3. EXTENSION OF A 2-SKEW-SYMMETRIC LINEAR FORM

Theorem. *Let* $\Lambda : M \to \mathbb{R}$ *be a* 2*-skew-symmetric form such that* $\Lambda(x,y) \le p(x,y)$ *for every* $(x,y) \in M$, $p : X^2 \to \mathbb{R}$ *be a* 2*-semi norm and* M *is a branch* 2*-subspace of the* 2*-space* X^2. *Let* M' *be an extension of* M *as in sub case 1 of case 2. Then there exists a* 2*-skew-symmetric linear form* $\Lambda' : M' \to \mathbb{R}$ *such that*

$\Lambda'/M = \Lambda$

$-p(-x,y) \le \Lambda(x,y) \le p(x,y)$. (*)

Proof. In this theorem, as before we will choose two arbitrary elements from the 2-subspace M, which at the same time belong also in the loop u. Let that be the elements $(\alpha_2 x_2 + \alpha_4 x_4, u)$ and $(\beta_2 x_2 + \beta_4 x_4, u)$. For 2-skew-symmetric form Λ, according to the conditions of the theorem we have that

$$\Lambda(\alpha_2 x_2 + \alpha_4 x_4, u) + \Lambda(\beta_2 x_2 + \beta_4 x_4, u) = \Lambda(\alpha_2 x_2 + \alpha_4 x_4 + \beta_2 x_2 + \beta_4 x_4, u) \le$$
$$\le p(\alpha_2 x_2 + \alpha_4 x_4 + \beta_2 x_2 + \beta_4 x_4, u) = p(\alpha_2 x_2 + \alpha_4 x_4 - v + \beta_2 x_2 + \beta_4 x_4 + v, u) \le$$ In
$$\le p(\alpha_2 x_2 + \alpha_4 x_4 - v, u) + p(\beta_2 x_2 + \beta_4 x_4 + v, u)$$

other words, the inequality holds

$$\Lambda(\alpha_2 x_2 + \alpha_4 x_4, u) - p(\alpha_2 x_2 + \alpha_4 x_4 - v, u) \le p(\beta_2 x_2 + \beta_4 x_4 + v, u) - \Lambda(\beta_2 x_2 + \beta_4 x_4, u).$$

Since $\alpha_2, \alpha_4 \in \mathbb{R}$ and $\beta_2, \beta_4 \in \mathbb{R}$ are arbitrary, we get that

$$\sup_{\alpha_2,\alpha_4} \Lambda(\alpha_2 x_2 + \alpha_4 x_4, u) - p(\alpha_2 x_2 + \alpha_4 x_4 - v, u) = d \le p(\beta_2 x_2 + \beta_4 x_4 + v, u) - \Lambda(\beta_2 x_2 + \beta_4 x_4, u)$$

According to that, for arbitrary $\alpha_2, \alpha_4, \beta_2, \beta_4 \in \mathbb{R}$, the inequalities hold

$$\Lambda(\alpha_2 x_2 + \alpha_4 x_4, u) - p(\alpha_2 x_2 + \alpha_4 x_4 - v, u) \le d$$
$$d \le p(\beta_2 x_2 + \beta_4 x_4 + v, u) - \Lambda(\beta_2 x_2 + \beta_4 x_4, u)$$

i.e.

$$\Lambda(\alpha_2 x_2 + \alpha_4 x_4, u) - d \le p(\alpha_2 x_2 + \alpha_4 x_4 - v, u) \quad (1)$$
$$\Lambda(\beta_2 x_2 + \beta_4 x_4, u) + d \le p(\beta_2 x_2 + \beta_4 x_4 + v, u) \quad (2)$$

Now, we will determine $\Lambda' : M' \to \mathbb{R}$ with

$$\Lambda'[A(\alpha_2 x_2 + \alpha_4 x_4 + \gamma v, u)] = (\det A)[\Lambda(\alpha_2 x_2 + \alpha_4 x_4, u) + \gamma d], \ \gamma \in \mathbb{R},$$
$$\Lambda'(x,y)] = \Lambda(x,y), \ (x,y) \in M.$$

According to this $\Lambda'/M = \Lambda$.

On the other hand, if in (1) instead α_2 and α_4 we choose $\frac{\alpha_2}{t}$ and $\frac{\alpha_4}{t}$, $t > 0$ and if we use the properties of Λ and p respectively, we get that

$$\Lambda(\alpha_2 x_2 + \alpha_4 x_4, u) - td \le p(\alpha_2 x_2 + \alpha_4 x_4 - tv, u). \quad (3)$$

Completely analogous, if in (2) instead β_2 and β_4 we choose $\frac{\beta_2}{t}$ and $\frac{\beta_4}{t}$, $t > 0$ respectively, and again if we use the properties of Λ and p, we get that

$$\Lambda(\alpha'_{i-1} x_{i-1} + \alpha'_{i+1} x_{i+1}, u) + td \le p(\alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1} + tv, u). \quad (4)$$

Now, from (3) and (4) we see that

$$\Lambda'(\beta_2 x_2 + \beta_4 x_4 + \gamma v, u) \le p(\beta_2 x_2 + \beta_4 x_4 + \gamma v, u),$$

where from it is clear that in general case $\Lambda' \leq p$ on M'. In other words the inequality (*) holds.

CONFLICT OF INTEREST

No conflict of interest was declared from the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

References

[1] R.Malčeski, A.Malčeski, n*-seminormed space*, Annuaire de l'Institute des Mathématiques, Faculté des Sciences de l'Université "Sv. Kiril et Metodij" - Skopje), 38(1997)

[2] A.Misiak: n*-inner product spaces*, Math.Nachr. 140 (1989)

[3] S.Gähler, *Lineare 2-normierte Raume,* Math.Nach. 28(1965)

[4] A.Malčeski, *Zabeleška za definicijata na 2-normiran prostor,* Mat. Bilten, Tom 26 (2002)

[5] A.Malčeski, V.Manova Erakovik, *Some 2-subspaces of 2-space*, Математички Билтен, 35(LXI), Makedonija, (2007)

[6] D.Mitrinović, *Polinomi i Matrici*, Naučna Knjiga, Beograd (1991)

[7] A.Malčeski, V. Manova Erakovikj. *An extended type of Hahn-Banach for Skew-Symmetric linear forms.* Mat. Bilten, 35(LXI)Tome, 2011, pp 41-49

[8] S.Brsakoska, A.Malčeski, *Extension of one sided branch 2-subspace and some extensions of Hahn-Banch type for skew-symmetric 2-linear functionsl defined on it*, Procedings od CODEMA 2020

[9]A,Malčeski, V. Manova Erakovik, *Algebraic structure of the kernel of the n-seminorm*, Matematički bilten, 31, (LVII), Makedonija, (2007)

[1)]Faculty for Natural Sciences and Mathematics, University "Sts. Cyril and Methodius", Skopje, Republic of N. Macedonia
E-mail address: sbrsakoska@gmail.com
[2)]Faculty Mechanical Engineering, University "Sts. Cyril and Methodius", Skopje, Republic of N. Macedonia
E-mail address: aleksa.malceski@gmail.com

SOME RECENT FIXED POINT RESULTS OF F-CONTRACTIVE MAPPINGS IN METRIC SPACES

UDC: 517.983:[515.124:515.126.4

Jelena Z. Vujaković[1], Ljiljana R. Paunović[2], Olga V. Taseiko[3]

Abstract. The opinion is that the most important result in the metrical theory of fixed points is the famous Banach contraction principle from 1922. It has been generalized and extended in several directions. One of the most interesting extensions was provided by Wardowski in 2012. He described a new contraction, so–called F-contraction and proved that every F-contraction has a unique fixed point, where $F:(0,+\infty)\to(-\infty,+\infty)$ satisfies conditions $(F1)$, $(F2)$ and $(F3)$. Several authors generalized his results by introducing the various types of F-contractions in other general metric spaces. In this paper we established some new fixed point results of F, Suzuki F and (φ,F)-contractive mappings in complete metric spaces. The goal was to improve the already published results but using only property $(F1)$ of strictly increasing mapping F. We believe that our approach significantly improves, complements, generalizes and enriches several known results in current literature.

1. INTRODUCTION AND PRELIMINARIES

In 2012., Wardowski [1] described a new contraction, so-called F-contraction and proved some new fixed point results that make a proper generalization of Banach contraction principle [2]. Wardowski's theorem play a significant role in the further research in the metrical fixed point theory. Several autors (for example [3]-[15]) generalized it by introducing the various types of F-contractions in other general metric spaces. Others have considered Wardowski's aproach in a multi-valued case for metric spaces and its generalizations.
In this section we provide some basic definitions and statements that will be used in sequel.

Definition 1.1. [1] Let (X,d) be a metric space, and let Φ denote the family of mappings $F:(0,+\infty)\to(-\infty,+\infty)$ satisfying the following condition:
(F1) F is strictly increasing, that is for all $x,y\in(0,+\infty)$, $x<y$ implies $F(x)<F(y)$;
(F2) for every sequence $\{x_n\}\subset(0,+\infty)$, $n\in\mathbb{N}$, $\lim_{n\to\infty}x_n=0$ if and only if $\lim_{n\to\infty}F(x_n)=-\infty$;

2010 *Mathematics Subject Classification.* Primary: 47H10O Secondary: 54H25.
Key words and phrases. metric space, fixed point, F-contraction, Suzuki F-contraction, (φ,F)-contraction

(F3) there is $k \in (0,1)$ such that $\lim_{x\to 0^+} x^k F(x) = 0$.

A mapping $T: X \to X$ is said to be an F-contraction on metric space (X,d) if there exists $F \in \Phi$ and $\theta > 0$ such that for all $x,y,z \in X$

$$d(Tx,Ty) > 0 \Leftrightarrow \theta + F(d(Tx,Ty)) \le F(d(x,y)). \qquad (1.1)$$

From conditions (F1) and (F2) it is easy to conclude that every F-contraction is a contraction. Based on condition (F1) and (1.1) one can obviously conclude that T is continuous.

Theorem 1.1 *[1] Let T be a self-mapping on a complete metric space (X,d). If T is an F-contraction then it has a unique fixed point, say $x^* \in X$, and for every $x \in X$ iterative sequence $\{T^n x\}, n \in \mathbb{N}$ converges to x^*.*

Recently, Proinov [13] proved that fixed point Theorem 1.1 is equivalent to a special case of the Skof's fixed point results from 1977, [16]. However, our approach to Wardowki's generalization differs from his.

In 2013, two interesting results appeared as generalizations of Theorem 1.1. Secelean [3] shown that condition (F2) in Definition 1.1 can be replaced by an equivalent condition

(S1) $\inf F = -\infty$, or

(S2) there exists a sequence $\{x_n\} \subset (0,+\infty), n \in \mathbb{N}$ such that $\lim_{n\to\infty} F(x_n) = -\infty$

while Turinci [5] observed that condition (F2) can be weakened as follows

$$F(t) = -\infty \text{ when } t \to 0^+, \text{ i.e. } F(0^+) = -\infty.$$

Considering Theorem 1.1, with conditions (S1), (S2) and a new condition

(PK) F is continuouns on $(0,+\infty)$

Piri and Kumam [6], in 2014, introduced a new type of F-contraction, a so-called Suzuki F-contraction, by which they generalized and extended results of Wardowski and Secelean.

Theorem 1.2. *[6] Let $T: X \to X$ be a given mapping in a complete metric space (X,d). Suppose that $F:(0,+\infty) \to (-\infty,+\infty)$ satisfies conditions (F1), (S1) and (PK) and let there exist $\theta > 0$ such that for all $x,y \in X$*

$$d(Tx,Ty) > 0 \Rightarrow \theta + F(d(Tx,Ty)) \le F(d(x,y)).$$

Then, T has a unique fixed point $x^* \in X$ and for every $x \in X$ iterative sequence $\{T^n x\}$, $n \in \mathbb{N}$ converges to x^*.

Definition 1.2. [6] A mapping $T: X \to X$ is said to be a Suzuki F-contraction in metric space (X,d) if there exists $\theta > 0$ such that for all $x, y \in X$

$$d(Tx,Ty) > 0,\ \frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow \theta + F\big(d(Tx,Ty)\big) \le F\big(d(x,y)\big)$$

where F satisfies conditions (F1), (S1), (PK).

Theorem 1.3. *[6] Let (X,d) be a complete metric space and $T: X \to X$ be an F-Suzuki contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the iterative sequence $\{T^n x\}$, $n \in \mathbb{N}$ converges to x^*.*

In 2018, Wardowski [11] introduced a concept of (φ, F)-contraction (or nonlinear F-contraction) and also genuinely generalized his Theorem 1.1

Definition 1.3. [11] A self-mapping T on metric space (X,d), for some functions $F \in \Phi$ and $\varphi:(0,+\infty) \to (0,+\infty)$, is said to be (φ, F)-contraction if the following conditions hold:

(W1) F satisfies (F1) and (S1);

(W2) $\liminf_{s \to t+} \varphi(s) > 0$ for all $t \ge 0$;

(W3) $\varphi\big(d(x,y)\big) + F\big(d(Tx,Ty)\big) \le F\big(d(x,y)\big)$ for all $x, y \in X$ with $Tx \ne Ty$.

Theorem 1.4. *[11] Let (X,d) be a complete metric space and let $T: X \to X$ be (φ, F)-contraction. Then T has a unique fixed point.*

In this article, we will prove Theorem 1.2, Theorem 1.3 and Theorem1.4 in the easier way: using only condition (F1) and the following two Lemmas.

Lemma 1.1 *[17] Let $\{x_n\}$, $n \in \mathbb{N}$ be a sequence in the metric space (X,d) such that $\lim_{n \to +\infty} d(x_n, x_{n+1}) = 0$. If sequence $\{x_n\}$ is not a Cauchy one in (X,d), then there exists $\varepsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that $n_k > m_k > k$ and the sequences*

$$\left\{d\left(x_{n_k}, x_{m_k}\right)\right\}, \left\{d\left(x_{n_k+1}, x_{m_k}\right)\right\}, \left\{d\left(x_{n_k}, x_{m_k-1}\right)\right\}, \left\{d\left(x_{n_k+1}, x_{m_k-1}\right)\right\}, \left\{d\left(x_{n_k+1}, x_{m_k+1}\right)\right\}, \ldots$$

tend to ε^+ as $k \to +\infty$.

Notice, that if the condition of Lemma 1.1 is satisfied, then the sequences $\left\{d\left(x_{n_k+s}, x_{m_k}\right)\right\}$ and $\left\{d\left(x_{n_k+s}, x_{m_k+1}\right)\right\}$ also converge to ε^+ when $k \to +\infty$, where

$s \in \mathbb{N}$.

Lemma 1.2. *[18] Let* $\{x_{n+1}\} = \{Tx_n\} = \{T^n x_0\}, n \in \mathbb{N} \cup \{0\}$, $T^0 x_0 = x_0$ *be a Picard sequence in metric space* (X, d) *induced by a mapping* $T : X \to X$ *and an initial point* $x_0 \in X$. *If* $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ *for all* $n \in \mathbb{N}$ *then* $x_n \neq x_m$ *whenever* $n \neq m$.

2. MAIN RESULTS

We begin this section with the theorem that generalizes and improves Theorem 1.2. In our result, the function $F : (0, +\infty) \to (-\infty, +\infty)$ satisfies only condition (F1).

Theorem 2.1 *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be an* $F-$*contraction mapping with property (F1), that is, let there exist* $\theta > 0$ *such that*

$$\theta + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{2.1}$$

for all $x, y \in X$ *with* $Tx \neq Ty$.

Then T *has a unique fixed point, say,* x^* *and for all* $x \in X$ *the sequence* $\{T^n x\}$, $n \in \mathbb{N}$ *converges to* x^*.

Proof. Considering condition (F1), there are both $\lim_{c \to t-} F(c) = F(t-0)$ and $\lim_{c \to t+} F(c) = F(t+0)$ for all $t \in (0, +\infty)$, because it is known from mathematical analysis that the following is true

$$F(t-0) \leq F(t) \leq F(t+0), \ t \in (0, +\infty). \tag{2.2}$$

Further, from the assumption that T is an $F-$contraction, it follows that T is contractive ($x \neq y$ implies $d(Tx, Ty) < d(x, y)$). This means that the mapping T is continuous. Besides, $F-$contractive condition (F1) implies the uniqueness of the fixed point if it exists.

We will show that T has a fixed point. Let x_0 be an arbitrary point in X. Consider the sequence $\{x_n\}$, $n \in \mathbb{N} \cup \{0\}$ with $x_{n+1} = Tx_n$. If $x_k = x_{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$ then x_k is a unique fixed point of T and the conclusion of the Theorem follows. Therefore, suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Based on $F-$contractive condition (2.1) of the mapping T, we get

$$F\left(d\left(x_n,x_{n+1}\right)\right)<\theta+F\left(d\left(x_n,x_{n+1}\right)\right)\leq F\left(d\left(x_{n-1},x_n\right)\right), \tag{2.3}$$

for all $n\in\mathbb{N}$, and then, in accordance with property (F1), it follows

$$d\left(x_n,x_{n+1}\right)<d\left(x_{n-1},x_n\right) \text{ for all } n\in\mathbb{N}.$$

The following result is related to generalization and the improvement of Theorem 1.3.

Theorem 2.2. *Let $\left(X,d\right)$ be a complete metric space and let $T:X\to X$ is a Suzuki $F-$contraction mapping where F satisfies condition (F1), that is, there exists $\theta>0$ such that $\frac{1}{2}d\left(x,Tx\right)<d\left(x,y\right)$ implies*

$$\theta+F\left(d\left(Tx,Ty\right)\right)\leq F\left(d\left(x,y\right)\right), \tag{2.4}$$

for all $x,y\in X$ with $d\left(Tx,Ty\right)>0$.

Then T has a unique fixed point $x^*\in X$ and for each $x\in X$ the iterative sequence $\left\{T^n x\right\}$, $n\in\mathbb{N}$ converges to x^*.

Proof. It is easily seen that relation (2.4) implies the uniqueness of the fixed point if it exists. Indeed, if we suppose that there are two distinct fixed points x^* and y^* of T, then it is clear that from $\frac{1}{2}d\left(x^*,Tx^*\right)<d\left(x^*,y^*\right)$ follows

$$\theta+F\left(d\left(Tx^*,Ty^*\right)\right)\leq F\left(d\left(x^*,y^*\right)\right), \text{ i.e.}$$

$$\theta+F\left(d\left(x^*,y^*\right)\right)\leq F\left(d\left(x^*,y^*\right)\right),$$

which is contradiction.

Now we show the existence of the fixed point. Let $x_0\in X$ be an arbitrary point and $\left\{x_n\right\}$, $n\in\mathbb{N}\cup\left\{0\right\}$ is the corresponding Picard's sequence, i.e. $x_n=T^n x_0$ with initial value $x_0=T^0x_0$. If $x_k=x_{k+1}$ for some $k\in\mathbb{N}\cup\left\{0\right\}$, then x_k is an unique fixed point and the proof is complete.

Therefore, suppose $x_n\neq x_{n+1}$ for all $n\in\mathbb{N}\cup\left\{0\right\}$. In this case $\frac{1}{2}d\left(x_n,x_{n+1}\right)<d\left(x_n,x_{n+1}\right)$ hold true for all $n\in\mathbb{N}\cup\left\{0\right\}$, so from (2.4) we have

$$\theta+F\left(d\left(x_{n+1},x_{n+2}\right)\right)\leq F\left(d\left(x_n,x_{n+1}\right)\right),$$

that is, according to condition (F1) we get

$$d\left(x_{n+1},x_{n+2}\right)<d\left(x_n,x_{n+1}\right), \text{ for all } n\in\mathbb{N}\cup\left\{0\right\}.$$

As in the proof of Theorem 2.1, we obtain that $d(x_n, x_{n+1}) \to 0$ as $n \to +\infty$. Since $d(x_n, x_{n_k+1}) \to 0$ and $d(x_{n_k}, x_{m_k}) \to \varepsilon^+$ as $k \to +\infty$ it follows that there is some $k_1 \in \mathbb{N}$ such that $\frac{1}{2} d(x_{n_k}, x_{m_k+1}) < d(x_{n_k}, x_{m_k})$, for all $k \in \mathbb{N}$ with $k \geq k_1$. Then, for $k \geq k_1$, we have

$$\theta + F\left(d\left(x_{n_k+1}, x_{m_k+1}\right)\right) \leq F\left(d\left(x_{n_k}, x_{m_k}\right)\right),$$

that is $\theta + F(\varepsilon^+ + 0) \leq F(\varepsilon^+ + 0)$, which is a contradiction because $\theta > 0$.

Hence we conclude that $\{x_n\}, n \in \mathbb{N} \cup \{0\}$ is a Cauchy sequence. The completeness of the metric space (X, d) guarantees the existence of some point $x^* \in X$ such that $\lim_{n \to +\infty} d(x_n, x^*) = 0$. The rest of proof is analogous to the proof of Theorem 1.3 from [6, page 7]. One can find that following inequalities hold

$$\frac{1}{2} d(x_n, Tx_n) < d(x_n, x^*) \text{ or } \frac{1}{2} d(Tx_n, T^2 x_n) < d(Tx_n, x^*),$$

for all $n \in \mathbb{N} \cup \{0\}$.

Further, from relation (2.4) it follows

$$\theta + F\left(d\left(Tx_n, Tx^*\right)\right) \leq F\left(d\left(x_n, x^*\right)\right), \text{ or } \theta + F\left(d\left(T^2 x_n, Tx^*\right)\right) \leq F\left(d\left(Tx_n, x^*\right)\right),$$

i.e.,

$$\theta + F\left(d\left(x_{n+1}, Tx^*\right)\right) \leq F\left(d\left(x_n, x^*\right)\right), \text{ or}$$

$$\theta + F\left(d\left(x_{n+2}, Tx^*\right)\right) \leq F\left(d\left(x_{n+1}, x^*\right)\right). \tag{2.5}$$

Finally, using condition (F1) inequalities (2.5) can be written in the form

$$d(x_{n+1}, Tx^*) \leq d(x_n, x^*) \text{ or } d(x_{n+2}, Tx^*) \leq d(x_{n+1}, x^*). \tag{2.6}$$

This proves that x^* is a unique fixed point of T, i.e., $Tx^* = x^*$.

In the following we present our second result. We assumed that the function $F \in \Phi$ satisfies condition (F1) only while the function $\varphi : (0, +\infty) \to (0, +\infty)$ has the (W2) property.

Theorem 2.3 *Let (X, d) be a complete metric space and let $T : X \to X$ be a (φ, F)-contraction satisfying*

$$\varphi(d(x, y)) + F\left(d(Tx, Ty)\right) \leq F\left(d(x, y)\right),$$

for all $x, y \in X$ with $Tx \neq Ty$. Then T has a unique fixed point.

Proof. First of all, condition (F1) implies that there exists both $\lim_{c\to t-} F(c) = F(t-0)$ and $\lim_{c\to t+} F(c) = F(t+0)$ for all $t \in (0,+\infty)$, and $F(t-0) \le F(t) \le F(t+0)$ holds true for each $t \in (0,+\infty)$. Further, considering the definition (φ, F)-contraction and assuming $\varphi : (0,+\infty) \to (0,+\infty)$, it follows that T is contractive mapping ($x \ne y$ implies $d(Tx,Ty) < d(x,y)$). This means that the mapping T is continuous. Also, (φ, F)-contraction gives the uniqueness of the fixed point if it exists.

In order to show that T has a fixed point, let x_0 be arbitrary point in X. Now, we define a sequence $\{x_n\}, n \in \mathbb{N} \cup \{0\}$, with $x_{n+1} = Tx_n$. If $x_k = x_{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$ then x_k is a unique fixed point and the proof of Theorem is completed. Therefore, suppose now that $x_n \ne x_{n+1}$ for every $n \in \mathbb{N} \cup \{0\}$. By the definition (φ, F)-contraction it follows that

$$F(d(x_n, x_{n+1})) < \varphi(d(x_{n-1}, x_n)) + F(d(x_n, x_{n+1})) \le F(d(x_{n-1}, x_n)), \tag{2.7}$$

for all $n \in \mathbb{N}$, that is, according to (F1) we have $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. This, further, means that $d(x_n, x_{n+1}) \to d^* \ge 0$, as $n \to +\infty$ as well as $d(x_n, x_{n+1}) > d^*$ for all $n \in \mathbb{N} \cup \{0\}$. Suppose that $d^* > 0$. Based on (W2) there exists $\tau > 0$ and an $n_1 \in \mathbb{N}$ such that for all $n \ge n_1$ we have

$$\tau + F(d(x_n, x_{n+1})) \le \varphi(d(x_{n-1}, x_n)) + F(d(x_n, x_{n+1})) \le F(d(x_{n-1}, x_n))$$

that is,

$$\tau + F(d(x_n, x_{n+1})) \le F(d(x_{n-1}, x_n)), \tag{2.8}$$

for all $n \ge n_1$. Hence we obtain

$$\tau + F(d^* + 0) \le F(d^* + 0), \tag{2.9}$$

which is a contradiction. Therefore, $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$.

Now, we will prove that $\{x_n\}, n \in \mathbb{N} \cup \{0\}$ is a Cauchy sequence by assuming the opposite. If we take $x = x_{n_k}$ and $y = x_{m_k}$ in (φ, F)-contraction (it is possible by Lemma 1.2), we get

$$\varphi(d(x_{n_k}, x_{m_k})) + F(d(x_{n_k+1}, x_{m_k+1})) \le F(d(x_{n_k}, x_{m_k})). \tag{2.10}$$

Since, according to Lemma 1.1, both $d(x_{n_k+1}, x_{m_k+1})$ and $d(x_{n_k}, x_{m_k})$ tend to ε^+ as $k \to +\infty$, we obtain

$$\lim_{d(x_{n_k}, x_{m_k}) \to \varepsilon^+} \inf \varphi(d(x_{n_k}, x_{m_k})) + \lim_{d(x_{n_k+1}, x_{m_k+1}) \to \varepsilon^+} \inf F(d(x_{n_k+1}, x_{m_k+1}))$$

$$\leq \lim \inf_{d\left(x_{n_k}, x_{m_k}\right) \to \varepsilon^+} F\left(d\left(x_{n_k}, x_{m_k}\right)\right),$$

i.e.,

$$\lim \inf_{d\left(x_{n_k}, x_{m_k}\right) \to \varepsilon^+} \varphi\left(d\left(x_{n_k}, x_{m_k}\right)\right) + F\left(\varepsilon^+ + 0\right) \leq F\left(\varepsilon^+ + 0\right). \tag{2.11}$$

This is in contradiction to $\lim \inf_{d\left(x_{n_k}, x_{m_k}\right) \to \varepsilon^+} \varphi\left(d\left(x_{n_k}, x_{m_k}\right)\right) > 0$. We conclude that $\{x_n\}, n \in \mathbb{N} \cup \{0\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, it follows that the sequence $\{x_n\}, n \in \mathbb{N} \cup \{0\}$ converges to a point $x^* \in X$. On the other hand, the continuity of the mapping T implies $Tx^* = x^*$, i.e. x^* is a unique fixed point of T and we have proved the theorem.

Remark 1. Taking $\varphi(t) = \theta > 0$ in Theorem 2.3, as a corollary we obtain the Theorem 1.2 from [1].

It is worth to mention that method we described in the proof of Theorem 2.3 improve, generalize, complement, unify and enrich the corresponding ones from Wardowski [11].

COMPETING INTERESTS

Authors have declared that no competing interests exist.

References

[1] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory and Applications, 2012:94 (2012)

[2] S. Banach, *Sur les operations dans les ensembles abstrait et leur application aux equations integrals*, Fund. Math., (1922), 3, 133--181.

[3] N. A. Secelean, *Iterated function system consisting of F -contractions*, Fixed Point Theory Appl., (2013), 2013, 277.

[4] N. A. Secelean, *Weak F -contractions and some fixed point results*, Bull. Iran. Math. Soc. (2016), 42, 779--798.

[5] M. Turinci, *Wardowski implicit contractions in metric spaces*, arXiv:1211.3164v2[math.CN] 15 Sep 2013

[6] H. Piri, P. Kumam, *Some fixed point theorems concerning F-contraction in complete metric spaces*, Fixed Point Theory and Applications, 2014:210 (2014)

[7] H. Piri, P. Kumam, *Fixed point theorems for generalized F -Suzuki-contraction mappings in complete b -.metric spaces*. Fixed Point Theory and

Application, art.num:90. Available at: https://doi.org/10.1186/s13663-016-0577-5.
[8] M. Cosentino, P. Vetro, *Fixed point result for F -contractive mappings of Hardy-Rogers-Type*, Filomat (2014), 28, 715--722.
[9] N. V. Dung, V. T. L. Hang, *A fixed point theorem for generalized F-contractions on complete metric space*. Vietnam Journal of Mathematics, (2015), art.num:13, pp.743-753. Available at: https://doi.org/10.1007/s10013-015-0123-5
[10] G. Minak, A. Helvaci, I. Altun, *Ćirić type generalized F -contractions on complete metric spaces and fixed point results*, Filomat, (2014), 28, 1143--1151.
[11] D. Wardowski, *Solving existence problems via F -.contractions*, Proceeding of American Mathematical Society (2018), 146(4), 1585-1598.
[12] D. Wardowski, *Solving existence problems via F -.contractions*, Amer. Math. Soc. (2018), 146, 1585--1598.
[13] P. D. Proinov, *Fixed point theorems for generalized contractive mappings in metric spaces*, J. Fixed point Theory Appl. (2020) 22:21
[14] J. Vujaković, S. Radenović, *On some F-contraction of Piri-Kumam-Dung type mappings in metric spaces*, Vojnotehnički glasnik/Military Technical Courier (2020), 68(4), 697-714.
[15] J. Vujaković, S. Mitrović, M. Pavlović, S. Radenović, *On recent results concerning F-contraction in generalized metric spaces*, Mathematics (2020) 8 (767).
[16] F. Skof, *Theoremi di punto fisso per applicazioni negli spazi metrici.* Atti. Acad. Sci. Torino Cl. Sci. Fis. Mat. Natur. (1977), 111, 323–329.
[17] S. Radenović, F. Vetro, J. Vujaković, *An alternative and easy approach to fixed point results via simulation functions*, Demonstratio mathematica (2017), 50(1), 223-230
[18] S. Radenović, Z. Kadelburg, D. Jandrlić, A. Jandrlić, *Some results on weakly contractive maps*, Bull. Iran. Math. Soc. (2012), 38, 625-645.

[1)]Faculty of Sciences, Department of Mathematics, University in Priština-Kosovska Mitrovica, Kosovska Mitrovica, Serbia.
E-mail address: jelena.vujakovic@pr.ac.rs
[2)]Teacher Education Faculty, University of Priština-Kosovska Mitrovica, Leposavić, Serbia
E-mail address: ljiljana.paunovic@pr.ac.rs
[3)]Reshetnev Siberian State University of Science and Technology, Krasnoyarsk
E-mail address: taseiko@gmail.com

OSTROWSKI-GRÜSS TYPE INEQUALITY OF CHEBYSHEV FUNCTIONAL WITH APPLICATION TO THE WEIGHT THREEPOINT INTEGRAL FORMULA

UDC: 517.587.512.13U

S. KOVAČ

Abstract. Recently there have been proven many results about error bounds for Chebyshev functional. The aim of our paper is to extend those results and give some new error estimation of the Chebyshev functional and applications to the threepoint weighted integral formulas.

1. Introduction

For two Lebesque integrable functions $f, g : [a, b] \to \mathbb{R}$ let us consider the Chebyshev functional:

$$T(f, g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b g(t)dt. \tag{1.1}$$

$L_p[a, b]$, $1 \le p < \infty$ stands for the space of the functions $f : [a, b] \to \mathbb{R}$ which are $p-$integrable, i.e. they are equipped by $p-$norm

$$\|f\|_p = \left[\int_a^b |f(t)|^p \, dt\right]^{\frac{1}{p}}$$

which is finite. $L_\infty[a, b]$ stands for space of the functions $f : [a, b] \to \mathbb{R}$ which are essentially bounded and $\infty-$norm defined by

$$\|f\|_\infty = \text{esssup}_{t\in[a,b]}|f(t)|$$

isi finite.

P. Cerone and S.S. Dragomir have delivered in [1] the following bounds for Chebyshev functional $T(\varphi, \varphi)$:

Lemma 1. *If $\varphi : [a, b] \to \mathbb{R}$ is an absolutely continuous function with*

$$(\cdot - a)(b - \cdot)(\varphi')^2 \in L[a, b],$$

then we have the inequality

$$T(\varphi, \varphi) \le \frac{1}{2(b-a)}\int_a^b (x-a)(b-x)\left[\varphi'(x)\right]^2 dx. \tag{1.2}$$

The constant $\frac{1}{2}$ is the best possible.

The following results of Grüss type have been obtained in [1]:

2000 *Mathematics Subject Classification.* 26D15,26D20, 26D99.
Key words and phrases. weight,function,Chebyshev functional, threepoint integral formula.

Theorem 1. *Let $f, g : [a, b] \to \mathbb{R}$ be two absolutely continuous functions on $[a, b]$ with*

$$(\cdot - a)(b - \cdot)(f')^2, (\cdot - a)(b - \cdot)(g')^2 \in L[a, b].$$

Then we have the inequality

$$\begin{aligned} |T(f,g)| &\leq \frac{1}{\sqrt{2}} [T(f,f)]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}} \left(\int_a^b (x-a)(b-x) \left[g'(x)\right]^2 dx \right)^{\frac{1}{2}} \quad (1.3) \\ &\leq \frac{1}{2(b-a)} \left(\int_a^b (x-a)(b-x) \left[f'(x)\right]^2 dx \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_a^b (x-a)(b-x) \left[g'(x)\right]^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

The constant $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}$ are best possible in (1.3).

Theorem 2. *Assume that $g : [a, b] \to \mathbb{R}$ is monotonic nondecreasing on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ is absolutely continuous with $f' \in L_\infty[a, b]$. Then we have the inequality*

$$|T(f,g)| \leq \frac{1}{2(b-a)} ||f'||_\infty \int_a^b (x-a)(b-x) dg(x). \quad (1.4)$$

The constant $\frac{1}{2}$ is the best possible.

In this paper we shall give the notion and new bounds for general open weighted threepoint formula by using upper results for Chebyshev functional.

2. General open weight threepoint integral formula

Let us recall the general integral formula obtained in [2]. Consider subdivision $\sigma = \{a = x_0 < x_1 < \ldots < x_m = b\}$ of the segment $[a, b]$,for some $m \in \mathbb{N}$. Let $w : [a, b] \to \mathbb{R}$ be an arbitrary integrable function.On each interval $[x_{k-1}, x_k]$, $k = 1, \ldots, m$ we consider different $w-$harmonic sequences of functions $\{w_{kj}\}_{j=1,\ldots,n}$, i.e. we have

$$\begin{aligned} w'_{k1}(t) &= w(t) && \text{for } t \in [x_{k-1}, x_k] \\ (w_{kj})'(t) &= w_{k,j-1}(t) && \text{for } t \in [x_{k-1}, x_k], \text{ for all } j = 2, 3, \ldots, n. \end{aligned} \quad (2.1)$$

Further, let us define

$$W_{n,w}(t, \sigma) = \begin{cases} w_{1n}(t) & \text{for } t \in [a, x_1], \\ w_{2n}(t) & \text{for } t \in (x_1, x_2], \\ \vdots & \\ w_{mn}(t) & \text{for } t \in (x_{m-1}, b]. \end{cases} \quad (2.2)$$

In order to obtain the weight threepoint integral formula, we shall use the following results about the general integral formula ([2]):

Theorem 3. *If $g : [a, b] \to \mathbb{R}$ is such that $g^{(n)}$ is a piecewise continuous on $[a, b]$, then the following identity holds*

$$\begin{aligned}\int_a^b w(t)g(t)dt &= \sum_{j=1}^{n}(-1)^{j-1}\Big[w_{mj}(b)g^{(j-1)}(b) \\ &+ \sum_{k=1}^{m-1}\big[w_{kj}(x_k) - w_{k+1,j}(x_k)\big]g^{(j-1)}(x_k) - w_{1j}(a)g^{(j-1)}(a)\Big] \\ &+ (-1)^n \int_a^b W_{n,w}(t,\sigma)g^{(n)}(t)dt.\end{aligned} \tag{2.3}$$

Theorem 4. *Let us suppose $w : [a, b] \to \mathbb{R}$ is an integrable function and $\{w_{kj}\}_{j=1,\ldots,n}$ $w-$harmonic sequences of functions on $[x_{k-1}, x_k]$, for $k = 1, \ldots, m$. If $g : [a, b] \to \mathbb{R}$ is a function such that $g^{(n-1)}$ is absolutely continuous and $g^{(n)} \in \mathrm{L_p}$ for some $1 \le p \le \infty$, then the following inequality holds*

$$\begin{aligned}\Big|\int_a^b w(t)g(t)dt &- \sum_{j=1}^{n}(-1)^{j-1}\Big[w_{mj}(b)g^{(j-1)}(b) \\ &+ \sum_{k=1}^{m-1}\big[w_{kj}(x_k) - w_{k+1,j}(x_k)\big]g^{(j-1)}(x_k) - w_{1j}(a)g^{(j-1)}(a)\Big]\Big| \\ &\le C(n,p,w)\cdot\|g^{(n)}\|_q,\end{aligned} \tag{2.4}$$

where

$$C(n,p,w) = \begin{cases} \left[\int_a^b |W_{n,w}(t,\sigma)|^q\,dt\right]^{\frac{1}{q}}, & 1 < p \le \infty, \quad \frac{1}{p}+\frac{1}{q}=1 \\ \sup_{t\in[a,b]} |W_{n,w}(t,\sigma)|, & p = 1. \end{cases} \tag{2.5}$$

The inequality is the best possible for $p = 1$ and sharp for $1 < p \le \infty$. Equality is attained for every function g such that

$$g(t) = M \cdot g_*(t) + p_{n-1}(t),$$

where $M \in \mathbf{R}$, p_{n-1} is an arbitrary polynomial of degree at most $n - 1$ and $g_*(t)$ is function on $[a, b]$ defined by

$$g_*(t) = \int_a^t \frac{(t-\xi)^{n-1}}{(n-1)!}\cdot \mathrm{sgn}W_{n,w}(\xi,\sigma)\cdot|W_{n,w}(\xi,\sigma)|^{\frac{1}{p-1}}d\xi, \tag{2.6}$$

for $1 < p < \infty$, and

$$g_*(t) = \int_a^t \frac{(t-\xi)^{n-1}}{(n-1)!}\cdot \mathrm{sgn}W_{n,w}(\xi,\sigma)d\xi, \tag{2.7}$$

for $p = \infty$.

We shall deliver now the general threepoint integral fomula with nodes x, $\frac{a+b}{2}$ and $a+b-x$, for some $x \in [a, \frac{a+b}{2})$. Let $n \in \mathbb{N}$ and $\{L_j\}_{j=0,1,\ldots,n}$ be some sequence of harmonic polynomials such that $\deg L_j \le j-1$ and $L_0 \equiv 0$. Let us consider subdivision of the segment $[a, b]$:

$$\sigma := \{x_0 < x_1 < x_2 < x_3 < x_4\}$$

where $x_0 = a$, $x_1 = x$, $x_2 = \frac{a+b}{2}$, $x_3 = a + b - x$ and $x_4 = b$. For $k = 1, \ldots, n$ we define functions $w_{jk}^{(3)} : [x_{j-1}, x_j] \to \mathbf{R}$, for $j = 1, 2, 3, 4$, in the following way:

$$\begin{aligned}
w_{1k}^{(3)}(t) &:= \frac{1}{(k-1)!}\int_a^t (t-s)^{k-1}w(s)\mathrm{d}s,\\
w_{2k}^{(3)}(t) &:= \frac{1}{(k-1)!}\int_x^t (t-s)^{k-1}w(s)\mathrm{d}s + L_k(t),\\
w_{3k}^{(3)}(t) &:= -\frac{1}{(k-1)!}\int_t^{a+b-x} (t-s)^{k-1}w(s)\mathrm{d}s + (-1)^k L_k(a+b-t),\\
w_{4k}^{(3)}(t) &:= -\frac{1}{(k-1)!}\int_t^b (t-s)^{k-1}w(s)\mathrm{d}s,
\end{aligned}$$

and $w_{j0}^{(3)}(t) := w(t)$. Further, let us define

$$W_{n,w}^{(3)}(t,x) = \begin{cases} w_{1n}^{(3)}(t), & \text{for } t \in [a,x],\\ w_{2n}^{(3)}(t), & \text{for } t \in (x, \frac{a+b}{2}],\\ w_{3n}^{(3)}(t), & \text{for } t \in (\frac{a+b}{2}, a+b-x]\\ w_{4n}^{(3)}(t), & \text{for } t \in (a+b-x, b]. \end{cases} \tag{2.8}$$

Remark 1. *Sequences* $\{w_{jk}^{(3)}\}_{k=0,1,\ldots,n}$ *are* $w-$*harmonic sequences of functions on* $[x_{j-1}, x_j]$, *for every* $j = 1.2, 3, 4$.

Remark 2. *If, in addition, we have* $w(t) = w(a+b-t)$, *for each* $t \in [a,b]$, *then the following symmetry conditions hold for* $k = 1, \ldots, n$:

$$w_{1k}^{(3)}(t) = (-1)^k w_{4k}^{(3)}(a+b-t), \qquad \text{for } t \in [a,x]$$

and

$$w_{2k}^{(3)}(t) = (-1)^k w_{3k}^{(3)}(a+b-t), \qquad \text{for } t \in (x, \frac{a+b}{2}].$$

Theorem 5. *Let* $f : [a,b] \to \mathbb{R}$ *be a function with piecewise continuous* $n-th$ *derivative, for some* $n \in \mathbf{N}$, $w : [a,b] \to \mathbb{R}$ *integrable function such that* $w(t) = w(a+b-t)$, *for each* $t \in [a,b]$ *and* $x \in [a, \frac{a+b}{2})$. *Further, let* $\{L_k\}_{k=0,1,\ldots,n}$ *be some sequence of harmonic polynomials such that* $\deg L_j \le j-1$ *and* $L_0(t) = 0$, *and* $W_{n,w}^{(3)}(t,x)$ *be defined by (2.8). Then the following identity holds:*

$$\begin{aligned}
\int_a^b w(t)f(t)\mathrm{d}t &= A_{1,w}^{(3)}(x)\left(f(x) + f(a+b-x)\right) + B_{1,w}^{(3)}(x) f\left(\frac{a+b}{2}\right) \\
&+ T_{n,w}^{(3)}(x) + (-1)^n \int_a^b W_{n,w}^{(3)}(t,x) f^{(n)}(t)\mathrm{d}t,
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
T_{n,w}^{(3)}(x) &= \sum_{k=2}^{n} A_k(x)\left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x)\right) \\
&+ \sum_{\substack{k=3\\ odd\ k}} B_k(x) f^{(k-1)}(\frac{a+b}{2}),
\end{aligned} \tag{2.10}$$

$$A_{k.w}^{(3)}(x) = (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_a^x (x-s)^{k-1}w(s)\mathrm{d}s - L_k(x)\right], \quad k \ge 1,$$

$$B_{k,w}^{(3)}(x) = 2\left[\frac{1}{(k-1)!}\int_x^{\frac{a+b}{2}}(\frac{a+b}{2}-s)^{k-1}w(s)\mathrm{d}s + L_k(\frac{a+b}{2})\right], \quad \textit{for odd } k \geq 1$$

and

$$B_{k,w}^{(3)}(x) = w_{2k}(\frac{a+b}{2}) - w_{3k}(\frac{a+b}{2}) = 0, \quad \textit{for even } k \geq 1.$$

Proof. We consider subdivision $x_0 = a$, $x_1 = x$, $x_2 = \frac{a+b}{2}$, $x_3 = a+b-x$ and $x_4 = b$ of the interval $[a,b]$ and apply formula (2.3) with $m = 4$.

□

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a function with a piecewise continuous $n-th$ derivative and $f^{(n)} \in L_p[a,b]$ for some $n \in \mathbb{N}$ and some $1 \leq p \leq \infty$. Then we have the following inequality*

$$\left|\int_a^b w(t)f(t)\mathrm{d}t - A_{1,w}^{(3)}(x)\left(f(x)+f(a+b-x)\right) - B_{1,w}^{(3)}(x)f\left(\frac{a+b}{2}\right) - T_{n,w}^{(3)}(x)\right|$$
$$\leq C_3(n,p,x,w)\cdot \|f^{(n)}\|_p, \tag{2.11}$$

where

$$C_3(n,p,x,w) = \begin{cases} 2^{1/q}\left[\int_a^{\frac{a+b}{2}}\left|W_{n,w}^{(3)}(t,x)\right|^q \mathrm{d}t\right]^{1/q}, & \frac{1}{p}+\frac{1}{q}=1, \quad 1 < p \leq \infty \\ \sup\{\left|W_{n,w}^{(3)}(t,x)\right|, t \in [a, \frac{a+b}{2}]\}, & p = 1, \end{cases} \tag{2.12}$$

The inequality is best possible for $p = 1$ and sharp for $1 < p \leq \infty$. Equality is attained for the function $f_* : [a,b] \to \mathbf{R}$ defined by

$$f_*(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}\left|W_{n,w}^{(3)}(s,x)\right|^{\frac{1}{p-1}} sgn W_{n,w}^{(3)}(s,x)\mathrm{d}s \tag{2.13}$$

for $1 < p < \infty$, while for $p = \infty$

$$f_*(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} sgn W_{n,w}^{(3)}(s,x)\mathrm{d}s \tag{2.14}$$

Proof. The proof follows from the Theorem 4 for $m = 4$ and subdivision $x_0 = a$, $x_1 = x$, $x_2 = \frac{a+b}{2}$, $x_3 = a+b-x$ and $x_4 = b$. □

Remark 3. *Using Theorem 5 and Theorem 6 for the uniform weight $w(t) = \frac{1}{b-a}$, we get the results obtained in [3], so these results are the generalization of the non-weight threepoint formula.*

Let us consider the special harmonic polynomials L_n and uniform weight $w(t) = \frac{1}{b-a}$:

$$\begin{aligned} L_{n,x}(t) \quad &:= \quad \frac{1}{n!(b-a)}\left[n(x-a-\frac{(b-a)^3}{6(2x-a-b)^2})(t-x)^{n-1}\cdot 1_{\{n\geq 1\}}\right. \\ &+ \quad \binom{n}{2}(x-a)^2(t-x)^{n-2}\cdot 1_{\{n\geq 2\}} + \binom{n}{3}(x-a)^3(t-x)^{n-3}\cdot 1_{\{n\geq 3\}} \\ &+ \quad \left.\binom{n}{4}(x-a)^4(t-x)^{n-4}\cdot 1_{\{n\geq 4\}}\right], \quad t \in [x, \frac{a+b}{2}]. \end{aligned} \tag{2.15}$$

After some calculation from Theorem 5 we get

$$A^{(3)}_{1,\frac{1}{b-a}}(x) = \frac{(b-a)^2}{6(2x-a-b)^2} \quad B^{(3)}_{1,\frac{1}{b-a}}(x) = 1 - \frac{(b-a)^2}{3(2x-a-b)^2},$$

$$A^{(3)}_{2,\frac{1}{b-a}}(x) = A^{(3)}_{3,\frac{1}{b-a}}(x) = A^{(3)}_{4,\frac{1}{b-a}}(x) = B^{(3)}_{3,\frac{1}{b-a}}(x) = 0.$$

Inserting polynomials $\{L_{n,x}\}$ in Theorem 5 and Theorem 6 we get:

$$\begin{aligned} &\Big|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - D(f,x) - T^{(3)}_{n,\frac{1}{b-a}}(x)\Big| \\ &\leq C_3(n,p,x,\frac{1}{b-a}) \cdot \|f^{(n)}\|_p, \end{aligned} \tag{2.16}$$

where

$$D(f,x) = \frac{(b-a)^2}{6(2x-a-b)^2}\left(f(x)+f(a+b-x)\right) + \left(1 - \frac{(b-a)^2}{3(2x-a-b)^2}\right) f(\frac{a+b}{2}), \tag{2.17}$$

$$\begin{aligned} T_{n,\frac{1}{b-a}}(f,x) &= \sum_{k=5}^{n} A^{(3)}_{k,\frac{1}{b-a}}(x)\left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x)\right) \\ &+ \sum_{\substack{k=5 \\ odd}}^{n} B^{(3)}_{k,\frac{1}{b-a}}(x) f^{(k-1)}(\frac{a+b}{2}) \end{aligned} \tag{2.18}$$

and

$$C_3(n,p,x,\frac{1}{b-a}) = \begin{cases} 2^{1/q}\left[\int_a^{\frac{a+b}{2}} \left|W^{(3)}_{n,\frac{1}{b-a}}(t,x)\right|^q \mathrm{d}t\right]^{1/q}, & \frac{1}{p}+\frac{1}{q}=1, \quad 1<p\leq\infty \\ \sup\{\left|W^{(3)}_{n,w}(t,x)\right|, t\in[a,\frac{a+b}{2}]\}, & p=1, \end{cases} \tag{2.19}$$

3. Ostrowski-Grüss type inequalities for the remainder of the weight threepoint formula

Let us apply identity (1.1) for $f \leftrightarrow (-1)^n W_{n,w}(\cdot,x)$ and $g \leftrightarrow f^{(n)}$:

$$\begin{aligned} T((-1)^n W^{(3)}_{n,w}(\cdot,x), f^{(n)}) &= \frac{1}{b-a}\int_a^b (-1)^n W^{(3)}_{n,w}(t,x) f^{(n)}(t)dt \\ &- \frac{1}{b-a}\int_a^b (-1)^n W^{(3)}_{n,w}(t,x)dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)dt. \end{aligned} \tag{3.1}$$

Theorem 7. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous function, $w:[a,b]\to[0,\infty)$ is weight function such that $w(t)=w(a+b-t)$, for each $t\in[a,b]$ and $x\in\left[a,\frac{a+b}{2}\right]$. Then the following identity holds:*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)w(t)dt - \frac{1}{b-a}\left[A^{(3)}_{1,w}(x)\left[f(x)+f(a+b-x)\right] + B^{(3)}_{1,w}(x) f(\frac{a+b}{2}) + T^{(3)}_{n,w}(x)\right] \\ &- \frac{(1+(-1)^n)A^{(3)}_{n,w}(x) + B^{(2)}_{n,w}(x)}{(b-a)^2}\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) = T((-1)^n W^{(3)}_{n,w}(\cdot,x), f^{(n)}). \end{aligned} \tag{3.2}$$

Proof. Divide relation (2.9) by $b-a$ and add to both sides of new identity term

$$- \frac{1}{b-a}\int_a^b (-1)^n W_{n,w}^{(3)}(t,x)dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)dt =$$
$$- \frac{(1+(-1)^n)A_{n,w}^{(3)}(x)+B_{n,w}^{(3)}(x)}{(b-a)^2}\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right).$$

Now the right side of the identity appears to be Chebyshev functional $T((-1)^n W_{n,w}^{(3)}(\cdot,x), f^{(n)})$ so the theorem is proved.

□

Now we establish error bounds for $T((-1)^n W_{n,w}^{(3)}(\cdot,x), f^{(n)})$..

Theorem 8. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous function with*

$$(\cdot-a)(b-\cdot)\left(f^{(n+1)}\right)^2 \in L[a,b],$$

$w:[a,b]\to[0,\infty)$ is weight function such that $w(t)=w(a+b-t)$, for each $t\in[a,b]$ and $x\in\left[a,\frac{a+b}{2}\right]$. Then we have

$$\left|T((-1)^n W_{n,w}^{(3)}(\cdot,x), f^{(n)})\right| \le \frac{1}{b-a}\left[(C_3(n,2,x,w))^2 - \frac{\left((1+(-1)^{n+1})A_{n+1,w}^{(3)}(x)+B_{n+1,x}^{(3)}(x)\right)^2}{b-a}\right]^{1/2}$$
$$\cdot \left[\frac{1}{2}\int_a^b (t-a)(b-t)\, f^{(n+1)}(t)^2 dt\right]^{1/2}. \qquad (3.3)$$

Proof. The proof follows from Cauchy-Schwartz's inequality

$$T(\varphi,\psi)^2 \le T(\varphi,\varphi)\cdot T(\psi,\psi)$$

applied to $T((-1)^n W_{n,w}^{(3)}(\cdot,x), f^{(n)})$. So we have

$$\left|T((-1)^n W_{n,w}^{(3)}(\cdot,x), f^{(n)})\right| \le \sqrt{T((-1)^n W_{n,w}^{(3)}(\cdot,x),(-1)^n W_{n,w}^{(3)}(\cdot,x))}\cdot\sqrt{T(f^{(n)},f^{(n)})}. \qquad (3.4)$$

We compute

$$T((-1)^n W_{n,w}^{(3)}(\cdot,x),(-1)^n W_{n,w}^{(3)}(\cdot,x)) = \frac{1}{b-a}\int_a^b (-1)^{2n} W_{n.w}^{(3)}(t,x)^2\, dt$$
$$- \frac{1}{(b-a)^2}\left[\int_a^b (-1)^n W_{n,w}^{(3)}(t,x)\, dt\right]^2$$
$$= \frac{[C_3(n,2,x,w)]^2}{b-a} - \frac{\left[(1+(-1)^{n+1})A_{n+1,w}^{(3)}(x)+B_{n+1,x}^{(3)}(x)\right]^2}{(b-a)^2}.$$

On the other side, according to the Lemma 1. we have

$$T\left(f^{(n)},f^{(n)}\right) \le \frac{1}{2(b-a)}\int_a^b (t-a)(b-t)\, f^{(n+1)}(t)^2 dt,$$

which finishes the proof. □

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be such that f' is absolutely continuous function with $(\cdot - a)(b - \cdot)(f'')^2 \in L[a,b]$ and $x \in \left[a, \frac{a+b}{2}\right]$. Then we have*

$$\left|T(-W^{(3)}_{1,\frac{1}{b-a}}(\cdot,x), f')\right| \leq \frac{1}{b-a}\left[\left(C_3\left(1,2,x,\frac{1}{b-a}\right)\right)^2 - \frac{\left(2A^{(3)}_{2,\frac{1}{b-a}}(x) + B^{(3)}_{2,\frac{1}{b-a}}(x)\right)^2}{b-a}\right]^{1/2} \cdot \left[\frac{1}{2}\int_a^b (t-a)(b-t) f''(t)^2 dt\right]^{1/2}. \tag{3.5}$$

Proof. We apply Theorem 8 for $w(t) = \frac{1}{b-a}$ and $n = 1$. □

Remark 4. *Inserting harmonic polynomials $L_{n,w}$ in (3.5) we get:*

$$\left|T(-W^{(3)}_{1,\frac{1}{b-a}}(\cdot,x), f')\right| \leq$$

$$\leq \frac{1}{b-a}\sqrt{\frac{b-a}{12}\left(1 - \frac{(b-a)^2}{(2x-a-b)^2} + \frac{4(x-a)^2}{(2x-a-b)^2} - \frac{(b-a)^3}{3(2x-a-b)^3}\right)}$$

$$\cdot \left[\frac{1}{2}\int_a^b (t-a)(b-t) f''(t)^2 dt\right]^{1/2}.$$

Specially, for $x = a$ we have the unequality related to of the Simpson quadrature formula:

$$\left|T(-W^{(3)}_{1,\frac{1}{b-a}}(\cdot,a), f')\right| \leq \frac{1}{6}\left[\frac{1}{2}\int_a^b (t-a)(b-t) f''(t)^2 dt\right]^{1/2}.$$

References

[1] P. Cerone, S.S. Dragomir, *Some new Ostrowski-type bounds for the Čebyšev functional and applications*,J. Math. Inequal.8(2014) (1) 159-170.

[2] S. Kovač, J. Pečarić, *Weighted version of general integral formula*,Math. Inequal. Appl., 13 (2010)(3) 579-599.

[3] S. Kovač, J. Pečarić, S. Tipurić-Spužević, *The Ostrowski Type Inequalities with the application to the three point integral formula*, Math. Iinequal. Appl., 22 (2019) (2) 401-420.

Faculty of Geotechnical Engineering, University of Zagreb, Hallerova aleja 7, 42 000 Varaždin, Croatia.

Email address: sanja.kovac@gfv.unizg.hr

A CONTRIBUTION TO THE LINEARIZATION

OF THE VEKUA EQUATION

UDC: 517.968.7:517.55

Slagjana Brsakoska

Abstract. In the paper it is given a small contribution to the linearization of the Vekua differential equation.

1. INTRODUCTION

The equation

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} + F \qquad (1)$$

where $A = A(z)$, $B = B(z)$ and $F = F(z)$ are given complex functions from a complex variable $z \in D \subseteq \mathbb{C}$ is the well known Vekua equation [1] according to the unknown function $W = W(z) = u + iv$. The derivative on the left side of this equation has been introduced by G.V. Kolosov in 1909 [2]. During his work on a problem from the theory of elasticity, he introduced the expressions

$$\frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] = \frac{\hat{d}W}{dz} \qquad (2)$$

and

$$\frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] = \frac{\hat{d}W}{d\bar{z}} \qquad (3)$$

known as operator derivatives of a complex function $W = W(z) = u(x,y) + iv(x,y)$ from a complex variable $z = x + iy$ and $\bar{z} = x - iy$ corresponding. The operating rules for this derivatives are completely given in the monograph of Г. Н.Положий [3] (page18-31). In the mentioned monograph are defined so cold operator integrals $\hat{\int} f(z)dz$ and $\hat{\int} f(z)d\bar{z}$ from $z = x + iy$ and $\bar{z} = x - iy$ corresponding (page 32-41). As for the complex integration in the same monograph is emphasized that it is assumed that all operator integrals can be solved in the area D.

In the Vekua equation (1) the unknown function $W = W(z)$ is under the sign of a complex conjugation which is equivalent to the fact that $B = B(z)$ is not identically equaled to zero in D. That is why for (1) the quadratures that we have for

2010 *Mathematics Subject Classification.* 34M45, 35Q74.
Key words and phrases. areolar derivative, areolar equation, analytic function, Vekua equation, generalized homogeneous differential equation.

the equations where the unknown function $W = W(z)$ is not under the sign of a complex conjugation, stop existing.

This equation is important not only for the fact that it came from a practical problem, but also because depending on the coefficients A, B and F the equation (1) defines different classes of generalized analytic functions. For example, for $F = F(z) \equiv 0$ in D the equation (1) i.e.

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} \tag{4}$$

which is called canonical Vekua equation, defines so cold generalized analytic functions from fourth class; and for $A \equiv 0$ and $F \equiv 0$ in D, the equation (1) i.e. the equation $\frac{\hat{d}W}{d\bar{z}} = B\overline{W}$ defines so cold generalized analytic functions from third class or the (r+is)-analytic functions [3], [4].

Those are the cases when $B \neq 0$. But if we put $B \equiv 0$, we get the following special cases. In the case $A \equiv 0$, $B \equiv 0$ and $F \equiv 0$ in the working area $D \subseteq \mathbb{C}$ the equation (1) takes the following expression $\frac{\hat{d}W}{d\bar{z}} = 0$ and this equation, in the class of the functions $W = u(x, y) + iv(x, y)$ whose real and imaginary parts have unbroken partial derivatives u'_x, u'_y, v'_x and v'_y in D, is a complex writing of the Cauchy - Riemann conditions. In other words it defines the analytic functions in the sense of the classic theory of the analytic functions. In the case $B \equiv 0$ in D is the so cold areolar linear differential equation [3] (page 39-40) and it can be solved with quadratures.

2. MAIN RESULT

Let`s consider the Vekua equation

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} + F \tag{1}$$

where $A = A(z), B = B(z)$ and $F = F(z)$ are given analytic functions from a complex variable $z \in D \subseteq \mathbb{C}$ and the areolar linear differential equation

$$\frac{\hat{d}W}{d\bar{z}} = AW + F\ . \tag{5}$$

As mentioned above, the equation (5) can be solved and its solution is given with the following formula:

$$W = e^{\hat{\int} A(z)d\bar{z}}[\Phi(z) + \hat{\int} F(z)e^{-\hat{\int} A(z)d\bar{z}} d\bar{z}]. \tag{6}$$

Here $\Phi = \Phi(z)$ is an arbitrary analytic function in the role of an integral constant.

If we have in mind that $A = A(z)$ and $F = F(z)$ are analytic functions, then they have the role of constants in the areolar integrals in (6) where the integration is by $\overline{z}$, we can write this solution in the following form

$$W = e^{A(z)\hat{\int} d\overline{z}}[\Phi(z) + F(z)\hat{\int} e^{-A(z)\hat{\int} d\overline{z}} d\overline{z}]$$

Now, $\hat{\int} d\overline{z} = \overline{z}$, so

$$W = e^{A\overline{z}}[\Phi(z) + F(z)\hat{\int} e^{-A\overline{z}} d\overline{z}]$$

and $\hat{\int} e^{-A\overline{z}} d\overline{z} = -\frac{1}{A} e^{-A\overline{z}}$, so we have

$$W = e^{A\overline{z}}[\Phi - \frac{F}{A} e^{-A\overline{z}}]$$

i.e.

$$W = \Phi e^{A\overline{z}} - \frac{F}{A}. \tag{7}$$

The function $\Phi = \Phi(z)$ is an arbitrary analytic function in the role of an integral constant for the areolar linear differential equation (5). Let`s consider the possibility the function (7) to be a solution of the Vekua equation (1) if we put $\Phi = f(z)g(\overline{z})$, where $f = f(z)$ is an analytic function and $g = g(\overline{z})$ is an antianalytic function. In other words we would like to find the condition that will make the function

$$W = f(z)g(\overline{z})e^{A\overline{z}} - \frac{F}{A} \tag{8}$$

a solution of the Vekua equation (1).

For that purpose, we have to find the areolar derivative of this function and substitute it in the equation (1). We have

$$\frac{\hat{d}W}{d\overline{z}} = f(z)\frac{\hat{d}}{d\overline{z}}\left(g(\overline{z})e^{A\overline{z}}\right) - 0$$

because A and F are analytic functions and

$$\frac{\hat{d}W}{d\overline{z}} = f(z)\left(\frac{\hat{d}g}{d\overline{z}}e^{A\overline{z}} + g(\overline{z})Ae^{A\overline{z}}\right) \text{ i.e.}$$

$$\frac{\hat{d}W}{d\overline{z}} = f(z)e^{A\overline{z}}\left(\frac{\hat{d}g}{d\overline{z}} + Ag(\overline{z})\right).$$

After substituting this derivative and the function (8) in (1), we get

$$f(z)e^{A\bar{z}}\left(\frac{\hat{d}g}{d\bar{z}}+Ag(\bar{z})\right)=A\left(f(z)g(\bar{z})e^{A\bar{z}}-\frac{F}{A}\right)+B\overline{\left(f(z)g(\bar{z})e^{A\bar{z}}-\frac{F}{A}\right)}+F$$

$$f(z)e^{A\bar{z}}\frac{\hat{d}g}{d\bar{z}}+Af(z)g(\bar{z})e^{A\bar{z}}=Af(z)g(\bar{z})e^{A\bar{z}}-F+B\overline{f(z)}\,\overline{g(\bar{z})}\,\overline{e^{A\bar{z}}}-B\frac{\bar{F}}{\bar{A}}+F$$

$$f(z)e^{A\bar{z}}\frac{\hat{d}g}{d\bar{z}}=B\overline{f(z)}\,\overline{g(\bar{z})}\,\overline{e^{A\bar{z}}}-B\frac{\bar{F}}{\bar{A}}. \qquad (9)$$

So, we can formulate the following

Theorem. *The function (8) is a solution of the Vekua equation (1) if the functions* $f=f(z)$ and $g=g(\bar{z})$ *are connected with the condition (9).*

What does this theorem gives us? We have the following method: if we have one Vekua equation (1), then, first we exclude the part with $\bar{W}$ and we solve the remaining linear areolar differential equation. Then, instead of the integral constant, we put a product of an analytic and an antianalytic function. If we consider this as a solution to the starting Vekua equation, then we can find the connection between the analytic and the antianalytic function. It is worth mentioning that if we put some concrete function or a class of functions in the place of $f=f(z)$, then for $g=g(\bar{z})$ we will get a new Vekua equation, where the unknown function is $g=g(\bar{z})$ which is not practical, and if we put some concrete function or a class of functions in the place of $g=g(\bar{z})$, then for $f=f(z)$ we can get an equation in which we have the functions f and $\bar{f}$.

Example. Let us consider the Vekua equation $\frac{\hat{d}W}{d\bar{z}}=zW+\bar{W}$. If we solve first the equation $\frac{\hat{d}W}{d\bar{z}}=zW$, we get the solution $W=\Phi e^{z\bar{z}}$. Now, if we proceed as described, we get $f(z)\frac{\hat{d}g}{d\bar{z}}=\overline{f(z)}\,\overline{g(\bar{z})}$. One solution are the functions $f=e^{z}$, $g=e^{\bar{z}}$.

Note: If $F=0$, i.e. instead the Vekua equation (1) we would like to consider the canonical Vekua equation (4), than the condition (9) would be a condition (10), where

$$f(z)e^{A\bar{z}}\frac{\hat{d}g}{d\bar{z}}=B\overline{f(z)}\,\overline{g(\bar{z})}\,\overline{e^{A\bar{z}}}. \qquad (10)$$

It is interesting that in this case, the modulo of B depends only from the antianalytic part of the integral constant, that is the function g and its areolar derivative, i.e.

$$|B|=\left|\frac{1}{\bar{g}}\cdot\frac{\hat{d}g}{d\bar{z}}\right|.$$

In [6] it is considered the case of accordance of the equation (1) and the generalized linear differential equation. Here we consider the accordance with the areolar linear differential equation, which at the same time is a process of linearization of the Vekua equation. Since we take some restrictions, the result is just a contribution in it.

References

[1] Н. И. Векуа, *Обобщение аналитические функции*, Москва,1988

[2] Г. В. Колосов, *Об одном приложении теории функции комплесного переменного к плоское задаче математическои упругости*, 1909

[3] Г. Н. Положии, *Обопштение теории аналитических фукции комплесного переменного*, Издателство Киевского Университета, 1965

[4] S. Brsakoska, *Operator differential equations from the aspect of the generalized analytic functions*, MSc thesis, Skopje, 2006

[5] B. Ilievski, S. Brsakoska, *Za soglasnost na homogena ravenka Vekua i obopštena diferencijalna ravenka od prv red* (*About an accordance of a homogenous Vekua equation and generalized differential equation of first order*), III Congress of Mathematicians of Macedonia, Struga, 2005

[6] S. Brsakoska, *About the accordance between the Vekua differential equation and the generalized linear differential equation*, Zbornik radova Međunarodne konferencije Matematičke i informacione tehnologije, MIT 2013, Vrnjačka Banja, Srbija, 2013, str. 95-98.

Faculty for Natural Sciences and Mathematics, University "Sts. Cyril and Methodius", Skopje, Republic of N.Macedonia
E-mail address: sbrsakoska@gmail.com