XXIII MMC 2020

Problem 1

Let a, b, c be positive real numbers such that $a+b+c=4$. Prove that

$$\frac{ab}{\sqrt[4]{3c^2+16}}+\frac{bc}{\sqrt[4]{3a^2+16}}+\frac{ca}{\sqrt[4]{3b^2+16}} \le \frac{4}{3}\sqrt[4]{12}.$$

Solution. Squaring both terms, we get

$$\left(\frac{ab}{\sqrt[4]{3c^2+1}}+\frac{bc}{\sqrt[4]{3a^2+1}}+\frac{ca}{\sqrt[4]{3b^2+1}}\right)^2 \le \frac{32}{9}\sqrt{3}.$$

Applying CBS to vectors $\vec{u}=\left(\sqrt{ab},\sqrt{bc},\sqrt{ca}\right)$ and

$$\vec{v}=\left(\sqrt{\frac{ab}{\sqrt{3c^2+16}}},\sqrt{\frac{bc}{\sqrt{3a^2+16}}},\sqrt{\frac{ca}{\sqrt{3b^2+16}}}\right)$$

we obtain

$$\left(\frac{ab}{\sqrt[4]{3c^2+16}}+\frac{bc}{\sqrt[4]{3a^2+16}}+\frac{ca}{\sqrt[4]{3b^2+16}}\right)^2$$
$$\le (ab+bc+ca)\left(\frac{ab}{\sqrt{3c^2+16}}+\frac{bc}{\sqrt{3a^2+16}}+\frac{ca}{\sqrt{3b^2+16}}\right)$$

From $a+b+c=4$ and the well-known inequality $(a+b+c)^2 \ge 3(ab+bc+ca)$ immediately follows that $ab+bc+ca \le 16/3$. Equality holds when $a=b=c=4/3$.

On the other hand,

$$\begin{aligned}\sqrt{3}\left(\frac{bc}{\sqrt{3a^2+16}}\right) &= \frac{bc}{\sqrt{a^2+16/3}} \le \frac{bc}{\sqrt{a^2+ab+bc+ca}}\\ &= \frac{bc}{\sqrt{(a+b)(a+c)}} \le \frac{1}{2}\left(\frac{bc}{a+b}+\frac{bc}{a+c}\right)\end{aligned}$$

The last inequality holds on account of AM-GM inequality. Likewise,

$$\sqrt{3}\left(\frac{ca}{\sqrt{3b^2+16}}\right) \le \frac{1}{2}\left(\frac{ca}{b+c}+\frac{ca}{b+a}\right)$$

and

$$\sqrt{3}\left(\frac{ab}{\sqrt{3c^2+16}}\right) \le \frac{1}{2}\left(\frac{ab}{c+a}+\frac{ab}{c+b}\right)$$

Adding the preceding inequalities, we obtain

$$\sqrt{3}\left(\frac{ab}{\sqrt{3c^2+16}}+\frac{bc}{\sqrt{3a^2+16}}+\frac{ca}{\sqrt{3b^2+16}}\right)$$

$$\leq \frac{1}{2}\left(\frac{ca}{b+c}+\frac{ca}{b+a}+\frac{bc}{a+b}+\frac{bc}{a+c}+\frac{ab}{c+a}+\frac{ab}{c+b}\right)=\frac{1}{2}(a+b+c)=2.$$

Therefore,

$$\frac{ab}{\sqrt{3c^2+16}}+\frac{bc}{\sqrt{3a^2+16}}+\frac{ca}{\sqrt{3b^2+16}}\leq \frac{2\sqrt{3}}{3}.$$

Equality holds when $a=b=c=4/3$. From the preceding, we have

$$\left(\frac{ab}{\sqrt[4]{3c^2+16}}+\frac{bc}{\sqrt[4]{3a^2+16}}+\frac{ca}{\sqrt[4]{3b^2+16}}\right)^2\leq \frac{16}{3}\cdot\frac{2\sqrt{3}}{3}=\frac{32}{9}\sqrt{3}.$$

from which

$$\frac{bc}{\sqrt[4]{3a^2+16}}+\frac{ca}{\sqrt[4]{3b^2+16}}+\frac{ab}{\sqrt[4]{3c^2+16}}\leq \frac{4}{3}\sqrt[4]{12}$$

follows. Equality holds when $a=b=c=4/3$, and we are done.

Problem 2

Let S be a set of $n \geq 2$ positive integers. Prove that there exist at least n^2 integers that can be written in the form $x + yz$ with $x, y, z \in S$.

Solution

Let m denote the largest number in S. We claim that for all pairs $(x_1, y_1), (x_2, y_2) \in S \times S$ with $(x_1, y_1) \neq (x_2, y_2)$ we have $x_1 + y_1 m \neq x_2 + y_2 m$. Indeed, suppose otherwise and rewrite the equation $x_1 + y_1 m = x_2 + y_2 m$ into

$$(y_1 - y_2)m \;=\; x_2 - x_1.$$

If $y_1 = y_2$, we get $x_2 - x_1 = 0$ and the contradiction $(x_1, y_1) = (x_2, y_2)$. If $y_1 \neq y_2$, we get $m = (x_2 - x_1)/(y_1 - y_2)$. Since $|y_1 - y_2| \geq 1$ and $|x_2 - x_1| \leq |m - 1|$, this inequality implies $m \leq m - 1$ which is ridiculous.

Consequently, the n^2 integers of the form $x + ym$ with $(x, y) \in S \times S$ are pairwise distinct.

Problem 3

Determine all integers $m \geq 2$ for which there exists an integer $n \geq 1$ with

$$\gcd(m, n) = 1 \quad \text{and} \quad \gcd(m, 4n+1) = 1.$$

Solution

The answer is all integers $m \geq 2$. We first discuss the cases where m is a prime number.

- If m is a prime with $m \neq 3$, then any number n with $n \equiv -1 \bmod m$ works: Indeed, let $n = km - 1$. Then $\gcd(m, n) = \gcd(m, km-1) = 1$ holds and furthermore

$$\gcd(m, 4n+1) \;=\; \gcd(m, 4km-3) \;=\; \gcd(m, 3) \;=\; 1.$$

- If $m = 3$, then any number n with $n \equiv 1 \bmod m$ works: Let $n = km + 1$. Then $\gcd(m, n) = \gcd(m, km+1) = 1$ and $\gcd(m, 4n+1) = \gcd(m, 4km+5) = \gcd(m, 5) = 1$.

Finally, let $m \geq 2$ be an arbitrary integer, and let $p_1 < p_2 < \cdots < p_s$ be an enumeration of the prime factors of m. We construct a system of congruences for n with $1 \leq i \leq s$: If $p_i = 3$, then we impose $n \equiv 1 \bmod p_i$. If $p_i \neq 3$, then we impose $n \equiv -1 \bmod p_i$. By the Chinese Remainder Theorem, there exists an integer n that simultaneously satisfies all these congruences and that by the above discussion satisfies $\gcd(m, n) = 1$ and $\gcd(m, 4n+1) = 1$ as desired.

Problem 4

Let P, Q and R three points on a circle k_1, such that PQ = PR and PQ>QR. Let k_2 *be the circle with centre P that passes through Q and R. Suppose that the circle with centre Q and passing through R intersect* k_1 again at X and k_2 *again at Y.*

The points X and R lie on different sides of the line PQ.

Prove that P, X and Y lie on a line.

<u>Solution 1</u>

All the angles to be directed. The result follows immediately from

$$\measuredangle YPQ = \measuredangle QPR = \measuredangle XPQ \ .$$

The first equality is due to QR = QY, and so these equal chords subtend equal angles in k_1. The second is due to QR = QX, and so these equal chords subtend equal angles at the centre of k_2 .■

<u>Solution 2</u>

We will need to prove that $\angle QPX = \angle QPY$ **.**

a) **Since Q, Y, P, R are concyclic,** $\angle QPY = \angle QRY$ and $\angle QPR = \angle QYR$**.**
b) **Since the triangle QRY is isosceles,** $\angle QRY = \angle QYR$ **.**
c) **Since the triangles QPR and QPX are congruent (3 equal sides),** $\angle QPX = \angle QPR$**.**

From the 3 previous items, we obtain

$$\angle QPY = \angle QRY = QYR = \angle QPR = \angle QPX \ ,$$

and we are done.■