7th Iranian Geometry Olympiad

Contest problems with solutions

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This booklet is prepared by Sina Qasemi Nezhad, Alireza Dadgarnia, Hesam Rajabzadeh, Siavash Rahimi, Mahdi Etesamifard and Morteza Saghafian.
With special thanks to Matin Yousefi and Alireza Danaei.

Contents

Elementary Level 3

Problems 3

Solutions 5

Intermediate Level 15

Problems 15

Solutions 17

Advanced Level 29

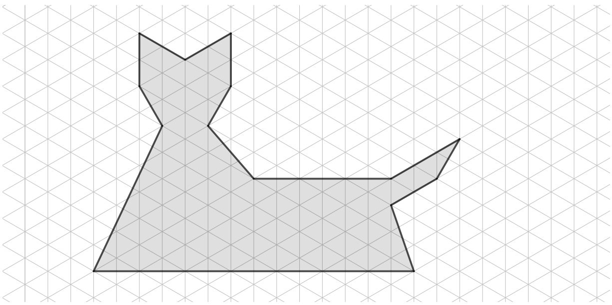
Problems 29

Solutions 31

Elementary Level

Problems

Problem 1. By a *fold* of a polygon-shaped paper, we mean drawing a segment on the paper and folding the paper along that. Suppose that a paper with the following figure is given. We cut the paper along the boundary of the shaded region to get a polygon-shaped paper.
Start with this shaded polygon and make a rectangle-shaped paper from it with at most 5 number of folds. Describe your solution by introducing the folding lines and drawing the shape after each fold on your solution sheet.
(Note that the folding lines do not have to coincide with the grid lines of the shape.)

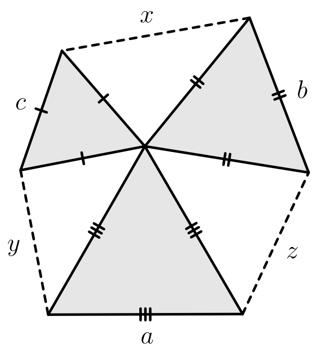


(→ p.5)

Problem 2. A parallelogram $ABCD$ is given ($AB \neq BC$). Points E and G are chosen on the line CD such that AC is the angle bisector of both angles $\angle EAD$ and $\angle BAG$. The line BC intersects AE and AG at F and H, respectively. Prove that the line FG passes through the midpoint of HE.

(→ p.8)

Problem 3. According to the figure, three equilateral triangles with side lengths a, b, c have one common vertex and do not have any other common point. The lengths x, y and z are defined as in the figure. Prove that $3(x+y+z) > 2(a+b+c)$.



(→ p.9)

Problem 4. Let P be an arbitrary point in the interior of triangle ABC. Lines BP and CP intersect AC and AB at E and F, respectively. Let K and L be the midpoints of the segments BF and CE, respectively. Let the lines through L and K parallel to CF and BE intersect BC at S and T, respectively; moreover, denote by M and N the reflection of S and T over the points L and K, respectively. Prove that as P moves in the interior of triangle ABC, line MN passes through a fixed point.

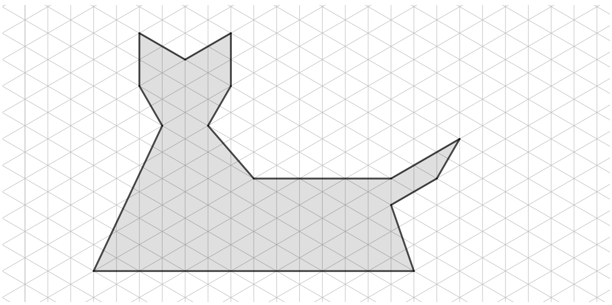
(→ p.10)

Problem 5. We say two vertices of a simple polygon are *visible* from each other if either they are adjacent, or the segment joining them is completely inside the polygon (except two endpoints that lie on the boundary). Find all positive integers n such that there exists a simple polygon with n vertices in which every vertex is visible from exactly 4 other vertices.
(A simple polygon is a polygon without hole that does not intersect itself.)

(→ p.11)

Solutions

Problem 1. By a *fold* of a polygon-shaped paper, we mean drawing a segment on the paper and folding the paper along that. Suppose that a paper with the following figure is given. We cut the paper along the boundary of the shaded region to get a polygon-shaped paper.
Start with this shaded polygon and make a rectangle-shaped paper from it with at most 5 number of folds. Describe your solution by introducing the folding lines and drawing the shape after each fold on your solution sheet.
(Note that the folding lines do not have to coincide with the grid lines of the shape.)



Proposed by Mahdi Etesamifard

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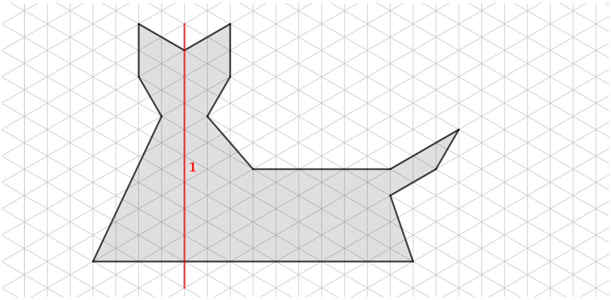
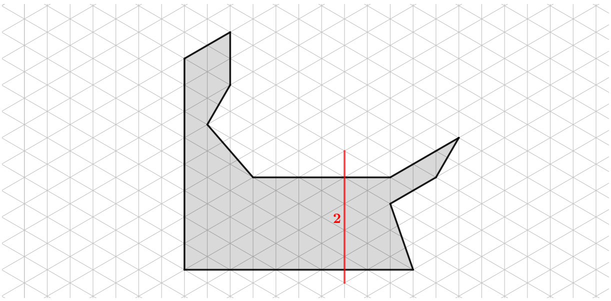
Solution. There are different ways of folding to get a rectangle. For instance, a solution can be given with only 4 number of folds, as following

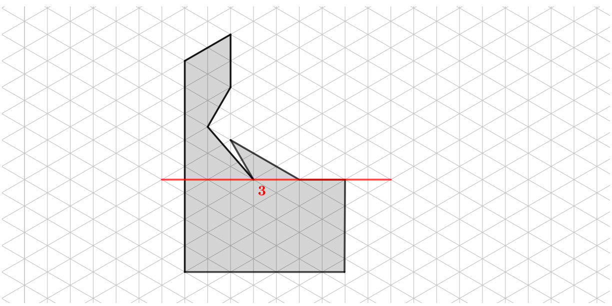
First fold:



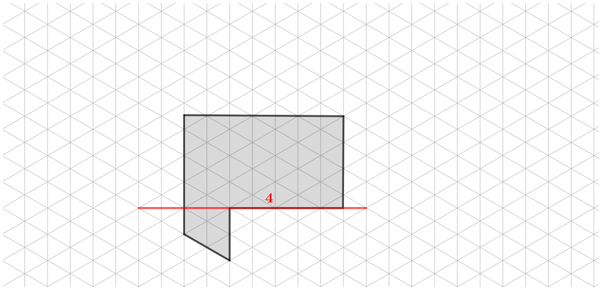
Second fold:



Third fold:



Fourth fold:



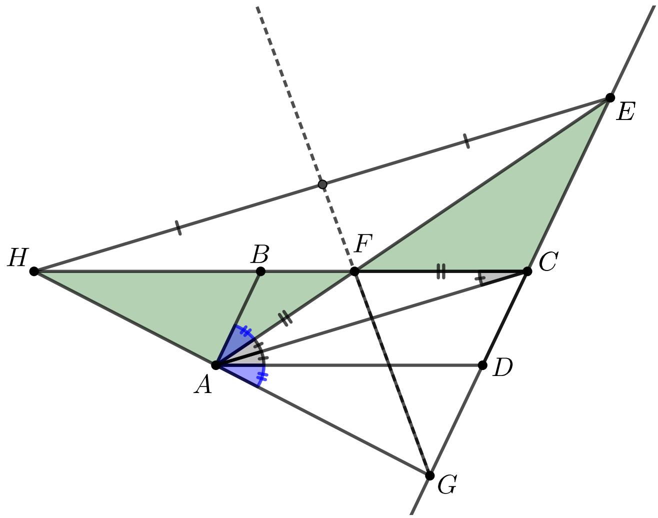
Comment. that by moving the folding lines slightly in 3rd and 4th folds (to down and up respectively), all the folding segments will be internal.

■

Problem 2. A parallelogram $ABCD$ is given ($AB \neq BC$). Points E and G are chosen on the line CD such that AC is the angle bisector of both angles $\angle EAD$ and $\angle BAG$. The line BC intersects AE and AG at F and H, respectively. Prove that the line FG passes through the midpoint of HE.

Proposed by Mahdi Etesamifard

Solution. Since AD and BC are parallel, we deduce that $\angle FCA = \angle DAC = \angle FAC$. So $FA = FC$. Similarly, $GA = GC$. So triangles $\triangle GAF$ and $\triangle GCF$ have a common side and two equal sides and are congruent. Resulting $\angle GAF = \angle GCF$ which leads to $\angle HAF = \angle ECF$ and $\angle AFH = \angle CFE$. Therefore triangles $\triangle AFH$ and $\triangle CFE$ are congruent as well and we get $FE = FH$. Similarly, $GE = GH$. So both points F and G lie on perpendicular bisector of segment HE. Resulting that FG is the perpendicular bisector of segment HE.



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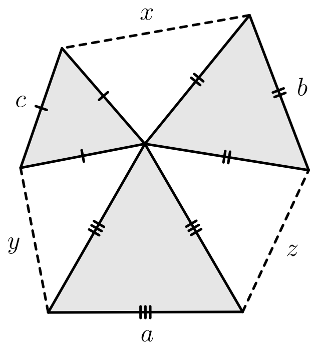
Problem 3. According to the figure, three equilateral triangles with side lengths a, b, c have one common vertex and do not have any other common point. The lengths x, y and z are defined as in the figure. Prove that $3(x+y+z) > 2(a+b+c)$.



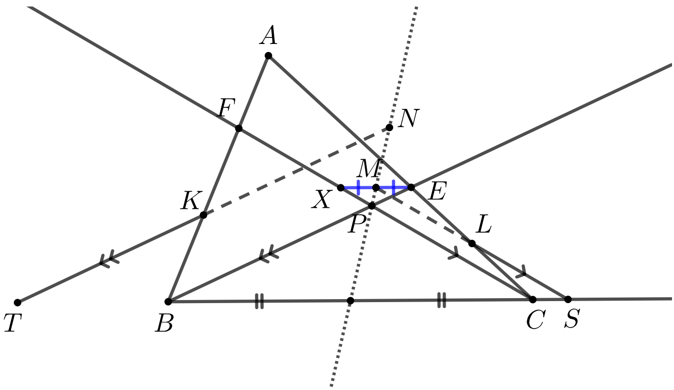
Proposed by Mahdi Etesamifard

Solution. Consider the three white triangles in the picture, rotating each triangle $60°$ degrees, clock-wise, will make a side of it coincide with another side of another triangle. So we can rotate one of them and glue it to the next, then rotating the glued figure a broken path will be formed between two points with distances like $2a$ which has length $x+y+z$. Thus $x+y+z > 2a$ summing up all three possible inequalities proves the desired. ■

Problem 4. Let P be an arbitrary point in the interior of triangle ABC. Lines BP and CP intersect AC and AB at E and F, respectively. Let K and L be the midpoints of the segments BF and CE, respectively. Let the lines through L and K parallel to CF and BE intersect BC at S and T, respectively; moreover, denote by M and N the reflection of S and T over the points L and K, respectively. Prove that as P moves in the interior of triangle ABC, line MN passes through a fixed point.

Proposed by Ali Zamani

Solution. Since in quadrilateral $EMCS$, diagonals bisect each other, this quadrilateral is a parallelogram. So, $EM \parallel BC$. Let X be the intersection of EM and CF. Note that $ML \parallel CX$ and L is the midpoint of CE, resulting that M is the midpoint of EX as well. Since $EX \parallel BC$, using parallel lines, one can find that MP passes through the midpoint of BC. Similarly, NP passes through the midpoint of BC. Hence proved.



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Problem 5. We say two vertices of a simple polygon are *visible* from each other if either they are adjacent, or the segment joining them is completely inside the polygon (except two endpoints that lie on the boundary). Find all positive integers n such that there exists a simple polygon with n vertices in which every vertex is visible from exactly 4 other vertices.
(A simple polygon is a polygon without hole that does not intersect itself.)

Proposed by Morteza Saghafian

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Solution.
First we prove there is no such polygon for $n > 6$. Let $A_1, A_2, \ldots, A_n$ be the vertices.

Lemma 1. *Let A_i be visible from $A_{i-1}, A_j, A_k, A_{i+1}$ in clockwise order (note that the first and the last one are the edge-neighbors). Then A_{i-1}, A_j can see each other, A_j, A_k can see each other and A_k, A_{i+1} can see each other.*

Proof. One can consider the triangulation of the three parts of polygon separated by A_iA_j and A_iA_k. □

Lemma 2. *Using the same naming as Lemma 1, A_jA_k is a side.*

Proof. Assume that A_jA_k is an internal diagonal. By Lemma 1, A_j can see A_{j-1}. But A_jA_i and A_jA_k are internal diagonals. So A_jA_{i-1} is a side. So there is only one vertex between A_i, A_j on the perimeter of polygon. Similarly, there is only one vertex between A_j, A_k and only one vertex between A_k, A_i on the perimeter of polygon. This contradicts $n > 6$. So A_jA_k is a side and $k = j - 1$. □

Now let i be such that A_{i-1}, A_{i+1} are visible from each other. We know that such i exists, for instance you cam take an ear triangle in the triangulation of the polygon. By Lemma 2, A_{i-1} can see A_{i+2}, A_{i+1} can see A_{i-2} and A_{i-2} can see A_{i+2}. So we found the four vertices visible from A_{i-1}, A_{i+1}. If A_i can see a vertex, then it is visible by either A_{i-1} or A_{i+1} (by Lemma 1). So A_i should see A_{i-2}, A_{i+2} and this means $A_{i-2}A_{i+2}$ is a side (by Lemma 2). Any convex pentagon is an example.
The only remaining case is $n = 6$ which means in Lemma 2 there are vertices A_i, A_j, A_k such that A_iA_j, A_jA_k, A_kA_i are internal diagonals. Let them be A_2, A_4, A_6 in the hexagon. So A_3 is not visible from A_6, meaning that one of the angles A_2, A_4 is larger that $180°$. But then A_3 cannot see either A_1 or A_5. Which contradicts the fact that A_3 is visible from 4 other vertices. So $n = 6$ is also not possible and the only possible n is 5.

■

Intermediate Level

Problems

Problem 1. A trapezoid $ABCD$ is given where AB and CD are parallel. Let M be the midpoint of the segment AB. Point N is located on the segment CD such that $\angle ADN = \frac{1}{2}\angle MNC$ and $\angle BCN = \frac{1}{2}\angle MND$. Prove that N is the midpoint of the segment CD.

(→ p.17)

Problem 2. Let ABC be an isosceles triangle ($AB = AC$) with its circumcenter O. Point N is the midpoint of the segment BC and point M is the reflection of the point N with respect to the side AC. Suppose that T is a point so that $ANBT$ is a rectangle. Prove that $\angle OMT = \frac{1}{2}\angle BAC$.

(→ p.18)

Problem 3. In acute-angled triangle ABC ($AC > AB$), point H is the orthocenter and point M is the midpoint of the segment BC . The median AM intersects the circumcircle of triangle ABC at X. The line CH intersects the perpendicular bisector of BC at E and the circumcircle of the triangle ABC again at F. Point J lies on circle ω, passing through X, E, and F, such that $BCHJ$ is a trapezoid ($CB \parallel HJ$). Prove that JB and EM meet on ω.

(→ p.19)

Problem 4. Triangle ABC is given. An arbitrary circle with center J, passing through B and C, intersects the sides AC and AB at E and F, respectively. Let X be a point such that triangle FXB is similar to triangle EJC (with the same order) and the points X and C lie on the same side of the line AB. Similarly, let Y be a point such that triangle EYC is similar to triangle FJB (with the same order) and the points Y and B lie on the same side of the line AC. Prove that the line XY passes through the orthocenter of the triangle ABC.

(→ p.21)

Problem 5. Find all numbers $n \geq 4$ such that there exists a convex polyhedron with exactly n faces, whose all faces are right-angled triangles.
(Note that the angle between any pair of adjacent faces in a convex polyhedron is less than $180°$.)

(→ p.23)

Solutions

Problem 1. A trapezoid $ABCD$ is given where AB and CD are parallel. Let M be the midpoint of the segment AB. Point N is located on the segment CD such that $\angle ADN = \frac{1}{2}\angle MNC$ and $\angle BCN = \frac{1}{2}\angle MND$. Prove that N is the midpoint of the segment CD.

Proposed by Alireza Dadgarnia

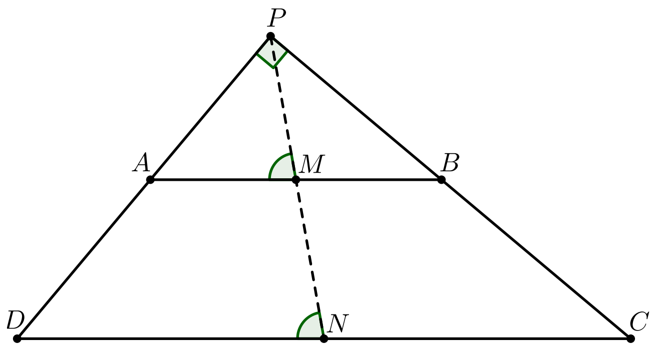
Solution. We have

$$\angle BCN + \angle ADN = \frac{1}{2}(\angle MND + \angle BDN) = 90^\circ.$$

Hence, AD and BC intersect in point P such that $\angle DPC = 90^\circ$. Since M is the midpoint of AB,

$$\angle PMA = 2\angle PBA = 2\angle PCD = \angle MND.$$

Note that AB and CD are parallel, therefore, PM and MN are parallel to and M, N and P lie on a straight line, hence N is the midpoint of segment CD.



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Problem 2. Let ABC be an isosceles triangle ($AB = AC$) with its circumcenter O. Point N is the midpoint of the segment BC and point M is the reflection of the point N with respect to the side AC. Suppose that T is a point so that $ANBT$ is a rectangle. Prove that $\angle OMT = \dfrac{1}{2}\angle BAC$.

Proposed by Ali Zamani

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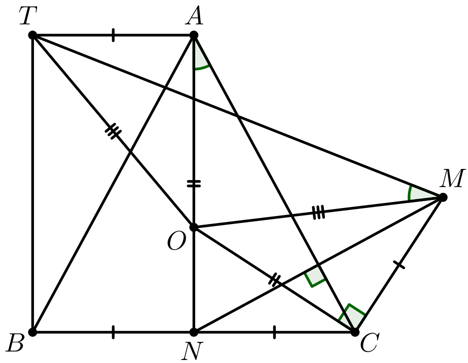
Since $\triangle ABC$ is an isosceles triangle, we have $\angle ANC = 90°$. Therefore,

$$\angle OCM = \angle OCA + \angle MCA = \angle OAC + \angle NCA = 90° = \angle TAO.$$

Also we have $CM = CN = BN = AT$ and $OC = OA$; So triangles $\triangle OCM$ and $\triangle OAT$ are congruent. Which leads to $OT = OM$ and

$$\angle AOT = \angle MOC \Longrightarrow \angle TOM = \angle AOC.$$

Thus, $\triangle AOC \sim \triangle MOT$ and $\angle OMT = \angle OAC = \frac{1}{2}\angle A$.



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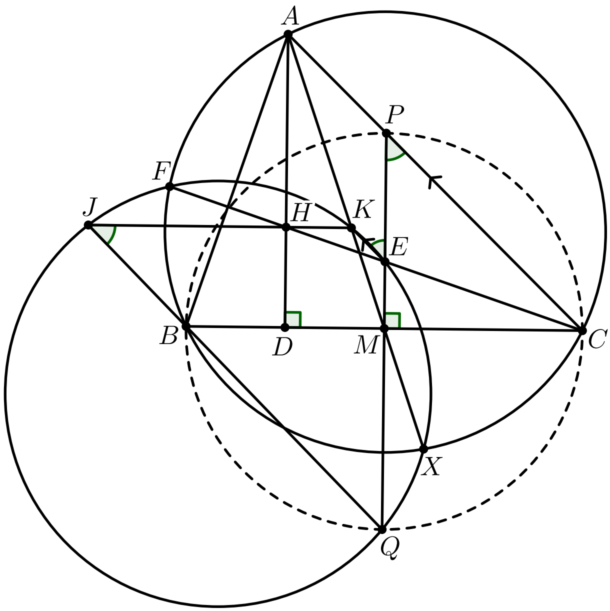
Problem 3. In acute-angled triangle ABC ($AC > AB$), point H is the orthocenter and point M is the midpoint of the segment BC . The median AM intersects the circumcircle of triangle ABC at X. The line CH intersects the perpendicular bisector of BC at E and the circumcircle of the triangle ABC again at F. Point J lies on circle ω, passing through X, E, and F, such that $BCHJ$ is a trapezoid ($CB \parallel HJ$). Prove that JB and EM meet on ω.

Proposed by Alireza Dadgarnia

Solution. Let D be the foot of altitude passing through A and P, K be the intersection of lines EM, AC and JH, AM, respectively.



From parallel lines, we have

$$\frac{ME}{EP} = \frac{DH}{HA} = \frac{MK}{KA} \Longrightarrow EK \parallel AC. \tag{1}$$

Note that $\angle XKE = \angle XAC = \angle XFE$. So K lies on ω. Let Q be the second intersection point of line EM and circle ω. We have

$$\angle KJQ = \angle KEP \overset{(1)}{=} \angle EPC = \angle QPC.$$

Note that it suffices to prove that $\angle KJQ = \angle CBQ$ or prove that $CPBQ$ is a cyclic quadrilateral. Which is equivalent to $MP \cdot MQ = MB \cdot MC$. Also, noting the parallel lines we can write $MA = \frac{MK \cdot MP}{ME}$. Using this equation and power of the point M with respect to the circumcircle of triangle $\triangle ABC$, we have

$$MB \cdot MC = MA \cdot MX = \frac{MK \cdot MX}{ME} \cdot MP = MQ \cdot MP.$$

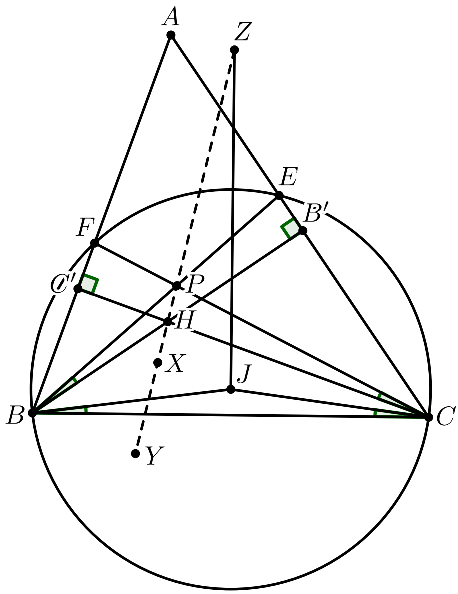
Where the last equation comes from power of the point M with respect to circle ω. Hence proved. ■

Comment. The same proof can be used to obtain the following generalised result:
In triangle ABC point P is an arbitrary point and point D lies on the line BC. The line AD intersects the circumcircle of triangle ABC at X. The line CP intersects the line parallel to AP through D at E and the circumcircle of triangle ABC again at F. Suppose that P lies inside of circle ω, passing through X, E, and F. Point J lies on ω such that $BCPJ$ is a trapezoid $(CB \parallel PJ)$. Then JB and ED meet on ω.

Problem 4. Triangle ABC is given. An arbitrary circle with center J, passing through B and C, intersects the sides AC and AB at E and F, respectively. Let X be a point such that triangle FXB is similar to triangle EJC (with the same order) and the points X and C lie on the same side of the line AB. Similarly, let Y be a point such that triangle EYC is similar to triangle FJB (with the same order) and the points Y and B lie on the same side of the line AC. Prove that the line XY passes through the orthocenter of the triangle ABC.

Proposed by Nguyen Van Linh - Vietnam

Solution. Let H be the orthocenter of triangle $\triangle ABC$, P be the intersection of BE and CF. PH cuts the perpendicular bisector of BC at Z.



We have

$$\angle HBP = \angle ABH - \angle ABP = 90^\circ - \angle BAC - \angle ABP = 90^\circ - \angle BEC = \angle JBC.$$

Then BH and BJ are isogonal lines with respect to angle $\angle PBC$. Similarly, CH and CJ are isogonal lines with respect to angle $\angle PCB$. From this, we deduce that H and J are isogonal conjugate with respect to triangle $\triangle BPC$. Then $\angle HPB = \angle JPC$. But $ZB = ZC$, $JF = JE$ and $\triangle PFE \sim \triangle PBC$. Therefore $\triangle PFE \cup \{J\} \sim \triangle PBC \cup \{Z\}$. Which follows that $\triangle JEF \sim$

$\triangle ZCB$.
Let B', C' be the intersections of BH and AC, CH and AB, repectively. We have

$$P^H_{(BE)} = HB \cdot HB' = HC \cdot HC' = P^H_{(CF)},$$

$$P^P_{(BE)} = PB \cdot PE = PC \cdot PF = P^P_{(CF)}.$$

We get Z lies on HP, which is the radical axis of circles with diameters BE and CF. Analogously, X, Y also lie on HP. Therefore XY passes through the orthocenter of triangle $\triangle ABC$. ■

Problem 5. Find all numbers $n \geq 4$ such that there exists a convex polyhedron with exactly n faces, whose all faces are right-angled triangles.
(Note that the angle between any pair of adjacent faces in a convex polyhedron is less than $180°$.)

Proposed by Hesam Rajabzadeh

Solution. If such a polyhedron exists for some n, the total number of sides of faces is from one hand equal to $3n$, and on the other is twice the number of edges. So $3n$ is divisible by 2 and n must be even. We will give an example of such a polyhedron for any even number $n \geq 4$.

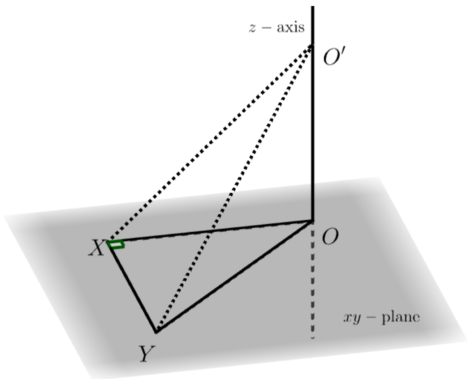
To this purpose, we need the following lemma.

Lemma 1. *Let O be the origin in the 3-dimensional space and suppose X, Y are two distinct points (different from O) in the xy-plane so that $\angle OXY = 90°$. Then for any point O' on the z-axis, the triangle $O'XY$ is right-angled (with $\angle O'XY = 90°$).*

Proof. The proof is based on the Pythagorean Theorem. If $O' = O$, there is nothing to prove. If $O' \neq O$, the line OO' (the z-axis) is perpendicular to the xy-plane and so is perpendicular to every line in this plane passing through O. In particular two triangles, $O'OX$ and $O'OY$ are right-angled. According to the Pythagorean Theorem in these two triangles together with triangle OXY, we have

$$O'Y^2 = O'O^2 + OY^2 = O'O^2 + OX^2 + XY^2 = O'X^2 + XY^2.$$

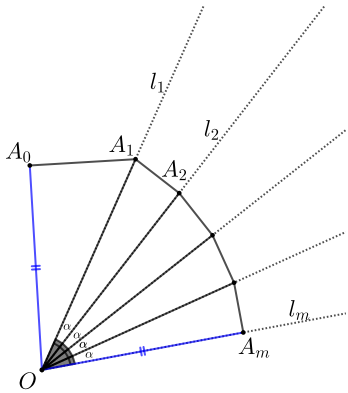
which implies $\angle O'XY = 90°$.



□

Now we return to the main problem. If $n = 4$, the tetrahedron with vertices O', O, X, Y as in the lemma works (above figure). So we can assume $n \geq 6$. Take $m = \frac{n-2}{2} \geq 2$. First, we construct a convex $(m+2)$-gon $OA_0A_1 \cdots A_m$ in the xy-plane (take O to be the origin) so that

- $OA_0 = OA_m$.
- All the triangles of the form OA_iA_{i+1} (for $0 \leq i \leq m-1$) are right-angled.



Consider m different rays with initial point O (denote them by $l_1, \ldots, l_m$ respectively in clockwise order) so that for a sufficiently small value of α,

$$\angle l_1Ol_2 = \angle l_2Ol_3 = \cdots = \angle l_{m-1}Ol_m = \alpha. \tag{1}$$

Take an arbitrary point on the ray l_1 and call it A_1. Start from A_1 and inductively by drawing perpendiculars from A_i to l_{i+1} define the points $A_2, A_3, \ldots, A_m$ so that

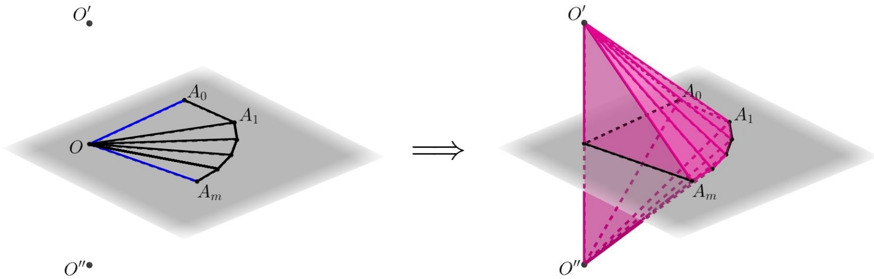
$$\angle OA_2A_1 = \angle OA_3A_2 = \cdots = OA_mA_{m-1} = 90^\circ. \tag{2}$$

By (1) and (2) all the triangles $OA_1A_2, OA_2A_3, \ldots, OA_{m-1}OA_m$ are similar. Therefore $\frac{OA_m}{OA_{m-1}} = \cdots = \frac{OA_3}{OA_2} = \frac{OA_2}{OA_1}$. We denote this common value by $r < 1$. Note that r can be arbitrarily close to 1 by taking α small. Now we have

$$OA_m = \frac{OA_m}{OA_{m-1}}. \cdots .\frac{OA_3}{OA_2}.\frac{OA_2}{OA_1}.OA_1 = r^m OA_1.$$

Note that since α is small all the points $A_2, A_3, \ldots, A_m$ are on the same side of the line OA_1. Take the point A_0 on the other side of this line so that $\angle OA_0A_1 = 90^\circ$ and $OA_0 = r^m.OA_1$ (A_0 is one of the intersection points of the circle with diameter OA_1 and the circle with center O and radius $r^m.OA_1$). If r is sufficiently close to 1 (equivalently α sufficiently close to zero), r^m will be close to one and we can ensure that $\angle A_0OA_1$ is small and so the polygon satisfies all desired properties.

After construction of the polygon, consider two points O', O'' on the z-axis (on different sides of the xy-plane) with $OO' = OO'' = OA_0 = OA_m$. Then the polyhedron with vertices $O', O'', A_0, A_1, \ldots, A_m$ (convex hull of these points) have exactly $n = 2m+2$ faces, and all are right-angled triangles. Indeed, it has $2m$ faces of the form $O'A_iA_{i+1}$ and $O''A_iA_{i+1}$ which are all right-angled according to the lemma and two faces $O'A_0O''$ and $O'A_mO''$ that are isosceles right triangles.



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Advanced Level

Problems

Problem 1. Let M, N, and P be the midpoints of sides BC, AC, and AB of triangle ABC, respectively. E and F are two points on the segment BC so that $\angle NEC = \frac{1}{2}\angle AMB$ and $\angle PFB = \frac{1}{2}\angle AMC$. Prove that $AE = AF$.

(→ p.31)

Problem 2. Let ABC be an acute-angled triangle with its incenter I. Suppose that N is the midpoint of the arc BAC of the circumcircle of triangle ABC, and P is a point such that $ABPC$ is a parallelogram. Let Q be the reflection of A over N, and R the projection of A on QI. Show that the line AI is tangent to the circumcircle of triangle PQR.

(→ p.33)

Problem 3. Assume three circles mutually outside each other with the property that every line separating two of them have intersection with the interior of the third one. Prove that the sum of pairwise distances between their centers is at most $2\sqrt{2}$ times the sum of their radii.
(A line separates two circles, whenever the circles do not have intersection with the line and are on different sides of it.)

Note. Weaker results with $2\sqrt{2}$ replaced by some other c may be awarded points depending on the value of $c > 2\sqrt{2}$.

(→ p.35)

Problem 4. Convex circumscribed quadrilateral $ABCD$ with incenter I is given such that its incircle is tangent to AD, DC, CB, and BA at K, L, M, and N. Lines AD and BC meet at E and lines AB and CD meet at F. Let KM intersects AB and CD at X and Y, respectively. Let LN intersects AD and BC at Z and T, respectively. Prove that the circumcircle of triangle XFY and the circle with diameter EI are tangent if and only if the circumcircle of triangle TEZ and the circle with diameter FI are tangent.

(→ p.37)

Problem 5. Consider an acute-angled triangle ABC ($AC > AB$) with its orthocenter H and circumcircle Γ. Points M and P are the midpoints of the segments BC and AH, respectively. The line AM meets Γ again at X and point N lies on the line BC so that NX is tangent to Γ. Points J and K lie on the circle with diameter MP such that $\angle AJP = \angle HNM$ (B and J lie on the same side of AH) and circle ω_1, passing through K, H, and J, and circle ω_2, passing through K, M, and N, are externally tangent to each other. Prove that the common external tangents of ω_1 and ω_2 meet on the line NH.

(→ p.43)

Solutions

Problem 1. Let M, N, and P be the midpoints of sides BC, AC, and AB of triangle ABC, respectively. E and F are two points on the segment BC so that $\angle NEC = \frac{1}{2}\angle AMB$ and $\angle PFB = \frac{1}{2}\angle AMC$. Prove that $AE = AF$.

Proposed by Alireza Dadgarnia

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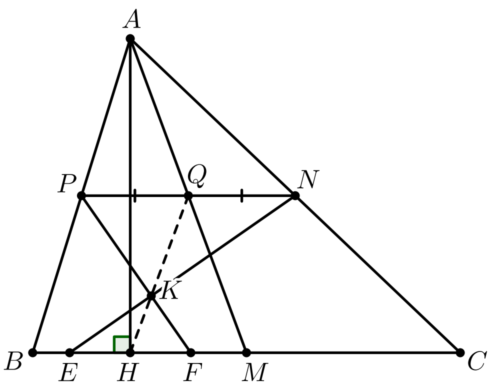
Solution. Let H be the foot of the altitude passing through Q, A be the midpoint of NP and K be the intersection point of NE and PF.



If we prove that points K, H and Q are collinear, using parallel lines ,we get that H is the midpoint of EF which is equivalent to the problem. Clearly, AM passes through Q and H is the reflection of A with respect to NP. Therefore, $\angle PQH = \angle AQP = \angle AMB$. So it suffices to show that $\angle PQK = \angle AMB$. Note that

$$\angle NEC + \angle PFB = \frac{1}{2}(\angle AMB + \angle AMC) = 90^\circ \Longrightarrow \angle EKF = 90^\circ.$$

So KQ is a median on the hypotenuse in triangle $\triangle PKN$ and we'll get

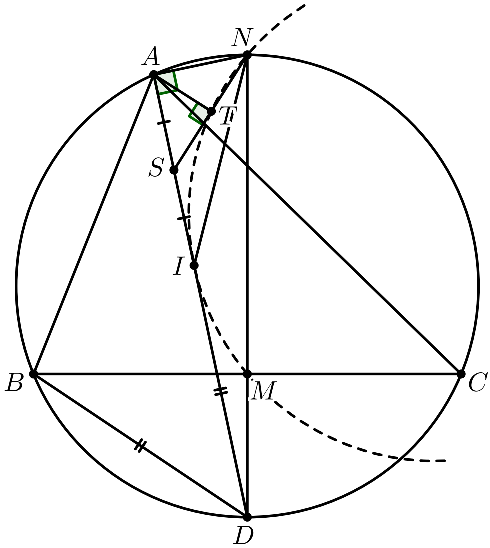
$$\angle PQK = 2\angle PNK = 2\angle NEC = \angle AMB$$

which completes the proof. $\blacksquare$

Problem 2. Let ABC be an acute-angled triangle with its incenter I. Suppose that N is the midpoint of the arc BAC of the circumcircle of triangle ABC, and P is a point such that $ABPC$ is a parallelogram. Let Q be the reflection of A over N, and R the projection of A on QI. Show that the line AI is tangent to the circumcircle of triangle PQR.

Proposed by Patrik Bak - Slovakia

Solution. Let M, S be the midpoint of segments BC, AI, respectively. By a homothety with center A and ratio $\frac{1}{2}$, P goes to M, Q to N and R to T; Where T is the projection of A on SN. So it suffices to show that the circumcircle of triangle $\triangle MNT$ is tangent to AI.



We claim that this circle is tangent to AI at point I. We know that $\angle NAS = 90°$, So by the similarity of two triangles $\triangle ASN$, $\triangle TSA$, we'll get

$$ST \cdot SN = SA^2 = SI^2.$$

Therefore, SI is tangent to the circumcircle of triangle $\triangle ITN$. Now if we show that SI is tangent to the circumcircle of triangle $\triangle NIM$ as well, our proof is completed; Because the circle passing through I and N and tangent to SI is unique. Let D be the second intersection point of AI and circumcircle of triangle $\triangle ABC$. Note that $\angle DBM = \angle DCB = \angle DNB$. Therefore,

$$DM \cdot DN = DB^2 = DI^2.$$

Thus, DI is tangent to the circumcircle of triangle $\triangle NIM$ and we're done. ■

Problem 3. Assume three circles mutually outside each other with the property that every line separating two of them have intersection with the interior of the third one. Prove that the sum of pairwise distances between their centers is at most $2\sqrt{2}$ times the sum of their radii.
(A line separates two circles, whenever the circles do not have intersection with the line and are on different sides of it.)

Note. Weaker results with $2\sqrt{2}$ replaced by some other c may be awarded points depending on the value of $c > 2\sqrt{2}$.

Proposed by Morteza Saghafian

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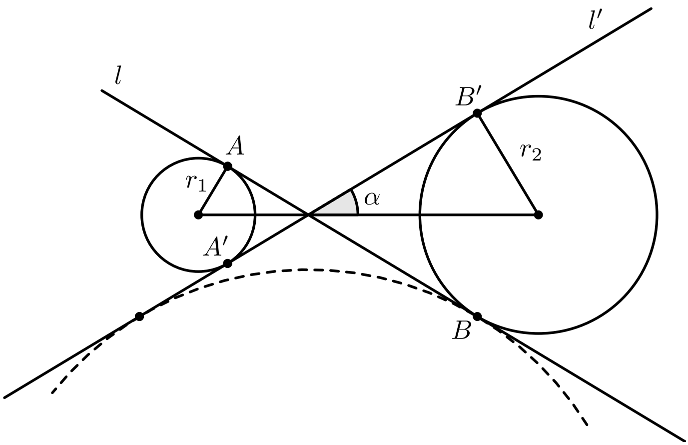
Solution. According to the figure, we denote the radii of the circles by r_1, r_2, r_3 and the distance O_iO_j by d_{ij}. Moreover, let l, l' be two interior common tangents of circles ω_1 and ω_2. We denote the tangency points of l and l' as in the figure. Obviously $d_{12} = \frac{r_1+r_2}{\sin\alpha}$ (α is defined in the figure). Withour loss of generality we assume that $r_1 \leq r_2$.



By assumption we can deduce that both lines l and l' must intersect the third circle (ω_3). If the intersection point of l and ω_3 lies outside between A and B, we can find a line separating ω_1 and ω_2 so which does not intersect ω_3 and this is a contradiction with the assumptions. We have similar arguments for l'. So we can assume that the intersection of ω_3 with l and l' is below B and A' respectively. Therefore, r_3 is at least the radius of the circle tangent to l at B and also is tangent to l' (why?). The radius of this circle is $r_2 \cot^2\alpha$. Hence

$$r_3 \geq r_2 \cot^2\alpha = r_2\left(\frac{1-\sin^2\alpha}{\sin^2\alpha}\right) \geq \frac{r_1+r_2}{2}\left(\frac{d_{12}^2}{(r_1+r_2)^2} - 1\right).$$

Consequently,

$$d_{12}^2 \leq (r_1 + r_2)^2 + 2r_3(r_1 + r_2), \quad (*)$$

We have similar equations for d_{13} and d_{23}. Summing these three together with Cauchy-Shwarz Inequality gives the assertion. Indeed,

$$\Big(\sum d_{ij}\Big)^2 \leq 3\sum d_{ij}^2 \leq 6\sum r_i^2 + 18\sum r_i r_j \leq 8\Big(\sum r_i\Big)^2$$

Here the first and third inequality are coming from Cauchy-Shwarz Inequality and the second inequality is the consequence of summing $(*)$ and two other similar inequlities.

Remark. Using upper bound $(r_1 + r_2 + r_3)^2$ for the right-hand side of $(*)$ gives $d_{12} \leq r_1 + r_2 + r_3$. Summing these, gives a weaker result with 3 replaced by $2\sqrt{2}$. ■

Problem 4. Convex circumscribed quadrilateral $ABCD$ with incenter I is given such that its incircle is tangent to AD, DC, CB, and BA at K, L, M, and N. Lines AD and BC meet at E and lines AB and CD meet at F. Let KM intersects AB and CD at X and Y, respectively. Let LN intersects AD and BC at Z and T, respectively. Prove that the circumcircle of triangle XFY and the circle with diameter EI are tangent if and only if the circumcircle of triangle TEZ and the circle with diameter FI are tangent.

Proposed by Mahdi Etesamifard

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Solution. First, let us prove these lemmas:

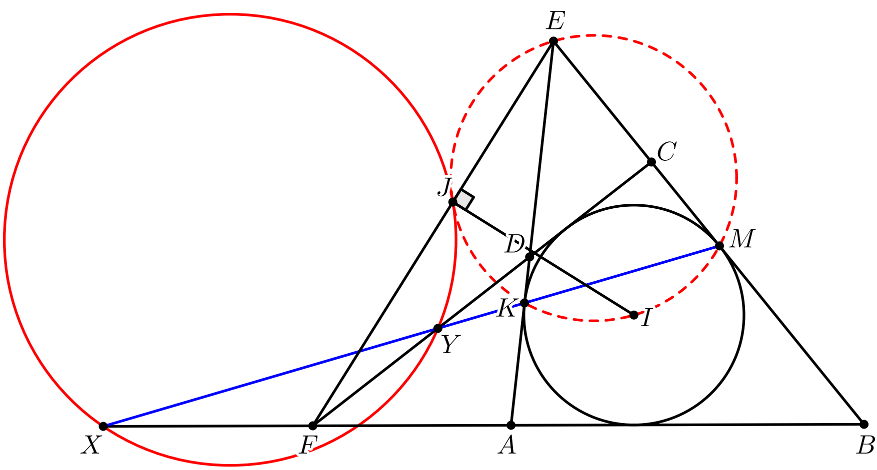
Lemma 1. *Lines AC, BD, KM and LN are concurrent.*

Proof. Using Brianchon's Theorem in quadrilateral $ABCD$, one can simply conclude the fact that AC, BD, KM and LN are concurrent. □

Lemma 2. *Let P be the point of concurrency of lines in Lemma 1. Therefore, P is also the intersection point of quadrilateral $ABCD$'s diagonals and we have $IP \perp EF$.*

Proof. We know that polar of point P is in fact line EF. Therefore, we'll get $IP \perp EF$. □

Lemma 3. *A circle with diameter EI and the circumcircle of triangle $\triangle XYJ$ are tangent.*



Proof. For the proof of tangency of circumcircle of triangle $\triangle XYJ$ to the circle with diameter EI (circle ω_2), it suffices that the equation of Casey's Theorem hold for points X, Y, J and circle ω_2.

$$\pm XY \cdot P^J_{\omega_2} \pm XJ \cdot P^Y_{\omega_2} \pm YJ \cdot P^X_{\omega_2} = 0.$$

Since $P^J_{\omega_2} = 0$, Therefore,

$$XJ\sqrt{YK \cdot YM} = YJ\sqrt{XK \cdot XM} \tag{1}$$

Since X, Y lie on the radical axis of two circles ω and ω_2, We have:

$$YK \cdot YM = YL^2 \quad , \quad XK \cdot XM = XN^2 \overset{(1)}{\Longrightarrow} XJ \cdot YL = YJ \cdot XN \tag{2}$$

So, we have to prove equation (2). Using Menelaus's Theorem for triangle $\triangle XFY$ and line LNP, We have:

$$\frac{XN}{FN} \cdot \frac{FL}{YL} \cdot \frac{YP}{XP} \overset{FN=FL}{\Longrightarrow} \frac{XN}{YL} = \frac{XP}{YP}.$$

From equation (2), we get:

$$\frac{XJ}{YJ} = \frac{XN}{YL} = \frac{XP}{YP}.$$

Therefore we need to prove that JP is the exterior angle bisector of angle $\angle XJY$. Since $JQ \perp JP$, we need to prove that $(XY, QP) = -1$.

$$(XY, PQ) = F(XY, PQ) \overset{NL}{=} (NL, PU) = -1.$$

And since point U lies on EF (polar of P), the last equation holds and we're done.

□

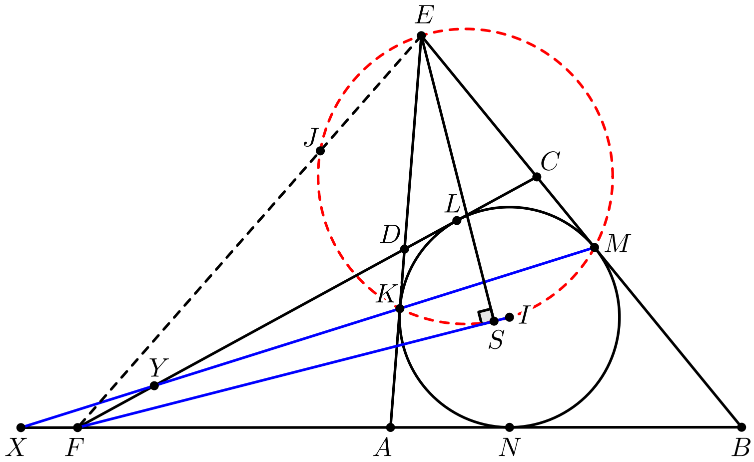
Lemma 4. *AK is tangent to the circumcircle of triangle $\triangle ABC$ if and only if*

$$\frac{BK}{KC} = \left(\frac{AB}{AC}\right)^2.$$

Proof. Using The Law of Sines and Ratio Lemma, one can simply get the desired results. □

Lemma 5. *If angle bisectors of angles $\angle E$ and $\angle F$ are perpendicular, then $ABCD$ is a cyclic quadrilateral.*

Proof. It's trivial. □



Now, Let's get back to the problem. First, we assume that two circles ω_1 and ω_2 are tangent to each other. Let S be the foot of the perpendicular line to FI passing through E. Using Casey's Theorem for points X, F, Y and circle ω_2, we have:

$$\begin{aligned} &\pm XF\sqrt{P^{Y}_{\omega_2}} \pm YF\sqrt{P^{X}_{\omega_2}} \pm XY\sqrt{P^{F}_{\omega_2}} = 0 \\ \Longrightarrow &\pm XF\sqrt{YK \cdot YM} \pm YF\sqrt{XK \cdot XM} \pm XY\sqrt{FS \cdot FI} = 0. \end{aligned} \tag{3}$$

Points X and Y lie on the radical axis of circles ω and ω_2. Therefore we have:

$$YK \cdot YM = YL^2 \quad , \quad XK \cdot XM = XN^2.$$

So equation (1) can be written as:

$$\pm XF \cdot YL \pm YF.XN \pm XY\sqrt{FS \cdot FI} = 0. \tag{4}$$

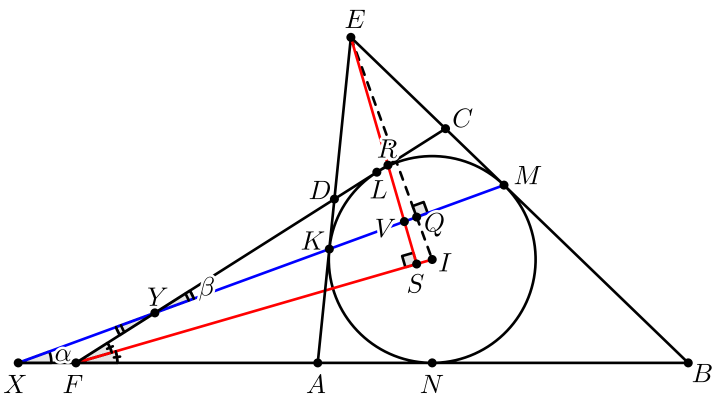
According to the figure, We have: $\angle F_1 = \angle F_2 = \frac{\alpha+\beta}{2}$.

$$YL = FL \pm FY = FI \cdot \cos(F_1) \pm FY = FI \cdot \cos\left(\frac{\alpha+\beta}{2}\right) \pm FY,$$

$$XN = FN \mp XF = FI \cdot \cos(F_2) \mp XF = FI \cdot \cos\left(\frac{\alpha+\beta}{2}\right) \mp XF.$$

Now, by putting them in equation (4), We'll get:

$$\begin{aligned}
&\pm XF \cdot \left(FI \cdot \cos\left(\frac{\alpha+\beta}{2}\right) \pm FY\right) \\
&\qquad\qquad \pm YF \cdot \left(FI \cdot \cos\left(\frac{\alpha+\beta}{2}\right) \mp XF\right) \pm XY\sqrt{FS \cdot FI} = 0 \\
&\Longrightarrow \pm FI\Big(XF + YF\Big)\cos\left(\frac{\alpha+\beta}{2}\right) = \pm XY\sqrt{FS \cdot FI} \\
&\Longrightarrow FI\left(\frac{XF+YF}{XY}\right)\cos\left(\frac{\alpha+\beta}{2}\right) = \sqrt{FS \cdot FI} \\
&\Longrightarrow \cos\left(\frac{\alpha+\beta}{2}\right) \cdot \left(\frac{\sin\alpha + \sin\beta}{\sin\alpha+\beta}\right) = \sqrt{\frac{FS}{FI}} \\
&\Longrightarrow \cos^2\left(\frac{\alpha-\beta}{2}\right) = \frac{FS}{FI}. \qquad (5)
\end{aligned}$$



Also, we have:

$$\angle FRS = 90^\circ - \left(\frac{\alpha+\beta}{2}\right) \Rightarrow \angle QVR = 90^\circ - \left(\frac{\alpha-\beta}{2}\right)$$
$$\Rightarrow \angle EIF = 90^\circ - \left(\frac{\alpha-\beta}{2}\right).$$

So, by equation (5), we have:

$$\sin^2(EIF) = \frac{FS}{FI}. \qquad (6)$$

We consider three cases for point S on line FI:

Case 1) $\angle EIF = 90^\circ$. Which gives us that S and I coincide.

$$\sin^2(EIF) = \frac{FS}{FI} = 1.$$

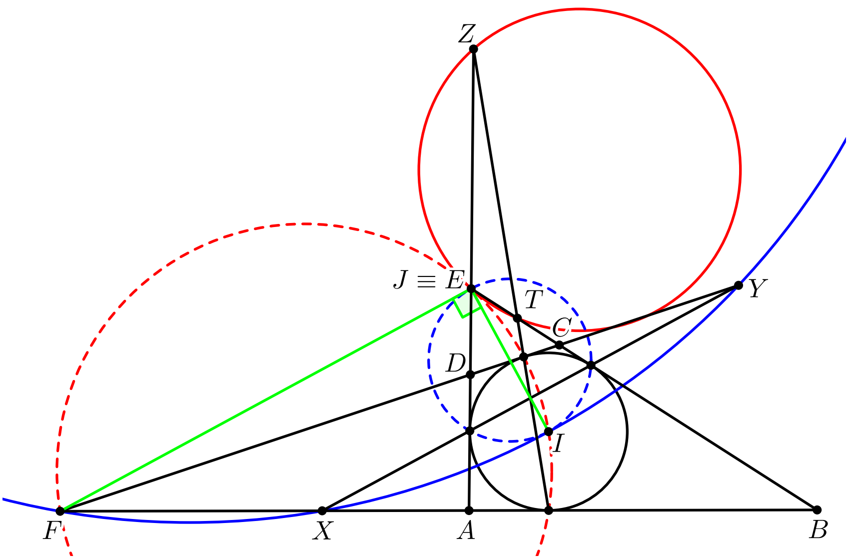
Now, by Lemma 5, $ABCD$ is a cyclic quadrilateral. On the other hand, $ABCD$ is circumscribed and every equation resulted from Casey's Theorem for the circumcircle of triangle $\triangle XFY$ and the circle with diameter EI, can be written for the circumcircle of triangle $\triangle TEZ$ and the circle with diameter FI as well. So by Casey's Theorem, these two circles are tangent to each other.

Case 2) $\angle EIF < 90^\circ$.

$$\sin^2(EIF) = \left(\frac{ES}{EI}\right) = \frac{FS}{FI}.$$

Now by Lemma 4, we get that EF is tangent to the circumcircle of triangle $\triangle ESI$ and

$$\angle FES = \angle FIF \Longrightarrow \angle IEF = 90^\circ.$$



Now since $\angle IEF = 90^\circ$, the foot of perpendicular line to EF passing through I, (Point J) coincides with point E. By Lemma 3, the circumcircle of triangle $\triangle TJZ$ (which is also the circumcircle of triangle

$\triangle TEZ$), will be tangent to the circle with diameter FI. In this case, tangency point of the circumcircle of triangle $\triangle TEZ$ and the circle with diameter EI, will be point I and tangency point of the circumcircle of triangle $\triangle TEZ$ and the circle with diameter FI, will be point E.

Case 3) $\angle EIF > 90°$. Since

$$\sin^2(EIF) = \frac{FS}{FI} > 1,$$

this case will never happen.

■

Problem 5. Consider an acute-angled triangle ABC ($AC > AB$) with its orthocenter H and circumcircle Γ. Points M and P are the midpoints of the segments BC and AH, respectively. The line AM meets Γ again at X and point N lies on the line BC so that NX is tangent to Γ. Points J and K lie on the circle with diameter MP such that $\angle AJP = \angle HNM$ (B and J lie on the same side of AH) and circle ω_1, passing through K, H, and J, and circle ω_2, passing through K, M, and N, are externally tangent to each other. Prove that the common external tangents of ω_1 and ω_2 meet on the line NH.

Proposed by Alireza Dadgarnia

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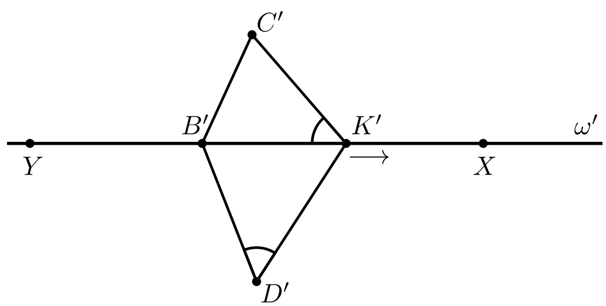
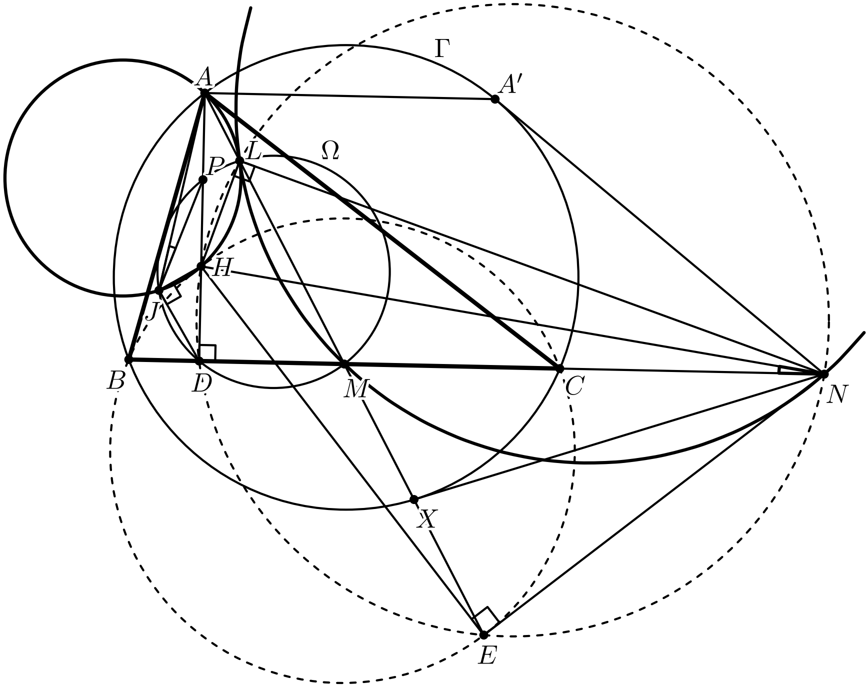
Solution 1. Let D be the intersection of AH and BC. Denote Ω by the circle with diameter PM. It's obvious that D lies on Ω. Also since ABC is acute, H lies on the segment PD and so inside of Ω. N lies on the extension of DM and so outside of Ω. We claim that there are at most two possible cases for K. The following lemma proves our claim.

Lemma. Given a circle ω and four points A, B, C, and D, such that A and B lie on the circle, C inside and D outside of the circle. There are exactly two points like K on ω so that the circumcircles of triangles ACK and BDK are tangent to each other.

Proof. Invert the whole diagram at center A with arbitrary radius, the images of points and circle are denoted by primes. Since A lies on ω, ω' is a line, passes through B' and K'. Notice that C' and D' lie on the different sides of ω'. Since the circumcircles of triangles ACK and BDK are tangent to each other, we have $C'K'$ is tangent to the circumcircle of triangle $B'D'K'$. It means $\angle C'K'B' = \angle B'D'K'$. Let X and Y be two arbitrary points, lie on ω' and the different sides of B'.



First assume that $K' \equiv B'$ so $\angle C'B'Y = \angle C'K'B' > 0 = \angle K'D'B'$ and when K' moves along the ray $\overrightarrow{B'X}$, $\angle C'K'B'$ decreases and $\angle K'D'B'$ increases. It yields there is exactly one point K' on the ray $\overrightarrow{B'X}$ so that $\angle C'K'B' = \angle B'D'K'$. In the same way we get there is only one possible case for K' on the ray $\overrightarrow{B'Y}$ and the result follows. $\square$



Denote ω_1 and ω_2 by the circumcircles of triangles AJP and HND. Let $\mathcal{H}$ be the indirect homothety that sends ω_1 to ω_2. Notice that J and N lie on the different sides of AH. Now since the arc AP of ω_1 is equal to the arc HD of ω_2 and $AP \parallel HD$, $\mathcal{H}$ sends A to D and P to H therefore (A, H) and (P, D) are anti-homologous pairs. Let L be the anti-homologous point of J under $\mathcal{H}$. It's well-known that the pairs of anti-homologous points lie on a circle so $ALHJ$ and $LPJD$ are cyclic quadrilaterals.

Let E be the reflection of A over the point M. We claim that $HDEN$ is cyclic. A' lies on Γ so that $AA' \parallel BC$. We know that $(A'X, BC) = -1$ hence NA' is tangent to Γ. Also by symmetry NE is tangent to the circumcircle of triangle CEB. Now since HE is the diameter of this circle, we have $\angle NEH = 90^\circ = \angle NDH$ and our claim is proved. The line AM meets the circumcircle of triangle PDM again at L'. We have

$$AL' \cdot AM = AP \cdot AD \Longrightarrow AL' \cdot AE = AH \cdot AD$$

it follows that $L'HDEN$ is cyclic so $L' \equiv L$. We have

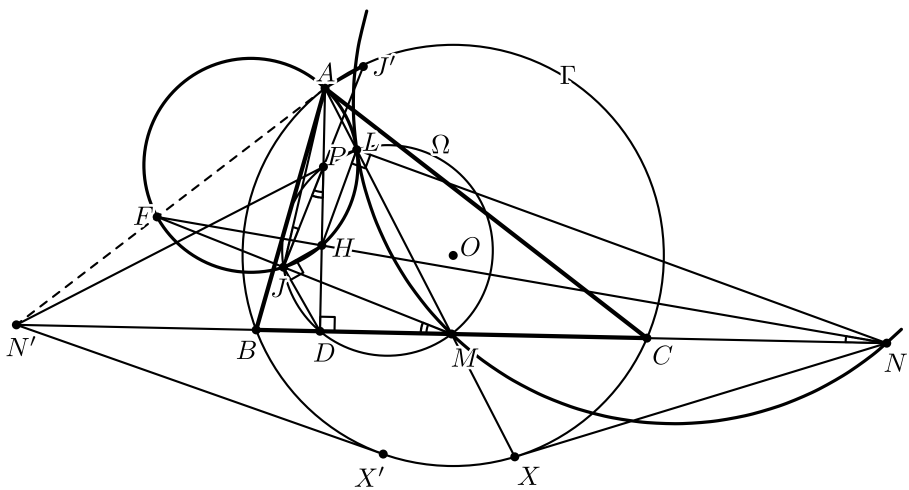
$$\begin{aligned}\angle PJH = \angle AJH - \angle AJP &= \angle HLM - \angle HND \\ &= \angle HLM - \angle HLD = \angle DLM = \angle DJM\end{aligned}$$

therefore $\angle HJD = 90^\circ$. From this we can conclude that the cirucmcircles of triangles DHJ and DMN are tangent to each other and the common external tangents of them are concurrent at H since the tangent line to the circumcircle of triangle DHJ through H is parallel to DMN. So the problem is proved for $K \equiv D$, now suppose that $K \neq D$. Since $\angle AHL = \angle LNM$ the circumcircles of triangles LHJ and LMN are tangent to each other. So $L \equiv K$. Denote O_1 and O_2 by the circumcenters of triangles LHJ and LMN. It's obvious that O_1, L, and O_2 are collinear so $\angle O_1LH + \angle O_2LN = 90^\circ$. It yields

$$\angle HO_1L = 180^\circ - 2\angle O_1LH = 2\angle O_2LN = 180^\circ - \angle LO_2N \Longrightarrow O_1H \parallel O_2N$$

therefore the direct homothety that sends (O_1) to (O_2), sends H to N and the conclusion follows. ■

Solution 2. Let D be the intersection of AH and BC. Denote Ω by the circle with diameter PM. It's obvious that D lies on Ω.



Let F be the intersection of NH and MJ. Since J and B lie on the same side of PD, J lies on the arc PD (the one that does not contain M) so J and H lie on the same side of BC. Also

$$\angle HNM = \angle AJP < \angle JPD = \angle JMD$$

therefore F and J lie on the same side of NM and we have $\triangle FMN \sim \triangle APJ$ since $\angle JPD = \angle JMD$. It follows that A, F, H, and J are concylic. Let J' and N' be the reflections of J and N over the points P and M, respectively. Since P is the midpoint of AH, $AJ'HJ$ is a parallelogram. The $A-$symmedian meets Γ again at X'. Since $XX' \parallel BC$, by symmetry $N'X'$ is tangent to Γ, too. Also we know that $(AX', BC) = -1$ so $N'A$ is tangent to Γ. Now $\triangle FMN \sim \triangle APJ$ yields $\triangle FMN' \sim \triangle APJ'$. It follows that

$$\angle N'FM = \angle J'AP = \angle AHJ = 180^\circ - \angle AFJ$$

hence A, F, and N' are collinear. Again from $\triangle FMN' \sim \triangle APJ'$ we get

$$\angle PJH = \angle AJ'P = \angle FN'M = 90^\circ - \angle PMN' = \angle DPM = \angle DJM$$

In the third equality we used that $MP \perp AN'$ (It's a well-known property, If we let O be the center of Γ then $APMO$ is a parallelogram). It yields $\angle HJD = \angle PJM = 90^\circ$. Like the first solution we know that there are at most two possible cases for K and we can conclude that D is one of them. Now we suppose that $K \neq D$. Let AM meets Ω again at L. We have

$$\angle LAH = 90^\circ - \angle LMD = \angle LJD - 90^\circ = \angle LJH$$

therefore $ALHJ$ is cyclic. Since $MP \perp AN'$ and $AP \perp MN'$, P is the orthocenter of triangle $AN'M$ and $N'P \perp AM$. It follows that N', P and L lie on a same line. Now since $\angle ALP = \angle N'LM = 90^\circ$ and $\angle APL = \angle N'ML$, we have $\triangle APL \sim \triangle N'ML$. It yields $\triangle LMN \sim \triangle LPH$. Hence

$$\angle MLN = \angle PLH \Longrightarrow \angle HLN = \angle PLM = 90^\circ$$

so $LNDH$ is cyclic and $\angle AHL = \angle LNM$. It follows that the circumcircles of triangles LHJ and LMN are tangent to each other. So $L \equiv K$. Denote O_1 and O_2 by the circumcenters of triangles LHJ and LMN. It's obvious that O_1, L, and O_2 are collinear so $\angle O_1LH + \angle O_2LN = 90^\circ$. It yields

$$\angle HO_1L = 180^\circ - 2\angle O_1LH = 2\angle O_2LN = 180^\circ - \angle LO_2N \Longrightarrow O_1H \parallel O_2N$$

therefore the direct homothety that sends (O_1) to (O_2), sends H to N and the conclusion follows. ■

Comment. We can also prove $LHDN$ is cyclic by angle-chasing. We have

$$\angle DLM = \angle DPM = 90^\circ - \angle PMD = \angle PJD - 90^\circ = \angle PJH$$

also $\angle HLM = \angle AJH$ so $\angle HLD = \angle AJP = \angle HND$ and it follows that $LHDN$ is cyclic.