ such that $2^k | x$. Then $\deg_2 T = \deg_2((a+1)(b-1)) = \deg_2((a+1)(b-1)a)$, since a is odd. Further, we have

$$\deg_2 T = \deg_2((a+1)(ab-a)) = \deg_2((a+1)(2^n - 1 - a)) = \deg_2(c(2^n - c)),$$

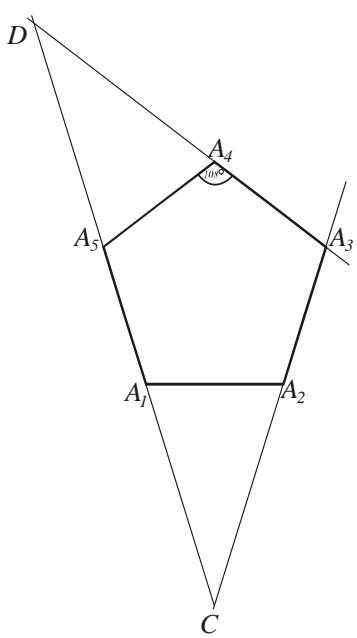
where $c = a+1$ is even, and $c < 2^n$. Let $\deg_2 c = k$. Then $\deg_2(2^n - c) = k$, and $\deg_2 T = \deg_2 c + \deg_2(2^n - c) = 2k$.

- Prove that if a convex pentagon satisfies the following conditions:

- (1) all interior angles are congruent; and
- (2) the lengths of all sides are rational numbers,

then it is a regular pentagon.

Solution. Let the pentagon $A_1A_2A_3A_4A_5$ have the mentioned properties. Then



the interior angles are equal to 108° . Let A_1A_5 intersect A_2A_3 and A_3A_4 in C and D respectively. Then the triangles A_1CA_2 , A_4DA_5 and CA_3D are isosceles. Therefore, $\overline{CA_3} = \overline{A_3D}$ and $2\overline{CA_2} \cos 72^\circ = \overline{A_1A_2}$, $2\overline{DA_4} \cos 72^\circ = \overline{A_4A_5}$. Hence, $\overline{CA_2} + \overline{A_2A_3} = \overline{A_3A_4} + \overline{A_4D}$. Therefore, $2\cos 72^\circ(\overline{A_2A_3} - \overline{A_3A_4}) = \overline{A_4A_5} - \overline{A_1A_2}$. If $\overline{A_2A_3} \neq \overline{A_3A_4}$, then it implies that $\cos 72^\circ$ is a rational number, false. (In fact, $4\cos 72^\circ \cos 36^\circ = 1$ and the number $a = \sin 18^\circ = \cos 72^\circ$ is a solution of the equation $4x(1-2x^2) = 1$ which is equivalent to $(2x-1)(4x^2+2x-1) = 0$. As far as $a \neq \frac{1}{2}$, the number a is irrational.)

Therefore, $\overline{A_2A_3} = \overline{A_3A_4}$ and $\overline{A_1A_2} = \overline{A_4A_5}$. Analogously, we can prove that all the sides of the pentagon are congruent. Hence, the pentagon is regular.

3. Let a, b, c be positive real numbers such that $a + b + c \geq abc$. Prove that $a^2 + b^2 + c^2 \geq \sqrt{3}abc$.

Solution. Assume by contradiction that $a^2 + b^2 + c^2 < \sqrt{3}abc$. Then $3\sqrt[3]{a^2b^2c^2} \leq a^2 + b^2 + c^2 < \sqrt{3}abc$. It follows that $abc > 3\sqrt{3}$. On the other hand we have: $\frac{a^2b^2c^2}{3} \leq \frac{(a+b+c)^2}{3} \leq a^2 + b^2 + c^2 < \sqrt{3}abc$, and obtain $abc < 3\sqrt{3}$; this is a contradiction.

4. A cube of dimensions $3 \times 3 \times 3$ is divided into 27 congruent unit cubical cells. One of these cells is empty and the others are filled with unit cubes labeled in an arbitrary manner with numbers 1, 2, ..., 26. An *admissible move* is the moving of a unit cube into an adjacent empty cell. Is there a finite sequence of admissible moves after which the unit cube labeled with k and the unit cube labeled with $27-k$ are interchanged, for each $k = 1, 2, \dots, 13$? (Two cells are said to be adjacent if they share a common face)

Solution. Colour the cells black or white, so that adjacent cells have different colours. Numerate the cells 1, 2, ..., 27 so that consecutive numbers correspond to adjacent cells (e.g. [I] upper level 1–9 ordered: row 1, from left to right; then row 2, from right to left; then row 3, from left to right. [II] middle level 10–18, where cell 10 is below cell 9, and the level is ordered similarly. [III] lower level 19–27, ordered similarly with cell 19 below cell 18).

Call the pair of cubes $\{m, n\}$ *inverted* if the cube with the greater number is in the cell with the less. Denote by $I(k)$ -th total amount of inverted unordered pairs of

cubes after $k > 0$ moves. A moved cube is always placed in a cell with a different colour. Then in the ordering of the cells, it passes even number (possibly 0) of other cubes. Passing a cube changes $I(\cdot)$ by one. Thus the moves preserve the parity of $I(\cdot)$. Suppose that we have reordered the cubes (as needed) in z moves. 26 cubes form $25 \cdot 13 = 325$ unordered pairs. A pair of cubes in two cells is inverted in the final ordering if and only if the pair of cubes in the same two cells has not been inverted in the initial ordering. Thus $I(z) = 325 - I(0)$ contradicting the equal parity of $I(z)$ and $I(0)$. So, it is impossible to reorder the cubes that way.

**МЕЃУНАРОДНА МАТЕМАТИЧКА ОЛИМПИЈАДА
Ј. КОРЕЈА, 2000**

- 1. (Русија).** Кружниците Γ_1 и Γ_2 се сечат во точките M и N . Правата l е заедничка тангента на Γ_1 и Γ_2 така, што точката M е поблиску до l од точката N . Правата l ги допира кружниците Γ_1 и Γ_2 во точките A и B , соодветно. Правата низ M паралелна со l по втор пат ги сече кружниците Γ_1 и Γ_2 во точките C и D , соодветно. Правите CA и DB се сечат во точката E , правите AN и CD во точката P и правите BN и CD во точката Q . Докажи дека $\overline{EP} = \overline{EQ}$.

Решение. Нека K е пресечната точка на AB и MN , направи цртеж. Од $\overline{AK}^2 = \overline{KM} \cdot \overline{KN} = \overline{BK}^2$ следува дека $\overline{AK} = \overline{BK}$ и како $PQ \parallel AB$, заклучуваме, дека M е средина на отсечката PQ . Понатаму, доволно е да докажеме, дека $EM \perp PQ$, од каде ќе следува дека ΔPQE е рамнокрак. За триаголниците ABM и ABE страната AB е заедничка, $\angle BAE = \angle MCA$, $(AB \parallel CD)$, $\angle MCA = \angle MAB$, (зашто?), т.е. $\angle BAE = \angle MAB$ и аналогно $\angle ABE = \angle MBA$. Според тоа, $\Delta ABM \cong \Delta ABE$, па затоа $EM \perp AB$ и како $AB \parallel PQ$ добиваме дека $EM \perp PQ$. Но M е средина на отсечката PQ , па затоа ΔPQE е рамнокрак, т.е. $\overline{EP} = \overline{EQ}$.

- 2. (САД).** Нека a, b се позитивни реални броеви такви што $abc = 1$. Докажи дека

$$(a - 1 + \frac{1}{b})(b - 1 + \frac{1}{c})(c - 1 + \frac{1}{a}) \leq 1.$$

Решение. Заради условот $abc = 1$ даденото неравенство може да се трансформира така, што ќе се искористат смените $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$, каде $x, y, z > 0$ и притоа ќе се добие неравенството

$$(x - y + z)(y - z + x)(z - x + y) \leq xyz.$$

Добиеното неравенство е парцијален случај на неравенството на Шур при $\lambda = 1$, кое за секои $x, y, z > 0$ и секој реален број λ гласи

$$(x - y)(x - z)x^\lambda + (y - z)(y - x)y^\lambda + (z - x)(z - y)z^\lambda \geq 0.$$

- 3. (Белорусија).** Нека $n \geq 2$ е даден природен број и нека на хоризонтална права се распоредени n болви така, што не сите се наоѓаат во една точка.

За позитивен реален број λ да дефинираме скок на следниот начин: Се избираат две болви, кои се наоѓаат во произволни точки A и B така, што A е лево од B , и болвата од A скока во точката C , која е на правата десно од B и така, што $\frac{\overline{BC}}{\overline{AB}} = \lambda$.

Да се најдат сите вредности на λ така, што за секоја точка M од правата и за произволен распоред на n -те болви постои конечна низа од скокови, после која сите болви се наоѓаат десно од точката M .

Решение. Нека после k -от скок имаме конфигурација, во која растојанието меѓу најлевата и најдесната болва го означуваме со d_k , а најмалото растојание меѓу соседните болви го означуваме со δ_k . Јасно, $d_k \geq (n-1)\delta_k$. Ја избирааме следната стратегија: во секој чекор најлевата болва ја прескокнува најдесната. Тогаш при $(k+1)$ -от чекор се појавува ново растојание меѓу соседните болви и тоа λd_k . Ако $\delta_{k+1} = \lambda d_k$, тогаш $\delta_{k+1} \geq \lambda(n-1)\delta_k$, а ако $\delta_{k+1} \neq \lambda d_k$, тогаш $\delta_{k+1} \geq \delta_k$. Да земеме $\lambda \geq \frac{1}{n-1}$. Тогаш, во секој случај $\delta_{k+1} \geq \delta_k$, за секој k и низата од најмалите растојанија е неопаѓачка. Тоа значи, дека во секој чекор најлевата болва се преместува во однос на најдесната на растојание кое не е помало од дадена константа, што значи дека при избраната стратегија сите болви ќе се преместат во десно на онаа далечина на која ќе посакаме.

Ќе покажеме обратно, дека при $\lambda < \frac{1}{n-1}$, како и да прескокнуваат болвите, тие не може да се најдат десно од претходно избрана точка M . Положбата на болвите ќе ја определиме со координатите во однос на координатен систем на дадената права. Нека s_k е збирот на координатите на n -те болви после k -от скок, а ω_k е координатата на најдесната болва после k -от скок. Јасно, $s_k \leq n\omega_k$. Ќе докажеме дека низата $\{\omega_k\}$ е ограничена.

Нека при $(k+1)$ -от скок болвата од точката A ја прескокнува болвата во точката B и паѓа на точката C . Координатите на точките A, B и C да ги означиме со a, b и c , соодветно. Тогаш, $s_{k+1} = s_k + c - a$, а според правилото за прескокнување имаме $c - b = \lambda(b - a)$, што е еквивалентно на $\lambda(c - a) = (1 + \lambda)(c - b)$. Според тоа, $s_{k+1} - s_k = c - a = \frac{\lambda}{1+\lambda}(c - b)$. Нека $c > \omega_k$. Тогаш, $\omega_{k+1} = c$ и како $b \leq \omega_k$ добиваме

$$s_{k+1} - s_k = \frac{\lambda}{1+\lambda}(c - b) \geq \frac{\lambda}{1+\lambda}(\omega_{k+1} - \omega_k).$$

Последното неравенство е точно и кога $c \leq \omega_k$, бидејќи притоа $\omega_{k+1} = \omega_k$ и $s_{k+1} - s_k = c - a > 0$. Сега да ја разгледаме низата

$$z_k = \frac{1+\lambda}{\lambda} \omega_k - s_k, k = 0, 1, 2, \dots.$$

Од претходната оценка следува, дека $z_{k+1} - z_k \leq 0$, т.е. разгледуваната низа не расте и затоа $z_k \leq z_0$, за секој k . Од $\lambda < \frac{1}{n-1}$ следува дека $1 + \lambda > n\lambda$, па затоа $z_k = (n + \mu)\omega_k - s_k$, каде $\mu = \frac{1+\lambda}{\lambda} - n > 0$. Според тоа, $z_k = \mu\omega_k + (n\omega_k - s_k) \geq \mu\omega_k$ односно $\omega_k \leq \frac{z_k}{\mu} \leq \frac{z_0}{\mu}$, за секој k . Значи, $\{\omega_k\}$ е ограничена и притоа нејзината ограниченост е докажана без разлика на избраната стратегија на прескокнување.

4. **(Унгарија).** Маѓионичар располага со 100 карти, нумериирани со броевите од 1 до 100. Тој ги става картите во три кутии: црвена, бела и сина така, што во секоја кутија има барем по една карта.

Човек од публиката избира две од трите кутии, вади од кутиите по една карта и го соопштува збирот на броевите од извадените карти. Дознавајќи го збирот, маѓионичарот ја определува кутијата, од која не е извадена карта.

На колку различни начини може да се распоредат сите карти во кутиите така, што маѓионичарот секогаш да е успешен?

(Два распореди се сметаат за различни, ако барем една карта е во различни кутии.)

Решение. Да ги означиме трите кутии со A, B и C и да ги разгледаме распоредите при кои маѓионичарот е успешен.

Прв случај. Постои i таков што $i, i+1, i+2$ се во различни кутии. Нека $i \in A, i+1 \in B$ и $i+2 \in C$. Од равенството $i+(i+3) = (i+1)+(i+2)$ следува дека треба $i+3 \in A$. Аналогно, $i+4 \in B$ и $i+5 \in C$. Според тоа, ако $1 \in A, 2 \in B$ и $3 \in C$, тогаш A ги содржи броевите од облик $3k+1$, B ги содржи броевите од облик $3k+2$ и C ги содржи броевите од облик $3k$. Јасно, при ваков распоред маѓионичарот е успешен и притоа имаме вкупно $3! = 6$ различни распореди.

Втор случај. Нека не постојат три последователни броеви кои се во различни кутии. Без ограничување на општоста можеме да земеме дека $1 \in A$ и нека i е најмалиот број кој не е во A . Без ограничување на општоста можеме да земеме дека $i \in B$. Нека најмалиот број во C е k . Бидејќи $i-1 \in A$ и $i \in B$, според претпоставката не смее $i+1 \in C$. Ќе докажеме, дека $k = 100$. Да претпоставиме, дека $k < 100$. Од равенството $i+k = (i-1)+(k+1)$ следува дека $k+1 \in A$. Но, $i+(k+1) = (i+1)+k$, па затоа $i+1 \in C$, што е противречност. Според тоа, $k = 100$. Ќе докажеме, дека секој $t = 2, 3, \dots, 99$ е во B . Нека претпоставиме дека постои $t \in \{2, 3, \dots, 99\}$ таков што $t \in A$. Сега од равенството $t+99 = (t-1)+100$ следува дека $t-1 \in C$, што е противречност. Лесно се проверува, дека при ваквиот распоред маѓионичарот е успешен. Јасно и во овој случај имаме вкупно $3! = 6$ распореди.

Конечно, вкупниот број на распореди при кои маѓионичарот е успешен е 12.

5. **(Русија).** Дали постои природен број n така, што n да има точно 2000 различни прости делители и бројот $2^n + 1$ да се дели со n ?

Решение. Прво ќе ја докажеме следната лема.

Лема. За секој природен број $n > 2$ постои прост број p таков што $p | n^3 + 1$, но $p \nmid n+1$.

Доказ. Нека претпоставиме дека постои $n > 2$ за кој тврдењето не е точно. Бидејќи $n^3 + 1 = (n+1)(n^2 - n + 1)$, тоа значи, дека секој прост број p , кој е делител на $n^2 - n + 1$ е делител и на $n+1$. Но,

$$n^2 - n + 1 = (n+1)(n-2) + 3$$

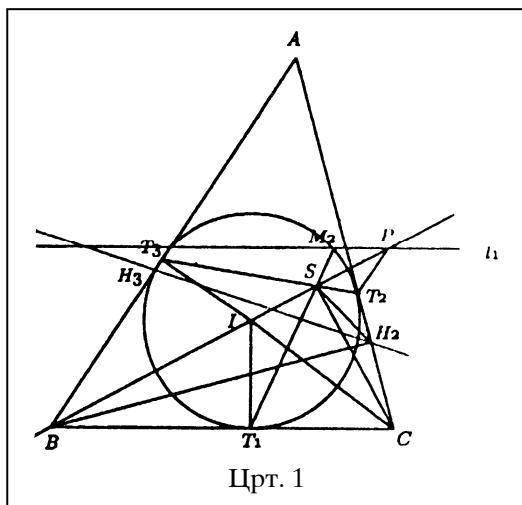
пз затоа $p | 3$, т.е. $p = 3$. Според тоа, $3 | n+1$ и како $n+1 \equiv n-2 \pmod{3}$ добиваме дека $3 | n-2$. Според тоа, $9 \nmid n^2 - n + 1$ и како $n^2 - n + 1$ е степен

на 3 заклучуваме дека $n^2 - n + 1 = 3$, што не е можно бидејќи за секој $n > 2$ важи $n^2 - n + 1 > 3$. ♦

Да се вратиме на задачата. Ќе докажеме поопшто тврдење, дека за секој природен број k постои n со k прости делители така, што $n \mid 2^n + 1$. Доказот ќе го спроведеме со индукција по k .

За $k=1$ доволно е да земеме $n=3$. Понатаму да допуштиме, дека за некој $k \geq 1$ постои број n од облик $n=3^l t$, каде $l \geq 1$ и $3 \nmid t$, таков, што $n \mid 2^n + 1$. Но, тогаш n е непарен број па затоа $3 \mid 2^{2n} - 2^n + 1$. Од друга страна $2^{3n} + 1 = (2^n + 1)(2^{2n} - 2^n + 1)$ и затоа $3 \mid 2^{3n} + 1$. Од претходно доказаната лема следува дека постои прост број p таков, што $p \mid 2^{3n} + 1$ и $p \nmid 2^n + 1$. Сега бројот $3pn$ ги исполнува барањата за $k+1$, со што индуктивниот доказ е завршен.

6. **(Русија).** Нека AH_1, BH_2, CH_3 се висините во остроаголниот триаголник ABC . Вписаната кружница во триаголникот ABC ги допира неговите страни BC, CA, AB во точките T_1, T_2, T_3 , соодветно. Правите l_1, l_2, l_3 се симетрични слики на правите H_2H_3, H_3H_1, H_1H_2 во донос на правите T_2T_3, T_3T_1, T_1T_2 , соодветно. Докажи дека правите l_1, l_2, l_3 формираат триаголни со чии темиња лежат на вписаната кружница на триаголникот ABC .



Црт. 1

$l_2 \parallel AC$ и $l_3 \parallel AB$.

Понатаму ќе докажеме, дека симетричната слика на H_2 во однос на правата T_2T_3 лежи на симетралата BI на аголот при темето B , каде I е центарот на вписаната кружница. Нека правата низ H_2 , која е нормална на T_2T_3 , ја сече BI во точка P , а S е пресечната точка на

Решение. Нека правата BC ги сече правите H_2H_3 и T_2T_3 во точките E и D , соодветно. Бидејќи $\angle H_2H_3A = \angle C$ е надворешен за $\triangle BEH_3$ добиваме

$$\angle BEH_3 = |\angle B - \angle C|.$$

Од друга страна

$$\angle T_2T_3A = \frac{180^\circ - \angle A}{2} = \frac{\angle B + \angle C}{2}$$

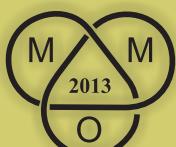
и е надворешен за $\triangle BDT_3$, па затоа $\angle BDT_3 = \frac{|\angle B - \angle C|}{2}$. Добиваме дека $\angle BEH_3 = \frac{1}{2} \angle BDT_3$, од што следува дека $l_1 \parallel BC$. Аналогно се докажува дека

T_2T_3 со BI . Доволно е да докажеме, дека $\angle PST_2 = \angle H_2ST_2$. Имаме $\angle PST_2 = \angle BST_3$. Но, $\angle T_2T_3A$ е надворешен за ΔBST_3 , па затоа

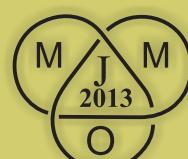
$$\angle BST_3 = (90^2 - \frac{1}{2}\angle A) - \frac{1}{2}\angle B = \frac{1}{2}\angle C.$$

Од друга страна, бидејќи T_1 и T_3 се симетрични во однос на BI важи $\angle BST_3 = \angle BST_1$, па затоа $\angle BST_1 = \frac{1}{2}\angle C = \angle ICT_1$. Од последното равенство следува дека околу четириаголникот IT_1CS може да се опише кружница. Но, тогаш $\angle BSC = 90^0$, бидејќи $IT_1 \perp BS$. Тоа значи дека и околу четириаголникот BCH_2S може да се опиша кружница со дијаметар BC . Тогаш, $\angle PSH_2 = \angle BCH_2 = \angle C$ и како $\angle PST_2 = \frac{1}{2}\angle C$ добиваме дека $\angle PST_2 = \angle H_2ST_2$.

Сега да забележиме, дека бидејќи околу четириаголникот BCH_2S може да се опише кружница, добиваме $\angle SH_2T_2 = \frac{1}{2}\angle B$, па затоа $\angle SPT_2 = \frac{1}{2}\angle B$. Понатаму, нека M_1, M_2 и M_3 се симетричните слики на T_1, T_2 и T_3 во однос на симетралите на аглите AI, BI и CI , соодветно. Јасно, M_1, M_2 и M_3 лежат на вписаната кружница. Притоа $\angle SPM_2 = \angle SPT_2 = \frac{1}{2}\angle B$ од што следува дека правата PM_2 е паралелна на правата BC . Но, претходно докажавме дека $l_1 \parallel BC$ и како $P \in l_1$ добиваме дека PM_2 и l_1 се совпаѓаат, т.е. l_1 минува низ M_2 . Аналогно се докажува дека l_1 минува низ M_3 . Со аналогни размислувања за правите l_2 и l_3 се докажува дека правите l_1, l_2 и l_3 се сечат на вписаната кружница.



20-та Македонска
математичка олимпијада



17-та JMMO

