Solutions of APMO 2016

Problem 1. We say that a triangle ABC is *great* if the following holds: for any point D on the side BC, if P and Q are the feet of the perpendiculars from D to the lines AB and AC, respectively, then the reflection of D in the line PQ lies on the circumcircle of the triangle ABC.

Prove that triangle ABC is great if and only if $\angle A = 90^\circ$ and $AB = AC$.

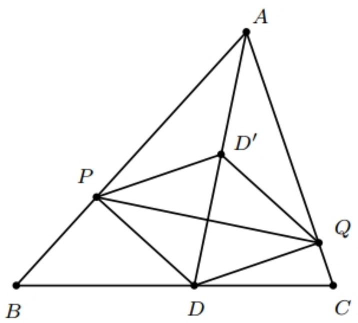
Solution. For every point D on the side BC, let D' be the reflection of D in the line PQ. We will first prove that if the triangle satisfies the condition then it is isosceles and right-angled at A.

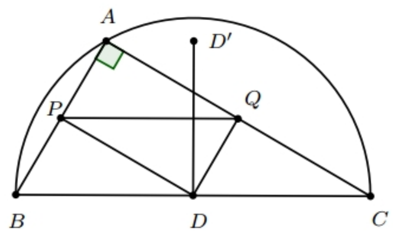
Choose D to be the point where the angle bisector from A meets BC. Note that P and Q lie on the rays AB and AC respectively. Furthermore, P and Q are reflections of each other in the line AD, from which it follows that $PQ \perp AD$. Therefore, D' lies on the line AD and we may deduce that either $D' = A$ or D' is the second point of the angle bisector at A and the circumcircle of ABC. However, since $APDQ$ is a cyclic quadrilateral, the segment PQ intersects the segment AD. Therefore, D' lies on the ray DA and therefore $D' = A$. By angle chasing we obtain

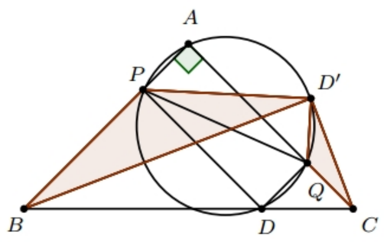
$$\angle PD'Q = \angle PDQ = 180^\circ - \angle BAC,$$

and since $D' = A$ we also know $\angle PD'Q = \angle BAC$. This implies that $\angle BAC = 90^\circ$.

Now we choose D to be the midpoint of BC. Since $\angle BAC = 90^\circ$, we can deduce that DQP is the medial triangle of triangle ABC. Therefore, $PQ||BC$ from which it follows that $DD' \perp BC$. But the distance from D' to BC is equal to both the circumradius of triangle ABC and to the distance from A to BC. This can only happen if $A = D'$. This implies that ABC is isosceles and right-angled at A.







We will now prove that if ABC is isosceles and right-angled at A then the required property in the problem holds. Let D be any point on side BC. Then $D'P = DP$ and we also have $DP = BP$. Hence, $D'P = BP$ and similarly $D'Q = CQ$. Note that $APDQD'$ is cyclic with diameter PQ. Therefore, $\angle APD' = \angle AQD'$, from which we obtain $\angle BPD' = \angle CQD'$. So triangles $D'PB$ and $D'QC$ are similar. It follows that $\angle PD'Q = \angle PD'C + \angle CD'Q = \angle PD'C + \angle BD'P = \angle BD'C$ and $\frac{D'P}{D'Q} = \frac{D'B}{D'C}$. So we also obtain that triangles $D'PQ$ and $D'BC$ are similar. But since DPQ and $D'PQ$ are congruent, we may deduce that $\angle BD'C = \angle PD'Q = \angle PDQ = 90^\circ$. Therefore, D' lies on the circle with diameter BC, which is the circumcircle of triangle ABC.

$\square$

Problem 2. A positive integer is called *fancy* if it can be expressed in the form

$$2^{a_1} + 2^{a_2} + \cdots + 2^{a_{100}},$$

where a_1, a_2, ..., a_{100} are non-negative integers that are not necessarily distinct.

Find the smallest positive integer n such that no multiple of n is a fancy number.

Answer: The answer is $n = 2^{101} - 1$.

Solution. Let k be any positive integer less than $2^{101} - 1$. Then k can be expressed in binary notation using at most 100 ones, and therefore there exists a positive integer r and non-negative integers a_1, a_2, ..., a_r such that $r \leq 100$ and $k = 2^{a_1} + \cdots + 2^{a_r}$. Notice that for a positive integer s we have:

$$\begin{aligned} 2^s k &= 2^{a_1+s} + 2^{a_2+s} + \cdots + 2^{a_{r-1}+s} + (1 + 1 + 2 + \cdots + 2^{s-1})2^{a_r} \\ &= 2^{a_1+s} + 2^{a_2+s} + \cdots + 2^{a_{r-1}+s} + 2^{a_r} + 2^{a_r} + \cdots + 2^{a_r+s-1}. \end{aligned}$$

This shows that k has a multiple that is a sum of $r + s$ powers of two. In particular, we may take $s = 100 - r \geq 0$, which shows that k has a multiple that is a fancy number.

We will now prove that no multiple of $n = 2^{101} - 1$ is a fancy number. In fact we will prove a stronger statement, namely, that no multiple of n can be expressed as the sum of at most 100 powers of 2.

For the sake of contradiction, suppose that there exists a positive integer c such that cn is the sum of at most 100 powers of 2. We may assume that c is the smallest such integer. By repeatedly merging equal powers of two in the representation of cn we may assume that

$$cn = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_r}$$

where $r \leq 100$ and $a_1 < a_2 < \ldots < a_r$ are distinct non-negative integers. Consider the following two cases:

- If $a_r \geq 101$, then $2^{a_r} - 2^{a_r-101} = 2^{a_r-101}n$. It follows that $2^{a_1} + 2^{a_2} + \cdots + 2^{a_{r-1}} + 2^{a_r-101}$ would be a multiple of n that is smaller than cn. This contradicts the minimality of c.
- If $a_r \leq 100$, then $\{a_1, \ldots, a_r\}$ is a proper subset of $\{0, 1, \ldots, 100\}$. Then

  $$n \leq cn < 2^0 + 2^1 + \cdots + 2^{100} = n.$$

  This is also a contradiction.

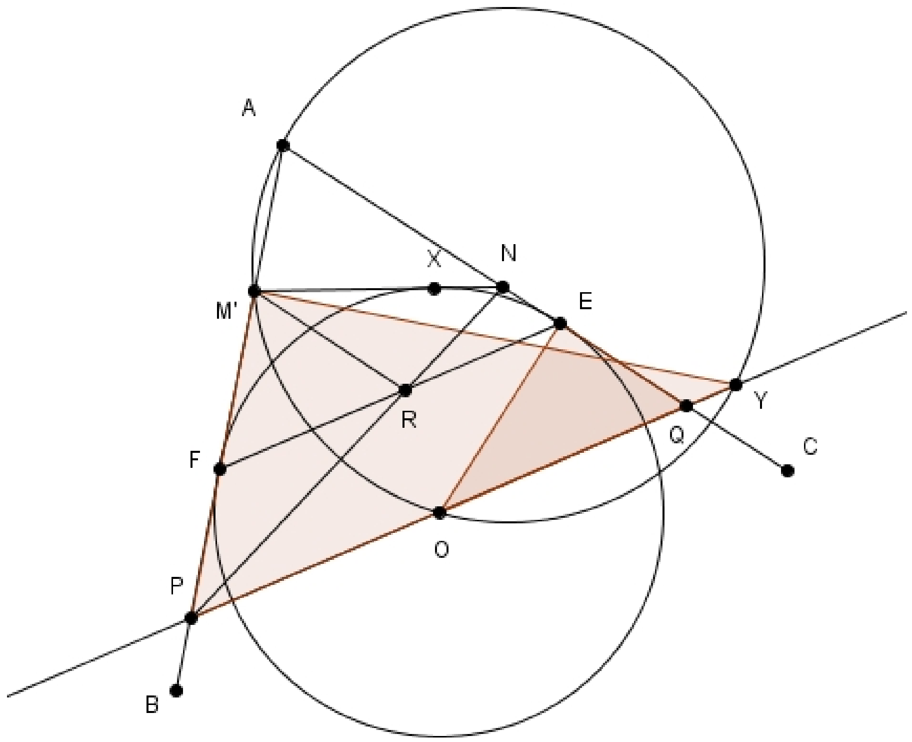
From these contradictions we conclude that it is impossible for cn to be the sum of at most 100 powers of 2. In particular, no multiple of n is a fancy number.

□

Problem 3. Let AB and AC be two distinct rays not lying on the same line, and let ω be a circle with center O that is tangent to ray AC at E and ray AB at F. Let R be a point on segment EF. The line through O parallel to EF intersects line AB at P. Let N be the intersection of lines PR and AC, and let M be the intersection of line AB and the line through R parallel to AC. Prove that line MN is tangent to ω.

Solution. We present two approaches. The first one introduces an auxiliary point and studies similarities in the figure. The second one reduces the problem to computations involving a particular exradius of a triangle. The second approach has two variants.

Solution 1.



Let the line through N tangent to ω at point $X \neq E$ intersect AB at point M'. It suffices to show that $M'R \parallel AC$, since this would yield $M' = M$.

Suppose that the line PO intersects AC at Q and the circumcircle of $AM'O$ at Y, respectively. Then

$$\angle AYM' = \angle AOM' = 90^\circ - \angle M'OP.$$

By angle chasing we have $\angle EOQ = \angle FOP = 90^\circ - \angle AOF = \angle M'AO = \angle M'YP$ and by symmetry $\angle EQO = \angle M'PY$. Therefore $\triangle M'YP \sim \triangle EOQ$.

On the other hand, we have

$$\angle M'OP = \angle M'OF + \angle FOP = \frac{1}{2}(\angle FOX + \angle FOP + \angle EOQ) =$$
$$= \frac{1}{2}\left(\frac{180^\circ - \angle XOE}{2}\right) = 90^\circ - \frac{\angle XOE}{2}.$$

Since we know that $\angle AYM'$ and $\angle M'OP$ are complementary this implies

$$\angle AYM' = \frac{\angle XOE}{2} = \angle NOE.$$

Therefore, $\angle AYM'$ and $\angle NOE$ are congruent angles, and this means that A and N are corresponding points in the similarity of triangles $\triangle M'YP$ and $\triangle EOQ$. It follows that

$$\frac{AM'}{M'P} = \frac{NE}{EQ} = \frac{NR}{RP}.$$

We conclude that $M'R \parallel AC$, as desired.

$\square$

Solution 2a.

As in Solution 1, we introduce point M' and reduce the problem to proving $\frac{PR}{RN} = \frac{PM'}{M'A}$.

Menelaus theorem in triangle ANP with transversal line FRE yields

$$\frac{PR}{RN} \cdot \frac{NE}{EA} \cdot \frac{AF}{FP} = 1.$$

Since $AF = EA$, we have $\frac{FP}{NE} = \frac{PR}{RN}$, so that it suffices to prove

$$\frac{FP}{NE} = \frac{PM'}{M'A}\,. \tag{1}$$

This is a computation regarding the triangle $AM'N$ and its excircle opposite A. Indeed, setting $a = M'N$, $b = NA$, $c = M'A$, $s = \dfrac{a+b+c}{2}$, $x = s - a$, $y = s - b$ and $z = s - c$, then $AE = AF = s$, $M'F = z$ and $NE = y$. From $\triangle OFP \sim \triangle AFO$ we have $FP = \frac{r_a^2}{s}$, where $r_a = OF$ is the exradius opposite A. Combining the following two standard formulas for the area of a triangle

$$|AM'N|^2 = xyzs \quad \text{(Heron's formula)} \quad \text{and} \quad |AM'N| = r_a(s-a),$$

we have $r_a^2 = \frac{yzs}{x}$. Therefore, $FP = \frac{yz}{x}$. We can now write everything in (1) in terms of x, y, z. We conclude that we have to verify

$$\frac{\frac{yz}{x}}{y} = \frac{z + \frac{yz}{x}}{x + y}\,,$$

which is easily seen to be true.

Note: Antoher approach using Menalaus theorem is to construct the tangent from M to create a point N' in AC and then prove, using the theorem, that P, R and N' are collinear. This also reduces to an algebraic identity.

Solution 2b.

As in Solution 1, we introduce point M'. Let the line through M' and parallel to AN intersect EF at R'. Let P' be the intersection of lines NR' and AM. It suffices to show that $P'O \parallel FE$, since this would yield $P = P'$, and then $R = R'$ and $M = M'$. Hence it is enough to prove that

$$\frac{AF}{FP'} = \frac{AD}{DO}\,, \tag{2}$$

where D is the intersection of AO and EF. Once again, this reduces to a computation regarding the triangle $AM'N$ and its excircle opposite A.

Let $u = P'F$ and x, y, z, s as in Solution 2a. Note that since $AE = AF$ and $M'R' \parallel AE$, we have $M'R' = M'F = z$. Since $M'R' \parallel AN$, we have $\frac{P'M'}{P'A} = \frac{M'R'}{NA}$, that is,

$$\frac{u+z}{u+x+y+z} = \frac{z}{x+z}\,.$$

From this last equation we obtain $u = \frac{yz}{x}$. Hence $\frac{AF}{FP'} = \frac{xs}{yz}$. Also, as in Solution 2a, we have $r_a^2 = \frac{yzs}{x}$.

Finally, using similar triangles ODF, FDA and OFA, and the above equalities, we have

$$\frac{AD}{DO} = \frac{AD}{DF} \cdot \frac{DF}{DO} = \frac{AF}{OF} \cdot \frac{AF}{OF} = \frac{s^2}{r_a^2} = \frac{s^2}{\frac{yzs}{x}} = \frac{xs}{yz} = \frac{AF}{FP'}\,,$$

as required.

Problem 4. The country Dreamland consists of 2016 cities. The airline Starways wants to establish some one-way flights between pairs of cities in such a way that each city has exactly one flight out of it. Find the smallest positive integer k such that no matter how Starways establishes its flights, the cities can always be partitioned into k groups so that from any city it is not possible to reach another city in the same group by using at most 28 flights.

Answer: 57

Solution. The flights established by Starways yield a directed graph G on 2016 vertices in which each vertex has out-degree equal to 1.

We first show that we need at least 57 groups. For this, suppose that G has a directed cycle of length 57. Then, for any two cities in the cycle, one is reachable from the other using at most 28 flights. So no two cities in the cycle can belong to the same group. Hence, we need at least 57 groups.

We will now show that 57 groups are enough. Consider another auxiliary directed graph H in which the vertices are the cities of Dreamland and there is an arrow from city u to city v if u can be reached from v using at most 28 flights. Each city has out-degree at most 28. We will be done if we can split the cities of H in at most 57 groups such that there are no arrows between vertices of the same group. We prove the following stronger statement.

Lemma: Suppose we have a directed graph on $n \geq 1$ vertices such that each vertex has out-degree at most 28. Then the vertices can be partitioned into 57 groups in such a way that no vertices in the same group are connected by an arrow.

Proof: We apply induction. The result is clear for 1 vertex. Now suppose we have more than one vertex. Since the out-degree of each vertex is at most 28, there is a vertex, say v, with in-degree at most 28. If we remove the vertex v we obtain a graph with fewer vertices which still satifies the conditions, so by inductive hypothesis we may split it into at most 57 groups with no adjacent vertices in the same group. Since v has in-degree and out-degree at most 28, it has at most 56 neighboors in the original directed graph. Therefore, we may add v back and place it in a group in which it has no neighbors. This completes the inductive step.

□

Problem 5. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$(z+1)f(x+y) = f(xf(z)+y) + f(yf(z)+x), \tag{3}$$

for all positive real numbers x, y, z.

Answer: The only solution is $f(x) = x$ for all positive real numbers x.

Solution. The identity function $f(x) = x$ clearly satisfies the functional equation. Now, let f be a function satisfying the functional equation. Plugging $x = y = 1$ into (3) we get $2f(f(z)+1) = (z+1)(f(2))$ for all $z \in \mathbb{R}^+$. Hence, f is not bounded above.

Lemma. Let a, b, c be positive real numbers. If c is greater than 1, a/b and b/a, then the system of linear equations

$$cu + v = a \quad u + cv = b$$

has a positive real solution u, v.

Proof. The solution is

$$u = \frac{ca - b}{c^2 - 1} \quad v = \frac{cb - a}{c^2 - 1}\,.$$

The numbers u and v are positive if the conditions on c above are satisfied.

□

We will now prove that

$$f(a)+f(b)=f(c)+f(d) \quad \text{for all } a,b,c,d \in \mathbb{R}^+ \text{ with } a+b=c+d. \tag{4}$$

Consider $a,b,c,d \in \mathbb{R}^+$ such that $a+b=c+d$. Since f is not bounded above, we can choose a positive number e such that $f(e)$ is greater than 1, a/b, b/a, c/d and d/c. Using the above lemma, we can find $u,v,w,t \in \mathbb{R}^+$ satisfying

$$\begin{aligned} f(e)u+v=a, &\quad u+f(e)v=b \\ f(e)w+t=c, &\quad w+f(e)t=d. \end{aligned}$$

Note that $u+v=w+t$ since $(u+v)(f(e)+1)=a+b$ and $(w+t)(f(e)+1)=c+d$. Plugging $x=u$, $y=v$ and $z=e$ into (3) yields $f(a)+f(b)=(e+1)f(u+v)$. Similarly, we have $f(c)+f(d)=(e+1)f(w+t)$. The claim follows immediately.

We then have

$$yf(x)=f(xf(y)) \quad \text{for all } x,y \in \mathbb{R}^+ \tag{5}$$

since by (3) and (4),

$$(y+1)f(x)=f\left(\frac{x}{2}f(y)+\frac{x}{2}\right)+f\left(\frac{x}{2}f(y)+\frac{x}{2}\right)=f(xf(y))+f(x).$$

Now, let $a=f(1/f(1))$. Plugging $x=1$ and $y=1/f(1)$ into (5) yields $f(a)=1$. Hence $a=af(a)$ and $f(af(a))=f(a)=1$. Since $af(a)=f(af(a))$ by (5), we have $f(1)=a=1$. It follows from (5) that

$$f(f(y))=y \quad \text{for all } y \in \mathbb{R}^+. \tag{6}$$

Using (4) we have for all $x,y \in \mathbb{R}^+$ that

$$\begin{aligned} &f(x+y)+f(1)=f(x)+f(y+1), \quad \text{and} \\ &f(y+1)+f(1)=f(y)+f(2). \end{aligned}$$

Therefore

$$f(x+y)=f(x)+f(y)+b \quad \text{for all } x,y \in \mathbb{R}^+, \tag{7}$$

where $b=f(2)-2f(1)=f(2)-2$. Using (5), (7) and (6), we get

$$4+2b=2f(2)=f(2f(2))=f(f(2)+f(2))=f(f(2))+f(f(2))+b=4+b.$$

This shows that $b=0$ and thus

$$f(x+y)=f(x)+f(y) \quad \text{for all } x,y \in \mathbb{R}^+.$$

In particular, f is strictly increasing.

We conclude as follows. Take any positive real number x. If $f(x)>x$, then $f(f(x))>f(x)>x=f(f(x))$, a contradiction. Similarly, it is not possible that $f(x)<x$. This shows that $f(x)=x$ for all positive real numbers x.

$\square$