PROBLEMS

Problem 1. Prove that for every $x \in (0,1)$ the following inequality holds:

$$\int_0^1 \sqrt{1+(\cos y)^2}\,dy > \sqrt{x^2+(\sin x)^2} \; .$$

Solution 1. Clearly

$$\int_0^1 \sqrt{1+(\cos y)^2}\,dy \geq \int_0^x \sqrt{1+(\cos y)^2}\,dy \, .$$

Define a function $F:[0,1] \to \mathbb{R}$ by setting:

$$F(x) = \int_0^x \sqrt{1+(\cos y)^2}\,dy - \sqrt{x^2+(\sin x)^2} \, .$$

Since $F(0)=0$, it suffices to prove $F'(x) \geq 0$. By the fundamental theorem of Calculus, we have

$$F'(x) = \sqrt{1+(\cos x)^2} - \frac{x+\sin x \cos x}{\sqrt{x^2+(\sin x)^2}} \, .$$

Thus, it is enough to prove that

$$(1+(\cos x)^2)(x^2+(\sin x)^2) \geq (x+\sin x \cos x)^2 \, .$$

But this is a straightforward application of the Cauchy-Schwarz inequality.

Solution 2. Clearly $\int_0^1 \sqrt{1+(\cos y)^2}\,dy \geq \int_0^x \sqrt{1+(\cos y)^2}\,dy$ for each fixed $x \in (0,1)$. Observe that $\int_0^x \sqrt{1+(\cos y)^2}\,dy$ is the arc length of the function $f(y) = \sin y$ on the interval $[0,x]$ which is clearly strictly greater than the length of the straight line between the points $(0,0)$ and $(x, \sin x)$ which in turn is equal to $\sqrt{x^2+(\sin x)^2}$.

Problem 2. For any positive integer n, let the functions $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_{n+1}(x) = f_1(f_n(x))$, where $f_1(x) = 3x - 4x^3$. Solve the equation $f_n(x) = 0$.

Solution. First, we prove that $|x| > 1 \Rightarrow |f_n(x)| > 1$ holds for every positive integer n. It suffices to demonstrate the validity of this implication for $n = 1$. But, by assuming $|x| > 1$, it readily follows that $|f_1(x)| = |x||3-4x^2| \geq |3-4x^2| > 1$, which completes the demonstration. We conclude that every solution of the equation $f_n(x) = 0$ lies in the closed interval $[-1,1]$. For an arbitrary such x, set $x = \sin t$ where $t = \arcsin x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. We clearly have $f_1(\sin t) = \sin 3t$, which gives

$$f_n(x) = \sin 3^n t = \sin(3^n \arcsin x).$$

Thus, $f_n(x) = 0$ if and only if $\sin(3^n \arcsin x) = 0$, i.e. only when $3^n \arcsin x = k\pi$ for some $k \in \mathbb{Z}$. Therefore, the solutions of the equation $f_n(x) = 0$ are given by

$$x = \sin \frac{k\pi}{3^n},$$

where k acquires every integer value from $\frac{1-3^n}{2}$ up to $\frac{3^n-1}{2}$.

Problem 3. For an integer $n > 2$, let $A, B, C, D \in M_n(\mathbb{R})$ be matrices satisfying:

$$AC - BD = I_n,$$
$$AD + BC = O_n,$$

where I_n is the identity matrix and O_n is the zero matrix in $M_n(\mathbb{R})$.

Prove that:

a) $CA - DB = I_n$ and $DA + CB = O_n$,

b) $\det(AC) \geq 0$ and $(-1)^n \det(BD) \geq 0$.

Solution. a) We have

$$AC - BD + i(AD + BC) = I_n \Leftrightarrow (A + iB)(C + iD) = I_n,$$

which implies that the matrices $A + iB$ and $C + iD$ are inverses to one another. Thus,

$$(C + iD)(A + iB) = I_n \Leftrightarrow CA - DB + i(DA + CB) = I_n$$
$$\Leftrightarrow CA - DB = I_n, DA + CB = O_n.$$

b) We have

$$\det((A + iB)C) = \det(AC + iBC)$$
$$\overset{AD+BC=O_n}{=} \det(AC - iAD)$$
$$= \det(A(C - iD)).$$

On the other hand,

$$\det C \overset{(C+iD)(A+iB)=I_n}{=} \det((C + iD)(A + iB)C) = \det((C + iD)A(C - iD))$$
$$= \det(A) \,|\det(C + iD)|^2.$$

Thus,

$$\det(AC) = (\det A)^2 |\det(C + iD)|^2 \geq 0.$$

Similarly

$$\det((A + iB)D) = \det(AD + iBD)$$
$$\overset{AD+BC=O_n}{=} \det(-BC + iBD)$$
$$= (-1)^n \det(B(C - iD)).$$

This implies that

$$\det D \overset{(C+iD)(A+iB)=I_n}{=} \det((C + iD)(A + iB)D) = (-1)^n \det((C + iD)B(C - iD))$$
$$= (-1)^n \det(B) \,|\det(C + iD)|^2.$$

Thus, $(-1)^n \det(BD) = (\det B)^2 \,|\det(C + iD)|^2 \geq 0$.

Problem 4. Let $I \subset \mathbb{R}$ be an open interval which contains 0, and $f : I \to \mathbb{R}$ be a function of class $C^{2016}(I)$ such that $f(0)=0, f'(0)=1, f''(0)=f'''(0)=...=f^{(2015)}(0)=0, f^{(2016)}(0)<0.$

i) Prove that there is $\delta > 0$ such that

$$0 < f(x) < x, \quad \forall x \in (0,\delta). \tag{1.1}$$

ii) With δ determined as in i), define the sequence (a_n) by

$$a_1 = \frac{\delta}{2},\ a_{n+1} = f(a_n),\ \forall n \geq 1.$$

(1.2)

Study the convergence of the series $\sum_{n=1}^{\infty} a_n^r,$ for $r \in \mathbb{R}$.

Solution. ***i)*** We claim that there exists $\alpha > 0$ such that $f(x) > 0$ for any $x \in (0,\alpha)$. For this, observe that, since f is of class C^1 and $f'(0)=1>0,$ there exists $\alpha > 0$ such that $f'(x) > 0$ on $(0,\alpha)$. Since $f(0)=0$ and f is strictly increasing on $(0,\alpha)$, the claim follows.

Next, we prove that there exists $\beta > 0$ such that $f(x) < x$ for any $x \in (0,\beta)$. Since $f^{(2016)}(0) < 0$ and f is of class C^{2016}, there is $\beta > 0$ such that $f^{(2016)}(t) < 0$, for any $t \in (0,\beta)$. By the Taylor's formula, for any $x \in (0,\beta)$, there is $\theta \in [0,1]$ such that

$$f(x) = f(0) + \frac{f'(0)}{1!}x + ... + \frac{f^{(2015)}(0)}{2015!}x^{2015} + \frac{f^{(2016)}(\theta x)}{2016!}x^{2016}, \tag{1.3}$$

hence

$$g(x) = \frac{f^{(2016)}(\theta x)}{2016!}x^{2016} < 0, \quad \forall x \in (0,\beta).$$

Taking $\delta = \min\{\alpha,\beta\} > 0$, the item *i)* is completely proven.

ii) We will prove first that the sequence (a_n) given by (1.2) converges to 0. Indeed, by relation (1.1) it follows that

$$0 < a_{n+1} < a_n, \ \forall n \geq 1,$$

hence the sequence (a_n) is strictly decreasing and lower bounded by 0. It follows that (a_n) converges to some $\ell \in [0,\frac{\delta}{2})$. Passing to the limit in (1.2) , one gets $\ell = f(\ell)$. Taking into account (1.1), we deduce that $\ell = 0$.

In what follows, we calculate

$$\lim_{n\to\infty} n a_n^{2015}.$$

From $a_n \downarrow 0$, using the Stolz-Cesàro Theorem, we conclude that

$$\lim_{n\to\infty} n a_n^{2015} = \lim_{n\to\infty} \frac{n}{\frac{1}{a_n^{2015}}} = \lim_{n\to\infty} \frac{(n+1)-n}{\frac{1}{a_{n+1}^{2015}} - \frac{1}{a_n^{2015}}} = \lim_{n\to\infty} \frac{1}{\frac{1}{f(a_n)^{2015}} - \frac{1}{a_n^{2015}}}$$

$$= \lim_{x\to 0} \frac{1}{\frac{1}{f(x)^{2015}} - \frac{1}{x^{2015}}} = \lim_{x\to 0} \frac{(xf(x))^{2015}}{x^{2015} - f(x)^{2015}}.$$

Observe that, by (1.3) $\dfrac{\left(xf\left(x\right)\right)^{2015}}{x^{2015}-f(x)^{2015}}=\dfrac{\left(x^2+\dfrac{f^{(2016)}(\theta x)}{2016!}x^{2017}\right)^{2015}}{-\dfrac{f^{(2016)}(\theta x)}{2016!}x^{2016}(x^{2014}+x^{2013}f(x)+\ldots+f(x)^{2014})}$.

Since f is of class C^{2016}, $\lim\limits_{x\to 0} f^{(2016)}(\theta x)=f^{(2016)}(0)$ and

$$\lim_{x\to 0}\frac{(xf(x))^{2015}}{x^{2015}-f(x)^{2015}}=-\frac{2016!}{2015f^{(2016)}(0)}>0.$$

It means, by the comparison criterion, that the series $\sum\limits_{n=1}^{\infty}a_n^r$ and $\sum\limits_{n=1}^{\infty}\frac{1}{n^{\frac{r}{2015}}}$ converge and/or diverge simultaneously, hence the series $\sum\limits_{n=1}^{\infty}a_n^r$ converges for $r>2015$, and diverges for $r\le 2015$.