Union of Mathematicians of Macedonia - ARMAGANKA

IX SEMINAR OF DIFFERENTIAL EQUATIONS AND ANALYSIS

and

1st CONGRESS OF DIFFERENTIAL EQUATIONS, MATHEMATICAL ANALYSIS AND APPLICATIONS

CODEMA 2020

Proceedings of the CODEMA 2020
Зборник на трудови од CODEMA 2020

Skopje, 2021

Union od Mathematicians of Macedonia – ARMAGANKA

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CONTENTS

FOR A CORRELATION BETWEEN A CLASS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS AND A CLASS OF SYSTEMS OF FIRST ORDER DIFFERENTIAL EQUATIONS

UDC: 517.926

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Abstract. In this paper, a class of second order linear differential equations and a class of systems of first order differential equations are considered. By a method of transformation, the results for a correlation between them are obtained.

1. INTRODUCTION

In this paper, we consider the second order linear differential equations

$$(t-a)(t-b)x''+(\alpha_1 t+\alpha_2)x'+\beta_1 x=0, a\neq b \tag{A}$$

and the system of first order differential equations

$$\begin{aligned}(t-a)x_1'+Ax_1+Bx_2&=0\\(t-b)x_2'+Cx_1+Dx_2&=0\end{aligned} \tag{B}$$

By replacing $x_1=\frac{1}{C}x_3, x_1'=\frac{1}{C}x_3', C\neq 0$ in the system (2), the system

$$\begin{aligned}(t-a)x_3'+Ax_3+BCx_2&=0\\(t-b)x_2'+x_3+Dx_2&=0\end{aligned} \tag{C}$$

is obtained.

On the 7th Macedonian symposium of the differential equations, Professor Boro Piperevski gave the following dilemma "Can a second order linear differential equation (A) transform in one or in more systems of first order differential equations (B)?" Sure, the second order linear differential equation (A) can transform in systems of differential equations from the type which is different from the systems (B). In mathematical literature until this moment does not have some result which is an answer to this question. In this paper, this dilemma will be solved. The consequence of this result is a sequence of differential equations of type (A) whose integrality depends on only one of them. An interesting case is when the class systems of differential equations from first-order of type (B) are considered as a linear matrix differential equation from first-order. This case is presented in more papers as [6,7,8,9,10,11,12]. Finally, a new dilemma for interpretation of the term reductability is presented.

2010 *Mathematics Subject Classification.* 34A30.
Key words and phrases. Second order linear differential equation, Systems of first order differential equations.

Considering a second order homogeneous linear differential equation of type $P_2(x)y''+P_1(x)y'+\lambda_n P_0(x)y=0$ where $P_i(x),(i=0,1,2)$ are polynomials and λ_n is a parameter. According to Brenke [4], this equation will have polynomial solutions of degree n for each $n\in\mathbb{N}$ with an appropriate value of the parameter λ_n, if $P_2(x), P_1(x)$, and $P_0(x)$ are polynomials of the second, first and zero degrees respectively. Also, the general formula for the series of polynomial solutions for the equation as well as certain conditions for their orthogonality with appropriate weight is shown.

Note that in the case when all members of the sequence $(\lambda_n), n=0,1,2,...$, are different, then λ_n are called their own values, and the polynomials y_n own functions.

Special cases of such known orthogonal polynomials are the polynomials of Legendre, Jacobi, Tschebyscheff, Hermite, Laguerre and others which are used in numerical mathematics.

Let us mention the classic results regarding polynomial solutions of the very important hypergeometric differential equation, as an equation with polynomial coefficients. Its solutions are special functions, especially the Jacobi, Legendre, Tschebyscheff polynomials, which belong to the class of classical orthogonal polynomials for which there are corresponding formulas, based on Rodrigues' famous formula.

In fact, this formula was obtained by Rodrigues O. in 1814 for a polynomial solution of a special differential equation of Legendre, but there are the other classical polynomials that are expressed in a similar way.

This special class of differential equations (1) is obtained when Laplace's partial differential equation is transformed into spherical coordinates and afterward, it is required its solution to be a product of functions that depend on only one variable.

In the theory of partial equations, the classical results for the solution of the internal problem of Dirihle for the contour problem for the Laplace's partial differential equation in the sphere are known. By transforming in spheres' coordinates and by using the Fourier method to the separation of the variables, the differential equations are obtained. Their solution is the classes of orthogonal

Legendre polynomials which are the own functions for the appropriate Sturm-Lowville task. Therefore, the solutions of the Laplace's partial differential equations in a type of homogeneous polynomials from the same degree are obtained. Spherical harmonic functions are called.

Remark 1.1. The term reductability of linear homogeneous differential equations has two interpretations. Reduction in a wider interpretation is a reduction of an equation of a system of linear homogeneous differential equations of lower order and that reduction can be more significant, i.e. it can be reduced to multiple classes of linear homogeneous differential equations systems from a lower order.

Definition 1.1. (Frobenius): A linear homogeneous differential equation whose coefficients are unambiguous functions is called more predictable according to Frobenius if there is no common solution with a linear homogeneous differential equation with coefficients unambiguous lower order functions. Otherwise, it is called reductive according to Frobenius. for example [1,2,3,4,5].

2. Mains Results

Let the system is given

$$\begin{aligned}(t-a)x_1' + Ax_1 + Bx_2 = 0\\ (t-b)x_2' + x_1 + Dx_2 = 0\end{aligned} \tag{1}$$

The equations which correspond to this system are

$$(t-a)(t-b)x_1'' + [(A+D+1)t-(A+1)b-Da]x_1' + (AD-B)x_1 = 0 \tag{2}$$

$$(t-a)(t-b)x_2'' + [(A+D+1)t-(D+1)a-Ab]x_2' + (AD-B)x_2 = 0 \tag{3}$$

The equation (2) is equivalent to equation

$$(t-a)(t-b)x_1'' + (\alpha_1 t+\alpha_2)x_1' + \beta_1 x_1 = 0,\ a \neq b \tag{4}$$

if the relations

$$a = a, b = b, A+D+1 = \alpha_1, -(A+1)b - Da = \alpha_2, AD-B = \beta_1 \tag{5}$$

are satisfied. The equation (3) is equivalent to equation

$$(t-a)(t-b)x_2'' + (\alpha_1^* t+\alpha_2^*)x_2' + \beta_1^* x_2 = 0,\ a \neq b \tag{6}$$

if the relations

$$\begin{aligned}a = a, b = b,\ A+D+1 = \alpha_1^*,\\ -(D+1)a - Ab = \alpha_2^*,\ AD-B = \beta_1^*\end{aligned} \tag{7}$$

are satisfied. The equation (3) or the equation (6) can be written in the form

$$(t-a)(t-b)x_2'' + (\alpha_1 t + b - a + \alpha_2)x_2' + \beta_1 x_2 = 0,\ a \neq b \tag{6*}$$

if an equation (4) is given.

Therefore, a system of type (1) corresponds to two differential equations (2) and (3) of type (4) i.e. (6). Therefore, if the coefficients of the system (1) with the relations (5) and (7) are known, then the equations (4) and (6) can be found, with the help of the connections (5) and (7).

Let, us ask the opposite question: how many systems of type (1) (if exist) suit the equation (4)?

Let the coefficients of equation (4) are known. From the relations (5), the equations

$$a = a, b = b, A = \frac{b-a+a\alpha_1+\alpha_2}{a-b}, D = \frac{-\alpha_2 - b\alpha_1}{a-b}, B = AD - \beta_1 \tag{8}$$

and the system

$$\begin{aligned}(t-a)x_1' + Ax_1 + Bx_2 &= 0\\ (t-b)x_2' + x_1 + Dx_2 &= 0\end{aligned} \tag{9}$$

are obtained i.e. the system (1).

From the relations (7), the equations

$$a=a, b=b, D^* = \frac{b-a-b\alpha_1-\alpha_2}{a-b}, A^* = \frac{\alpha_2 + a\alpha_1}{a-b}, B^* = A^*D^* - \beta_1 \tag{10}$$

and the system

$$\begin{aligned}(t-a)y_1' + A^*y_1 + B^*y_2 &= 0\\ (t-b)y_2' + y_1 + D^*y_2 &= 0\end{aligned} \tag{11}$$

are obtained.

Let us see which two equations correspond to this new system (11). By using (1), (2) and (3), the equations

$$(t-a)(t-b)y_1'' + [(A^*+D^*+1)t-(A^*+1)b-D^*a]y_1' + (A^*D^*-B^*)y_1 = 0 \tag{12}$$

$$(t-a)(t-b)y_2'' + [(A^*+D^*+1)t-(D^*+1)a-A^*b]y_2' + (A^*D^*-B^*)y_2 = 0 \tag{13}$$

are obtained or according to (10), the equations

$$(t-a)(t-b)y_1'' + (\alpha_1 t + a - b + \alpha_2)y_1' + \beta_1 y_1 = 0, a \neq b \tag{14}$$

$$(t-a)(t-b)y_2'' + (\alpha_1 t + \alpha_2)y_2' + \beta_1 y_2 = 0, a \neq b \tag{15}$$

are obtained.

So, the equation (4) corresponds to two systems: the system (9), which is the same as the system (1), and the system (11).

The equation (4) is the same as the equation (15) for $y_2 = x_1$. Equation (14) appears as a new equation.

By using (10) and (8) the system (11) can be write as

$$\begin{aligned}(t-a)y_1' + (A+1)y_1 + (-A+D-1+B)y_2 &= 0\\ (t-b)y_2' + y_1 + (D-1)y_2 &= 0\end{aligned} \tag{11*}$$

Let, the same procedure for the system (11*) is repeated i.e. for the new equation (12), i.e. (14), putting $A^* = A+1, B^* = -A+D-1+B, D^* = D-1$. By the same procedure is obtained the following system

$$\begin{aligned}(t-a)z_1' + (A^*+1)z_1 + (-A^*+D^*-1+B^*)z_2 &= 0\\ (t-b)z_2' + z_1 + (D^*-1)z_2 &= 0\end{aligned} \tag{16}$$

The following equations correspond to the system (16),

$$(t-a)(t-b)z_1'' + [(A^*+D^*+1)t-(A^*+2)b-(D^*-1)a]z_1' + (A^*D^*-B^*)z_1 = 0 \tag{17}$$

$$(t-a)(t-b)z_2'' + [(A^*+D^*+1)t-(A^*+1)b-D^*a]z_2' + (A^*D^*-B^*)z_2 = 0 \tag{18}$$

Obviously $y_1 = z_2$, but a new equation is an equation (17) for the function z_1.

In accordance with the corresponding shifts, the equations (17) and (18) are

$$(t-a)(t-b)z_1'' + [(A+D+1)t-(A+3)b-(D-2)a]z_1' + (AD-B)z_1 = 0 \quad (17^*)$$

$$(t-a)(t-b)z_2'' + [(A+D+1)t-(A+2)b-(D-1)a]z_2' + (AD-B)z_2 = 0 \quad (18^*)$$

i.e.

$$(t-a)(t-b)z_1'' + (\alpha_1 t + 2a - 2b + \alpha_2)z_1' + \beta_1 z_1 = 0, a \neq b \quad (17^{**})$$

$$(t-a)(t-b)z_2'' + (\alpha_1 t + a - b + \alpha_2)z_2' + \beta_1 z_2 = 0, a \neq b \quad (18^{**})$$

The equations (17) and (18) correspond to the system

$$\begin{aligned} &(t-a)z_1' + (A+2)z_1 + (2D-2A+B-4)z_2 = 0 \\ &(t-b)z_2' + z_1 + (D-2)z_2 = 0 \end{aligned} \quad (16^*)$$

Theorem 1. Let equation (4) be given. Then the equation (4) is reductive to only two systems of first order differential equations of type (1).

Proof. Let the differential equation (4) is given. By the method of transformations of a unique way from the formulas (5) and (7), the two systems (1) and (11) i.e. (11*) are obtained.

It is interesting that the free member of the polynomial coefficient before the first derivative of the equations (4), (6^*), (14), (17**) changes with changing of the systems and that

$$-(a-b)+\alpha_2, \quad \alpha_2, \quad a-b+\alpha_2, \quad 2(a-b)+\alpha_2, \dots \quad n(a-b)+\alpha_2, n \in \mathbb{Z}$$

After a series of transformations, we can obtain an equation of type

$$(t-a)(t-b)x'' + [\alpha_1 t + n(a-b) + \alpha_2]x' + \beta_1 x = 0, a \neq b \quad (19)$$

Theorem 2. Let the differential equation (4) is given. Then there is a sequence of differential equations of type (19).

Proof. Let a differential equation is given that is reductive on the systems (1) and (11*). The system (1) corresponds to the differential equation of type (6*). By the method of transformations from equation (4) with the formulas (7), we obtained the system (11*) to which corresponds the equation (14). By continuing the method of mathematical induction, the sequence of differential equations of type (19) is obtained.

We summarize: Let the equation

$$(t-a)(t-b)x_1'' + (\alpha_1 t + \alpha_2)x_1' + \beta_1 x_1 = 0, a \neq b$$

is given.

Let x_{10} be one of its particular solutions. From the system (9)

$$Bx_{20} = -[(t-a)x_{10}' + Ax_{10}]$$

is obtained, where x_{20} is a particular solution to the equation

$$(t-a)(t-b)x_2'' + (\alpha_1 t + b - a + \alpha_2)x_2' + \beta_1 x_2 = 0, a \neq b$$

and

$$a = a, b = b, A = \frac{b - a + a\alpha_1 + \alpha_2}{a-b}, D = \frac{-\alpha_2 - b\alpha_1}{a-b}, B = AD - \beta_1$$

From the system (11*)

$$y_{10} = -[(t-b)x_{10}' + (D-1)x_{10}]$$

is obtained, where y_{10} is a particular solution to the equation

$$(t-a)(t-b)y_1'' + (\alpha_1 t + a - b + \alpha_2)y_1' + \beta_1 y_1 = 0, a \neq b$$

From the system (16*)

$$z_{10} = -[(t-b)y_{10}' + (D-2)y_{10}]$$

is obtained, where z_{10} is a particular solution to the equation

$$(t-a)(t-b)z_1'' + (\alpha_1 t + 2(a-b) + \alpha_2)z_1' + \beta_1 z_1 = 0, a \neq b$$

By continuing the procedure, the following result is obtained.

A particular solution to an equation of type

$$(t-a)(t-b)x'' + (\alpha_1 t + n(a-b) + \alpha_2)x' + \beta_1 x = 0, a \neq b$$

will be obtained by the formula

$$x_0 = -[(t-b)x_{n-1,0}' + (D-n)x_{n-1,0}]$$

where $x_{n-1,0}$ is a particular solution to the equation

$$(t-a)(t-b)x_{n-1}'' + (\alpha_1 t + (n-1)(a-b) + \alpha_2)x_{n-1}' + \beta_1 x_{n-1} = 0, a \neq b, n \in \mathbb{Z}.$$

Lemma. Let the sequence of differential equations of type (19) is given and let a particular solution $x_{k-1,0}$ of the equation

$$(t-a)(t-b)x_{k-1}'' + (\alpha_1 t + (k-1)(a-b) + \alpha_2)x_{k-1}' + \beta_1 x_{k-1} = 0, a \neq b, k \in \mathbb{Z}$$

is known. Then by formula

$$x_{k,0} = -[(t-b)x_{k-1,0}' + (D-k)x_{k-1,0}], k \in \mathbb{Z}$$

the particular solution of the equation

$$(t-a)(t-b)x'' + (\alpha_1 t + k(a-b) + \alpha_2)x' + \beta_1 x = 0, a \neq b, k \in \mathbb{Z}$$

is given.

Proof. By the principle of mathematical induction, the result is obtained.

3. EXAMPLE

The result obtained for correlation between a class of second order linear differential equations and a class of systems of first order differential equations, we will shows via example.

Let the second order differential equation

$$(t-1)(t-2)x'' + (-3t+1)x' + 3x = 0 \qquad (1')$$

and the system of first order differential equations

$$\begin{aligned} (t-1)x_1' + x_1 + 2x_2 &= 0 \\ (t-2)x_2' - 4x_1 - 5x_2 &= 0 \end{aligned} \qquad (2')$$

i.e.

$$\begin{aligned} (t-1)x_1' + x_1 - 8x_2 &= 0 \\ (t-2)x_2' + x_1 - 5x_2 &= 0 \end{aligned} \qquad (2'')$$

are given. By corresponding transformations of (2'), the equations are obtained

$$(t-1)(t-2)x_1'' + (-3t+1)x_1' + 3x_1 = 0 \quad (3')$$

$$(t-1)(t-2)x_2'' + (-3t+2)x_2' + 3x_2 = 0 \quad (4')$$

where $x_1 = x$. The solution is the functions $x_1 = 12t-4, x_2 = 3t-2$. The new system from (3') is

$$\begin{aligned}(t-1)y_1' + 2y_1 - 15y_2 = 0\\ (t-2)y_2' + y_1 - 6y_2 = 0\end{aligned} \quad (5')$$

The equations which suit the system (5') are

$$(t-1)(t-2)y_1'' - 3t\,y_1' + 3y_1 = 0 \quad (6')$$

$$(t-1)(t-2)y_2'' + (-3t+1)\,y_2' + 3y_2 = 0 \quad (7')$$

where $y_2 = x_1$. The solution is the functions

$$y_2 = 12t-4, y_1 = 60t \text{ or } y_2 = 3t-1, y_1 = 15t.$$

By the same procedure, a new system from the system (5') is obtained

$$\begin{aligned}(t-1)z_1' + 3z_1 - 24z_2 = 0\\ (t-2)z_2' + z_1 - 7z_2 = 0\end{aligned} \quad (8')$$

The equations which suit the system (8') are

$$(t-1)(t-2)z_1'' - (3t+1)z_1' + 3z_1 = 0$$

$$(t-1)(t-2)z_2'' - 3t\,z_2' + 3z_2 = 0$$

where $z_2 = y_1$. The solution is the functions $z_1 = 360t + 120, z_2 = 60t$ or $z_1 = 6t+2, z_2 = t$.

4. CONCLUSIONS

We concluded that exists a correlation between the class of second order linear differential equations (A) and the class of systems of first order differential equations (C). That means that the second order linear differential equation (4) is reductive with two systems of first order differential equations of type (1). But also for the second order linear differential equation (4) exists a sequence of differential equations of type (19).

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THREE THEOREMS ABOUT FIXED POINT FOR CONTRACTIONS IN A COMPLETE METRIC SPACE

UDC: 515.126.4

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Abstract. In this paper are presented some generalizations of the R. Kannan, S. K. Chatterjea and P.V. Koparde and B.B. Waghmode theorems about common fixed points in a complete metric space (X,d). In doing so, we defined a continuous, injection and subsequentially convergent mapping T, and a function f. The function belongs to class Θ continuous monotony non-decreasing functions $f:[0,+\infty)\to[0,+\infty)$ such that $f^{-1}(0)=\{0\}$. In some results f is additionally defined as sub additive.

1. Introduction

The Banach principle for fixed point is well known in the literature. That is:

Let (X,d) be a metric space. The mapping $S:X\to X$ is said to be a contraction if there exists $\lambda\in(0,1)$ such that for all $x,y\in X$ holds that

$$d(Sx,Sy)\le\lambda d(x,y)\,. \tag{1}$$

If the metric space (X,d) is a complete metric space, then the mapping T for the condition (1) is satisfied has a unique fixed point.

R. Kannan, 1968 ([4]) generalized the Banach principle about a fixed point, as the following:

Theorem 1. If the mapping $S:X\to X$ for a complete metric space (X,d), satisfies the inequality

$$d(Sx,Sy)\le\lambda(d(x,Sx)+d(y,Sy))\,, \tag{2}$$

for $\lambda\in(0,\frac{1}{2})$ and $x,y\in X$, then S has a unique fixed point. □

If S satisfies the condition (2), then S is said to be Kannan type mapping.

S. K. Chatterjea, 1972 ([7]), defined similar conditions for contraction as the following:

Theorem 2. If the mapping $S:X\to X$ for a complete metric space (X,d) satisfies the inequality

$$d(Sx,Sy)\le\lambda(d(x,Sy)+d(y,Sx))\,, \tag{2}$$

2010 *Mathematics Subject Classification.* Primary: 47H10 Secondary: 54H25.
Key words and phrases. subsequentially convergent mapping, fixed point

for $\lambda \in (0,\frac{1}{2})$ and $x, y \in X$, then S has a unique fixed point. □

If S satisfies the condition (2), then S is said to be Chatterjea type of mapping.

P. V. Koparde and B. B. Waghmode, 1991 ([3]), presented a new generalize-tion of the Banach principle for a fixed point as the following:

Theorem 3. If the mapping $S: X \to X$ for a complete metric space (X,d) satisfies the inequality

$$d^2(Sx,Sy) \leq \lambda(d^2(x,Sx) + d^2(y,Sy)), \tag{3}$$

for $\lambda \in (0,\frac{1}{2})$ and $x, y \in X$, then S has a unique fixed point. □

If S satisfies the condition (3), then S is Koparde-Waghmode type of mapping.

S. Moradi and D. Alimohammadi [9] generalized the R. Kannan result, using the sequentially convergent mappings. Some generalizations of The Kannan, Chatterjea and Koparde-Waghmode theorems are proven in [1], by using the sequentially convergent mappings, defined as the following:

Definition 1 ([8]). Let (X,d) be a metric space. A mapping $T: X \to X$ is said sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ also is convergence. A mapping T is said sub-sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ has a convergent subsequence.

S. Moradi и A. Beiranvand, [8] introduce the concept for T_f contractive mapping, by using Θ class of continuous monotony non-decreasing functions $f:[0,+\infty) \to [0,+\infty)$ such that $f^{-1}(0) = \{0\}$, defined as the following.

Definition 2 ([8]). Let (X,d) be a metric space, $S,T: X \to X$ and $f \in \Theta$. A mapping S is said T_f −contraction if there exist $\lambda \in (0,1)$ such that

$$f(d(TSx,TSy)) \leq \lambda f(d(Tx,Ty)),$$

for all $x, y \in X$.

We must notice that, if $f \in \Theta$, then $f^{-1}(0) = \{0\}$ implies that $f(t) > 0$, for all $t > 0$. S. Moradi and A. Beiranvand proved that if S is T_f contractive mapping, then S has a unique fixed point. Then, M. Kir and H. Kiziltunc, [2] generalized the S. Moradi and A. Beiranvand result, for Kannan and Chatterjea type of mapping.

In our further consideration we will generalize the Kir and Kiziltunc results and will elaborate its application to the Koparde-Waghmoden type of mapping.

2. Mains Results

Theorem 4. Let (X,d) be a complete metric space $S:X\to X$, $f\in\Theta$ and the mapping $T:X\to X$ be continuous, injection and subsequentially convergent. If there exist $\alpha>0,\beta\geq 0$ such that $\alpha+2\beta\in(0,1)$ and

$$f(d(TSx,TSy))\leq(\alpha+\beta)f(d(Tx,TSx))+\beta f(d(Ty,TSy)) \tag{4}$$

for all $x,y\in X$, then S has a unique fixed point.

Proof. Let x_0 be any point on X and let the sequence $\{x_n\}$ be defined as $x_{n+1}=Sx_n$, $n=0,1,2,3,\ldots$. For $\lambda=\frac{\alpha+2\beta}{2-(\alpha+2\beta)}$ and since $\alpha+2\beta\in(0,1)$, $\alpha,\beta\geq 0$, we get that $\lambda\in(0,1)$. The inequality (4) implies

$$\begin{aligned}f(d(Tx_{n+1},Tx_n))&=f(d(TSx_n,TSx_{n-1}))\\&\leq(\alpha+\beta)f(d(Tx_{n-1},TSx_{n-1}))+\beta f(d(Tx_n,TSx_n))\\&=(\alpha+\beta)f(d(Tx_{n-1},Tx_n))+\beta f(d(Tx_n,Tx_{n+1})).\end{aligned}$$

Analogously,

$$f(d(Tx_{n+1},Tx_n))\leq\beta f(d(Tx_{n-1},Tx_n))+(\alpha+\beta)f(d(Tx_n,Tx_{n+1})).$$

By adding the last two inequalities , we get the following

$$f(d(Tx_{n+1},Tx_n))\leq\lambda f(d(Tx_n,Tx_{n-1})), \tag{5}$$

for each $n=1,2,3,\ldots$. The inequality (5) implies that

$$f(d(Tx_{n+1},Tx_n))\leq\lambda^n f(d(Tx_1,Tx_0)), \tag{6}$$

for each $n=1,2,3,\ldots$. Further, the inequalities (4) and (6) imply that for all $m,n\in\mathbb{N}$ $n>m$

$$\begin{aligned}f(d(Tx_n,Tx_m))&=f(d(TSx_{n-1},TSx_{m-1}))\\&\leq(\alpha+\beta)f(d(TSx_{n-1},Tx_{n-1}))+\beta f(d(TSx_{m-1},Tx_{m-1}))\\&=(\alpha+\beta)f(d(Tx_n,Tx_{n-1}))+\beta f(d(Tx_m,Tx_{m-1}))\\&\leq[(\alpha+\beta)\lambda^{n-1}+\beta\lambda^{m-1}]f(d(Tx_1,Tx_0))\end{aligned}$$

holds true. Analogously,

$$f(d(Tx_n,Tx_m))\leq[(\alpha+\beta)\lambda^{m-1}+\beta\lambda^{n-1}]f(d(Tx_1,Tx_0))].$$

By adding the last two inequalities, we get that

$$f(d(Tx_n,Tx_m))\leq\frac{\alpha+2\beta}{2}(\lambda^{m-1}+\lambda^{n-1})f(d(Tx_1,Tx_0))].$$

The last inequality implies that

$$\lim_{m,n\to\infty} f(d(Tx_n,Tx_m))=0,$$

and since $f\in\Theta$ we get that $\lim_{m,n\to\infty} d(Tx_n,Tx_m)=0$. Therefore, the sequence $\{Tx_n\}$ is Caushy sequence. But, X is complete metric space, and therefore the sequence $\{Tx_n\}$ is a convergent sequence. The mapping $T:X\to X$ is subsequentially convergent, therefore the sequence $\{x_n\}$ consists a convergent subsequence $\{x_{n(k)}\}$, i.e. it exists $u\in X$ so that $\lim_{k\to\infty} x_{n(k)}=u$. The continuity of T implies that $\lim_{k\to\infty} Tx_{n(k)}=Tu$. Further, $\{Tx_{n(k)}\}$ is a subsequence of the convergent sequence $\{Tx_n\}$, therefore $\lim_{n\to\infty} Tx_n=\lim_{k\to\infty} Tx_{n(k)}=Tu$.

It will be proven that $u\in X$ is fixed point for the mapping S. Now,

$$\begin{aligned} f(d(TSu,Tx_{n+1}))&=f(d(TSu,TSx_n))\le(\alpha+\beta)f(d(TSu,Tu))+\beta f(d(TSx_n,Tx_n))\\ &=(\alpha+\beta)f(d(TSu,Tu))+\beta f(d(Tx_{n+1},Tx_n))\end{aligned}$$

holds true. Analogously,

$$f(d(TSu,Tx_{n+1}))\le\beta f(d(TSu,Tu))+(\alpha+\beta)f(d(Tx_{n+1},Tx_n)),$$

therefore

$$f(d(TSu,Tx_{n+1}))\le\tfrac{\alpha+2\beta}{2}[f(d(TSu,Tu))+f(d(Tx_{n+1},Tx_n))]$$

For $n\to\infty$, in the inequality above, $\lim_{n\to\infty} Tx_n=Tu$ and the properties of f and the metrics imply that

$$f(d(TSu,Tu))\le\tfrac{\alpha+2\beta}{2}[f(d(TSu,Tu))+f(0)]$$

holds true. But $1-\frac{\alpha+2\beta}{2}>0$ and $f^{-1}(0)=\{0\}$. Therefore, the above inequality implies that $d(TSu,Tu)=0$, i.e. $TSu=Tu$. Finally, T is injection, and therefore $Su=u$, that is the mapping S has a fixed point.

Let $u,v\in X$ be two fixed points for S, i.e. $Su=u$ and $Sv=v$. So, (4) implies that

$$f(d(Tu,Tv))=f(d(TSu,TSv))\le(\alpha+\beta)[f(d(Tu,TSu))+\beta f(d(Tv,TSv))]=0,$$

holds true, that is $d(Tu,Tv)=0$. Therefore, $Tu=Tv$. But, T is injection, and therefore $u=v$, that is T has a unique fixed point. ■

Corollary 1. Let (X,d) be a complete metric space, $S:X\to X$ and $f\in\Theta$. If there exist $\alpha,\beta\ge0$ such that $\alpha+2\beta\in(0,1)$ and

$$f(d(Sx,Sy))\le(\alpha+\beta)f(d(x,Sx))+\beta f(d(y,Sy)),$$

for all $x,y\in X$, then S has a unique fixed point and for each $x_0\in X$ the sequence $\{S^n x_0\}$ converges to the fixed point.

Proof. The mapping $Tx = x$, for each $x \in X$ is continuous, injection and sequentially convergent. Therefore, the corollary is directly implied by Theorem 4 for $Tx = x$. ■

Corollary 2. Let (X,d) be a complete metric space, $S: X \to X$ and the mapping $T: X \to X$ is continuous, injection and subsequentially convergent. If there exist $\alpha, \beta \geq 0$ such that $\alpha + 2\beta \in (0,1)$ and

$$d(TSx, TSy) \leq (\alpha + \beta) d(Tx, TSx) + \beta d(Ty, TSy)$$

for all $x, y \in X$, then S has a unique fixed point.

Proof. The function $f(t) = t$, $t \geq 0$ is monotony increasing and $f^{-1}(0) = \{0\}$. Therefore, the corollary is a direct implication of Theorem 4 for $f(t) = t$. ■

Comment 1. 1) For $\alpha = 0$ и $\beta = \lambda$, the Theorem 4 is transformed as the Theorem2.1 [2].

2) If we take into a consideration that the mapping $Tx = x$, for all $x \in X$ is continuous, injection and subsequentially convergent, the Corollary 2 implies that if for the mapping $S: X \to X$ exist $\alpha, \beta \geq 0$ such that $\alpha + 2\beta \in (0,1)$ and

$$d(Sx, Sy) \leq (\alpha + \beta) d(x, Sx) + \beta d(y, Sy), \tag{7}$$

for all $x, y \in X$, then S has a unique fixed point.

3) For $\alpha = 0$ and $\beta = \lambda$ in (7) , we get that the Theorem 4 implies the Theorem 1.

Theorem 5. Let (X,d) be a complete metric space $S: X \to X$, $f \in \Theta$ and the mapping $T: X \to X$ be continuous, injection and subsequentially convergent. If it exist $\alpha > 0, \beta \geq 0$ such that $\alpha + 2\beta \in (0,1)$ and

$$f(d^2(TSx, TSy)) \leq (\alpha + \beta) f(d^2(Tx, TSx)) + \beta f(d^2(Ty, TSy))$$

for all $x, y \in X$, then S has a unique fixed point.

Proof. It is obvious that $g(t) = t^2, t \geq 0$ is a function of the Θ class. Further, if $f, g \in \Theta$, then $f \circ g \in \Theta$, therefore the truth of the theorem is implied by the Theorem 4. ■

Corollary 3. Let (X,d) be a complete metric space, $S: X \to X$ and $f \in \Theta$. If there exist $\alpha, \beta \geq 0$ such that $\alpha + 2\beta \in (0,1)$ and

$$f(d^2(Sx, Sy)) \leq (\alpha + \beta) f(d^2(x, Sx)) + \beta f(d^2(y, Sy)),$$

for all $x, y \in X$, then S has a unique fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to the fixed point.

Proof. The mapping $Tx = x$, for each $x \in X$ is continuous, injection and sequentially convergent. Therefore, the corollary is directly implied by Theorem 5 for $Tx = x$. ■

Corollary 4. Let (X,d) be a complete metric space, $S: X \to X$ and the mapping $T: X \to X$ is continuous, injection and subsequentially convergent. If there exist $\alpha, \beta \geq 0$ such that $\alpha + 2\beta \in (0,1)$ and

$$d^2(TSx, TSy) \leq (\alpha+\beta) d^2(Tx, TSx) + \beta d^2(Ty, TSy)$$

for all $x, y \in X$, then S has a unique fixed point.

Proof. The function $f(t) = t, t \geq 0$ is monotony increasing and $f^{-1}(0) = \{0\}$. Therefore, the corollary is a direct implication of Theorem 4 for $f(t) = t$. ■

Comment 2. 1) For $\alpha = 0$ and $\beta = \lambda$, in the Theorem 5 we get that in a complete metric space (X,d) if $S: X \to X$, $f \in \Theta$ and the mapping $T: X \to X$ is continuous, injection and subsequentially convergent and if it exists $\lambda \in (0, \frac{1}{2})$ is such that

$$f(d^2(TSx, TSy)) \leq \lambda (f(d^2(Tx, TSx)) + f(d^2(Ty, TSy)))$$

for all $x, y \in X$, then S has a unique fixed point.

2) If we take consideration that the mapping $Tx = x$, for all $x \in X$ is continuous, injection and subsequentially convergent, the Corollary 4 implies that if for the mapping $S: X \to X$ there exist $\alpha, \beta \geq 0$ such that $\alpha + 2\beta \in (0,1)$ and

$$d^2(Sx, Sy) \leq (\alpha+\beta) d^2(x, Sx) + \beta d^2(y, Sy), \tag{8}$$

for all $x, y \in X$, then S has a unique fixed point.

3) For $\alpha = 0$ and $\beta = \lambda$ in (8) we get that the Theorem5 implies the Theorem3.

Theorem 6. Let (X,d) be a complete metric space $S: X \to X$, the mapping $T: X \to X$ be continuous, injection and subsequentially convergent and $f \in \Theta$ is such that $f(a+b) \leq f(a) + f(b)$, for all $a, b \geq 0$. If there exist $\alpha > 0, \beta \geq 0$ such that $\alpha + 2\beta \in (0,1)$ and

$$f(d(TSx, TSy)) \leq (\alpha+\beta) f(d(Tx, TSy)) + \beta f(d(Ty, TSx)) \tag{9}$$

for all $x, y \in X$, then S has a unique fixed point.

Proof. Let x_0 be any point on X and the sequence $\{x_n\}$ be defined as the following $x_{n+1} = Sx_n$, $n = 0,1,2,3,...$. The inequality (9) and the property of f imply the followings

$$f(d(Tx_{n+1}, Tx_n)) \leq \beta f(d(Tx_{n-1}, Tx_n)) + \beta f(d(Tx_n, Tx_{n+1}))$$

and

$$f(d(Tx_{n+1},Tx_n)) \le (\alpha+\beta)f(d(Tx_{n-1},Tx_n)) + (\alpha+\beta)f(d(Tx_n,Tx_{n+1})).$$

By summarizing the last two inequalities we obtain the following

$$f(d(Tx_{n+1},Tx_n)) \le \lambda f(d(Tx_n,Tx_{n-1})), \qquad (10)$$

for each $n=1,2,3,...$, for $\lambda = \frac{\alpha+2\beta}{2-(\alpha+2\beta)} < 1$. Further, by applying the inequality (10), analogously as the proof in theorem 4, we get that the sequence $\{Tx_n\}$ is convergent. Therefore, the sequence $\{x_n\}$ consists of convergent subsequence, i.e. it exists $u \in X$ and a subsequence $\{x_{n(k)}\}$ of the sequence $\{x_n\}$ such that $\lim_{k\to\infty} x_{n(k)} = u$. The continuity of T implies that $\lim_{k\to\infty} Tx_{n(k)} = Tu$, that is $\lim_{n\to\infty} Tx_n = Tu$. Further, the inequality (9), analogously as the proof in theorem 4, implies

$$f(d(TSu,Tx_{n+1})) \le \frac{\alpha+2\beta}{2}[f(d(Tu,Tx_{n+1})) + f(d(Tx_n,TSu))]$$

For $n\to\infty$ in the above inequality, the continuity of f and T and the properties of the metric, imply

$$f(d(TSu,Tu)) \le \frac{\alpha+2\beta}{2}[f(d(TSu,Tu)) + f(0)].$$

Therefore, analogously as the proof in theorem 4, we conclude that $Su=u$, that is the mapping S has a fixed point.

Let $u,v \in X$ be fixed point on S, i.e. $Su=u$ and $Sv=v$. Then, (9) implies

$$\begin{aligned} f(d(Tu,Tv)) &= f(d(TSu,TSv)) \\ &\le (\alpha+\beta)f(d(Tu,TSv)) + \beta f(d(Tv,TSu)) \\ &= (\alpha+2\beta)f(d(Tu,Tv)). \end{aligned}$$

Therefore, $u=v$, i.e. S has a unique fixed point. Finally, if T e sequentially convergent, then by substituting the sequence $\{n(k)\}$ with the sequence $\{n\}$ and arbitrarily of $x_0 \in X$ and the above stated, imply that for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to the unique fixed point on S. ■

Corollary 5. Let (X,d) be a complete metric space, $S: X \to X$ and $f \in \Theta$ be such that $f(a+b) \le f(a)+f(b)$, for all $a,b \ge 0$. If it exist $\alpha,\beta \ge 0$ such that $\alpha+2\beta \in (0,1)$ and

$$f(d(Sx,Sy)) \le (\alpha+\beta)f(d(x,Sy)) + \beta f(d(y,Sx)),$$

for all $x,y \in X$, then S has a unique fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ converges to that point.

Proof. For $Tx = x$, in theorem 6 we get the corollary. ■

Corollary 6. Let (X,d) be a complete metric space, $S:X\to X$ and the mapping $T:L\to L$ be continuous, injection and subsequentially convergent. If it exists $\alpha,\beta\geq 0$ such that $\alpha+2\beta\in(0,1)$ and

$$d(TSx,TSy)\leq(\alpha+\beta)\,|\,d(Tx,TSy)+\beta d(Ty,TSx)$$

for all $x,y\in X$, then S has a unique fixed point.

Proof. For $f(t)=t$, in theorem 6 we get the corollary. ■

Comment 3. 1) For $\alpha=0$ and $\beta=\lambda$, the Theorem6 implies the Theorem 2.2 [2].

When expressing the Theorem 2.2 [2] is missing the condition of sub-additivity of the function $f\in\Theta$ which is applied at the beginning of the proof of the Theorem.

2) Having in mind that $Tx=x$, for all $x\in X$ is continuous, injection and subsequentially convergent, the Corollary 6 implies that if for the mapping $S:X\to X$ exist $\alpha,\beta\geq 0$ such that $\alpha+2\beta\in(0,1)$ and

$$d(Sx,Sy)\leq(\alpha+\beta)d(x,Sx)+\beta d(y,Sy), \qquad (11)$$

for all $x,y\in X$, then S has a unique fixed point.

3) For $\alpha=0$ and $\beta=\lambda$ in (11), the Theorem6 implies the Theorem2.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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N-TUPLE ORBITS TENDING TO INFINITY

UDC: 517.984.3

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Abstract. In this paper we prove some results on the existence of a dense set of vectors each having an n-tuple orbit tending to infinity for sequences of mutually commuting bounded linear operators acting on an infinite dimensional complex Banach space.

1. INTRODUCTION

Let X be a complex infinite-dimensional Banach space and $B(X)$ the algebra of all bounded linear operators acting on X. For an operator $T \in B(X)$, $\sigma(T)$, $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{ap}}(T)$ and $r(T)$ will denote the spectrum, the point spectrum, the approximate point spectrum and the spectral radius of the operator T, respectively.

If $T_1, T_2, \ldots, T_n \in B(X)$ are mutually commuting operators, the *n-tuple orbit* (or the orbit under the *n*-tuple $\mathbf{T} = (T_1, T_2, \ldots, T_n)$) of the vector $x \in X$ is the set

$$\mathrm{Orb}(\{T_i\}_{i=1}^{n}, x) = \mathrm{Orb}(\mathbf{T}, x) = \left\{ T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} x : k_i \geq 0; 1 \leq i \leq n \right\}. \tag{1.1}$$

The *n*-tuple orbit *tends to infinity* if

$$\lim_{k_i \to \infty} \left\| T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} x \right\| = \infty, \text{ for all } k_j \geq 0,\ j \neq i,\ 1 \leq i, j \leq n .$$

For $n = 1$, the *n*-tuple orbit (1.1) reduces to a simple sequence of form

$$\mathrm{Orb}(T, x) = \left\{ T^n x : n = 0, 1, 2, \ldots \right\}.$$

usually referred as *single orbit* (or simply *orbit*) of the vector $x \in X$ under the operator T. Regardless of the dimension of the space, single orbits tending to infinity may exist only when T is power unbounded operator, i.e. when $\sup_n \|T^n\| = \infty$. In this case, by the Banach-Steinhaus theorem, the space will contain a dense G_δ-set of vectors each having an unbounded orbit under T (i.e. orbit with $\sup_n \|T^n x\| = \infty$). But, unlike the case of an operator T acting on a finite-dimensional space where the only unbounded orbits for T are those tending to infinity and may exist if, and only if, $r(T) > 1$, in the case of an

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2010 *Mathematics Subject Classification.* Primary: 47A05; Secondary: 47A11, 47A25.

Key words and phrases. Banach spaces, orbits tending to infinity, n-tuple orbits, sequences of operators

infinite-dimensional space, the structure of the set of all vectors with orbits tending to infinity can be quite different. Clearly, if $\sigma_{\mathrm{p}}(T)$ contains a point λ such that $|\lambda|>1$, this set will contain all the elements of $\mathrm{Ker}(T-\lambda I)\setminus\{0\}$. Furthermore, the set of all vectors with orbits tending to infinity can be dense in the whole space, even if the point spectrum of the operator is empty. The results obtained by B. Beauzamy for operators on infinite-dimensional Hilbert or reflexive Banach space X (cf. [1, Ch. III]) imply that for any operator $T\in B(X)$ for which $\sigma_{\mathrm{ap}}(T)\setminus\sigma_{\mathrm{p}}(T)$ contains a point λ with $|\lambda|>1$, the space will contain a dense set D such that $\|T^n x\|\to\infty$ as $n\to\infty$, for all $x\in D$. The results obtained by V. Müller ([7] and [8]) imply that such set exists for any operator T on arbitrary infinite-dimensional Banach space as long as $r(T)>1$. In general, this set is not a G_δ-set since the space may contain another dense G_δ-set of vectors with unbounded orbits: vectors for which $\mathrm{Orb}(T,x)$ itself is dense in the whole space (cf. [9, Theorem 1] or [1, III.0.C]).

Under the assumption that T_1 and T_2 are bounded linear operators on infinite-dimensional Hilbert or reflexive Banach space satisfying

$$(\sigma_{\mathrm{ap}}(T_1)\setminus\sigma_{\mathrm{p}}(T_1))\cap\{\lambda\in\mathbb{C}:|\lambda|>1\}\neq\varnothing,$$
$$(\sigma_{\mathrm{ap}}(T_2)\setminus\sigma_{\mathrm{p}}(T_2))\cap\{\lambda\in\mathbb{C}:|\lambda|>1\}\neq\varnothing.$$

in [2] and [3] the authors have shown that the space contains a dense set D such that

$$\|T_1^n x\|\to\infty \text{ and } \|T_2^n x\|\to\infty, \text{ for all } x\in D.$$

If, in addition, T_1 and T_2 are commuting operators, each bounded bellow, then for every $x\in D$ the corresponding 2-tuple orbit tends to infinity ([10, Theorem 1.4]):

$$\left\|T_1^{k_1}T_2^{k_2}x\right\|\to\infty \text{ as } k_1\to\infty, \text{ for every } k_2\geq 0,$$

and

$$\left\|T_1^{k_1}T_2^{k_2}x\right\|\to\infty \text{ as } k_2\to\infty, \text{ for every } k_1\geq 0.$$

Using the following three results, in the next section we are going to generalize this result for n-tuple orbits and sequences of mutually commuting operators each bounded bellow.

Theorem 1.1. ([8, Theorem V.37.14]) *Let X and Y be Banach spaces and let $(T_n)_{n\geq 1}$ be a sequence of operators in $B(X,Y)$. Then for every sequence of positive numbers $(a_n)_{n\geq 1}$ with $\sum_{n\geq 1}a_n<\infty$, in every open ball in X with* *radius strictly larger than* $\sum_{n\geq 1} a_n < \infty$, *there is a vector* $x \in X$ *satisfying* $\|T_n x\| \geq a_n \|T_n\|$, *for all* $n \geq 1$.

Lemma 1.2. ([8, Lemma V.37.15]) *Let* $\varepsilon > 0$ *and* $(a_n)_{n\geq 1}$ *be a sequence of positive numbers satisfying* $\sum_{n\geq 1} a_n < \varepsilon$. *Then there is a sequence of positive numbers* $(b_n)_{n\geq 1}$ *such that* $b_n \to \infty$ *as* $n \to \infty$ *and* $\sum_{n\geq 1} a_n b_n < \varepsilon$.

Corollary 1.3. ([8, Corollary V.37.16]) *If* $T \in B(X)$ *satisfies* $\sum_{n=1}^{\infty} \|T^n\|^{-1} < \infty$, *then there is a dense set* $D \subset X$ *such that* $\operatorname{Orb}(T, x)$ *tends to infinity for every* $x \in D$.

2. MAIN RESULTS

Throughout the rest of this paper we assume that the spaces are complex and infinite-dimensional.

Lemma 2.1. *Let* X *be a Banach space and* $T_1, T_2, \ldots T_n \in B(X)$ *are mutually commuting operators with at least one of the following properties:*

(P.1) *the operator* T_i *is bounded bellow, for every* i;

(P.2) $(T_i^k - T_j^k)_{k\geq 0}$ *is a norm bounded sequence, for every* i *and* j.

If $x \in X$ *is such that* $\operatorname{Orb}(T_i, x)$ *tends to infinity for every* $i \in \{1, 2, \ldots, n\}$, *then for every* $1 \leq m \leq n$ *and every* $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ *the m-tuple orbit* $\operatorname{Orb}(\{T_{i_j}\}_{j=1}^m, x)$ *tends to infinity.*

Proof. If the operators $T_1, T_2, \ldots T_n$ have the property (P.1), then there are positive numbers $C_1, C_2, \ldots C_n$ such that

$$\|T_i x\| \geq C_i \|x\|, \text{ for all } x \in X,\ 1 \leq i \leq n.$$

Hence, if $1 \leq m \leq n$, $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ and $k_j \geq 0$, $j \in \{1, 2, \ldots, m\}$, then for every $s \in \{1, 2, \ldots, m\}$ and fixed values for k_j, $j \in \{1, 2, \ldots, m\} \setminus \{s\}$

$$\left\|T_{i_1}^{k_1} T_{i_2}^{k_2} \ldots T_{i_m}^{k_m} x\right\| \geq \left(\prod_{\substack{l=1 \\ l \neq s}}^{m} C_{i_l}^{k_l} \right) \cdot \left\|T_{i_s}^{k_s} x\right\| \to \infty, \text{ as } k_s \to \infty.$$

Now, assume that the operators $T_1, T_2, \ldots T_n$ have the property (P.2). For $i, j \in \{1, 2, \ldots, n\}$, let $M_{i,j} > 0$ is such that $\left\| T_i^k - T_j^k \right\| \le M_{i,j}$, for every $k \ge 0$.

We continue by induction. Let $m = 2$ and $1 \le i_1 < i_2 \le n$. Then

$$\begin{aligned}\left\| T_{i_1}^{k_1+k_2} x \right\| &\le \left\| T_{i_1}^{k_1+k_2} x - T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \\ &= \left\| T_{i_1}^{k_1} (T_{i_1}^{k_2} - T_{i_2}^{k_2}) x \right\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \\ &\le \left\| T_{i_1}^{k_1} \right\| \cdot \left\| T_{i_1}^{k_2} - T_{i_2}^{k_2} \right\| \cdot \|x\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \\ &\le \left\| T_{i_1} \right\|^{k_1} \cdot M_{i_1, i_2} \cdot \|x\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\|.\end{aligned}$$

Since $\left\| T_{i_1}^{n} x \right\| \to \infty$ as $n \to \infty$ (and hence $\left\| T_{i_1}^{k_1+k_2} x \right\| \to \infty$ as $k_2 \to \infty$, for all $k_1 \ge 0$), the above inequalities imply that

$$\left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \to \infty, \text{ as } k_2 \to \infty, \text{ for all } k_1 \ge 0.$$

Similarly, the following inequalities

$$\begin{aligned}\left\| T_{i_2}^{k_1+k_2} x \right\| &\le \left\| T_{i_2}^{k_1+k_2} x - T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \\ &= \left\| T_{i_2}^{k_2} (T_{i_2}^{k_1} - T_{i_1}^{k_1}) x \right\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \\ &\le \left\| T_{i_2}^{k_2} \right\| \cdot \left\| T_{i_2}^{k_1} - T_{i_1}^{k_1} \right\| \cdot \|x\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \\ &\le \left\| T_{i_2} \right\|^{k_2} \cdot M_{i_2, i_1} \cdot \|x\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\|.\end{aligned}$$

imply that

$$\left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \to \infty, \text{ as } k_1 \to \infty, \text{ for all } k_2 \ge 0.$$

To complete the proof, it is enough to show the claim for $m = n$, under the assumption that $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^{n-1}, x)$ tends to infinity for all $1 \le i_1 < \ldots < i_{n-1} \le n$.

For a fixed $i \in \{1, \ldots, n\}$, arbitrary $j \in \{1, \ldots, n\} \setminus \{i\}$ and $k_1, \ldots, k_n \ge 0$ we have

$$\begin{aligned}
&\left\|T_1^{k_1}\ldots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\ldots T_n^{k_n}x\right\| \\
&\quad\le \left\|T_1^{k_1}\ldots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\ldots T_n^{k_n}x - T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x\right\| + \left\|T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x\right\| \\
&\quad= \left\|T_1^{k_1}\ldots T_{i-1}^{k_{i-1}}T_{i+1}^{k_{i+1}}\ldots T_n^{k_n}(T_j^{k_i} - T_i^{k_i})x\right\| + \left\|T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x\right\| \\
&\quad\le \left\|T_1^{k_1}\ldots T_{i-1}^{k_{i-1}}T_{i+1}^{k_{i+1}}\ldots T_n^{k_n}\right\|\cdot\left\|T_j^{k_i} - T_i^{k_i}\right\|\cdot\|x\| + \left\|T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x\right\| \\
&\quad\le \left(\prod_{\substack{l=1\\ l\ne i}}^{n}\|T_l\|^{k_l}\right)\cdot M_{i,j}\cdot\|x\| + \left\|T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x\right\|.
\end{aligned}$$

Since $j\in\{1,2,\ldots,n\}\setminus\{i\}$,

$$T_1^{k_1}\ldots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\ldots T_n^{k_n}x\in \mathrm{Orb}(\{T_1\ldots T_{i-1}T_{i+1}\ldots T_n\},x),$$

and, by assumption, this (n-1)-tuple orbit tents to infinity,

$$\left\|T_1^{k_1}\ldots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\ldots T_n^{k_n}x\right\|\to\infty \text{ as } k_i\to\infty, \text{ for all } k_j\ge 0,\ j\ne i.$$

This, together with the above inequalities implies that

$$\left\|T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x\right\|\to\infty \text{ as } k_i\to\infty, \text{ for all } k_j\ge 0,\ j\ne i,$$

which completes the proof. □

Theorem 2.2. *If X is a Banach space and $T_1,T_2,\ldots T_n\in B(X)$ are operators with $r(T_i)>1$, $1\le i\le n$, then there is a dense set $D\subset X$ such that $\mathrm{Orb}(T_i,x)$ tends to infinity for every $x\in D$ and every $1\le i\le n$. If, in addition, the operators are mutually commuting and have at least one of the properties (P.1) and (P.2) in Lemma 2.1, then the m-tuple orbit $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^{m},x)$ tends to infinity for every $x\in D$, $1\le m\le n$ and $1\le i_1<i_2<\ldots<i_m\le n$.*

Proof. By Lemma 2.1, it is sufficient to prove the first assertion in the theorem.

Let $z\in X$ and $\varepsilon>0$. Since $r(T_i)>1$ there is $\lambda_i\in\sigma(T_i)$ such that $|\lambda_i|>1$, $1\le i\le n$. If $q,C\in\mathbb{R}$ are chosen such that

$$1<q<\min\{|\lambda_1|,|\lambda_2|,\ldots,|\lambda_n|\},$$

and

$$0<C<\frac{\varepsilon(q-1)^2}{2(n+1)},$$

then the sequences of positive numbers $\{(a_{i,k})_{k\ge1}:1\le i\le n\}$ defined with

$$a_{i,k} = Cq^{-(i+k)}\,,\ 1 \le i \le n\,,\ k \ge 1\,,$$

will satisfy

$$\sum_{1\le i\le n}\sum_{k\ge 1} a_{i,k} < \frac{\varepsilon}{2}\,. \tag{2.1}$$

If the sequence of operators $(S_j)_{j\ge 1}$ and the sequence of positive numbers $(a_j)_{j\ge 1}$ are defined with

$$S_{(k-1)n+i} = T_i^k \text{ and } a_{(k-1)n+i} = a_{i,k}\,, \text{ for } 1 \le i \le n\,,\ k \ge 1\,, \tag{2.2}$$

then

$$\sum_{j\ge 1} a_j = \sum_{1\le i\le n}\sum_{k\ge 1} a_{i,k}\,,$$

and hence, by Theorem 1.1 (applied on $(S_j)_{j\ge 1}$ and $(a_j)_{j\ge 1}$), the Spectral Mapping Theorem and (2), the open ball with center z and radius ε will contain a vector $x \in X$ such that for every $1 \le i \le n$ and $k \ge 1$,

$$\begin{aligned}\left\|T_i^k x\right\| &= \left\|S_{(k-1)n+i}x\right\| \ge a_{(k-1)n+i}\left\|S_{(k-1)n+i}\right\| = a_{i,k}\left\|T_i^k\right\| \\ &\ge Cq^{-(i+k)}\left|\lambda_i\right|^k = Cq^{-i}\left|q^{-1}\lambda_i\right|^k.\end{aligned}$$

Since, by the choice of q, $|q^{-1}\lambda_i|^k \to \infty$ as $k \to \infty$, for every $1 \le i \le n$, the above inequalities imply that

$$\left\|T_i^k x\right\| \to \infty \text{ as } k \to \infty\,, \text{ for all } 1 \le i \le n\,,$$

which completes the proof. □

By Theorem 1.1 and Lemma 2.1 alone we can obtain similar result for sequence of operators $(T_i)_{i\ge 1}$ in $B(X)$.

Theorem 2.3. *If X is a Banach space and $(T_i)_{i\ge 1}$ is a sequence of operators in $B(X)$ for which there is $\beta > 0$ such that $r(T_i) > 1+\beta$, for all $i \ge 1$, then there is a dense set $D \subset X$ such that* $\mathrm{Orb}(T_i, x)$ *tends to infinity for every $x \in D$ and $i \ge 1$. If, in addition, the operators are mutually commuting and have at least one of the properties (P.1) and (P.2) in Lemma 2.1, then for every $n \ge 1$ and every positive integers $i_1 < i_2 < \ldots < i_n$ the n-tuple orbit* $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^n, x)$ *tends to infinity for every $x \in D$.*

The proof of the first assertion in Theorem 2.3 is given in [6].

The requirement "*there is* $\beta>0$ *such that* $r(T_i)>1+\beta$, *for all* $i\geq 1$" in Theorem 2.3 can be replaced with the following one: "$r(T_i)>1$, *for all* $i\geq 1$". In order to show this, first we are going to give an appropriate generalization of Corollary 1.3.

Theorem 2.4. *If* X *is a Banach space and* $T_1,T_2,\ldots T_n\in B(X)$ *are operators satisfying* $\sum_{n=1}^{\infty}\left\|T_i^n\right\|^{-1}<\infty$, *for all* $1\leq i\leq n$, *then there is a dense set* $D\subset X$ *such that* $\mathrm{Orb}(T_i,x)$ *tends to infinity for every* $x\in D$ *and every* $1\leq i\leq n$. *If, in addition, the operators are mutually commuting and have at least one of the properties (P.1) and (P.2) in Lemma 2.1, then the m-tuple orbit* $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^{m},x)$ *tends to infinity for every* $x\in D$, $1\leq m\leq n$ *and* $1\leq i_1<i_2<\ldots<i_m\leq n$.

Proof. Once again, by Lemma 2.1, it is sufficient to prove the first assertion in the theorem. Let $z\in X$ and $\varepsilon>0$. For $1\leq i\leq n$, let $\varepsilon_i>0$ be such that

$$\varepsilon_i\left(\sum_{k=1}^{\infty}\left\|T_i^k\right\|^{-1}\right)<\frac{\varepsilon}{2(n+1)}.$$

By Lemma 1.2 there are a sequences of positive numbers $(b_{i,k})_{k\geq 1}$ so that $b_{i,k}\to\infty$ as $k\to\infty$ and

$$\sum_{k=1}^{\infty}\varepsilon_i b_{i,k}\left\|T_i^k\right\|^{-1}<\frac{\varepsilon}{2(n+1)}.$$

For $1\leq i\leq n$ and $k\in\mathbb{N}$, let $a_{i,k}=\varepsilon_i b_{i,k}\left\|T_i^k\right\|^{-1}$. If the sequence of operators $(S_j)_{j\geq 1}$ and the sequence of positive numbers $(a_j)_{j\geq 1}$ are defined with (2.2), then $\sum_{j\geq 1}a_j<\varepsilon/2$. Hence, by Theorem 1.1, there is a vector $x\in X$ satisfying $\|x-z\|<\varepsilon$ and for every $1\leq i\leq n$ and $k\geq 1$,

$$\begin{aligned}\left\|T_i^k x\right\|&=\left\|S_{(k-1)n+i}x\right\|\\&\geq a_{(k-1)n+i}\left\|S_{(k-1)n+i}\right\|=a_{i,k}\left\|T_i^k\right\|=\varepsilon_i b_{i,k}\left\|T_i^k\right\|^{-1}\left\|T_i^k\right\|=\varepsilon_i b_{i,k}.\end{aligned}$$

This implies that

$$\left\|T_i^k x\right\|\to\infty \text{ as } k\to\infty\text{, for all } 1\leq i\leq n,$$

which completes the proof. □

Theorem 2.5. *If* X *is a Banach space and* $(T_i)_{i\geq 1}$ *is a sequence of operators in* $B(X)$ *such that* $\sum_{k=1}^{\infty}\left\|T_i^k\right\|^{-1}<\infty$, *for all* $i\geq 1$, *then there is a dense set* $D\subset X$ *so that* $\mathrm{Orb}(T_i,x)$ *tends to infinity for every* $x\in D$ *and* $i\geq 1$. *If, in addition, the operators are mutually commuting and have at least one of the properties (P.1) and (P.2) in Lemma 2.1, then for every* $n\geq 1$ *and every positive integers* $i_1<i_2<\ldots<i_n$ *the n-tuple orbit* $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^{n},x)$ *tends to infinity for every* $x\in D$.

The proof of the first assertion in Theorem 2.5 the is given in [6].

Corollary 2.6. *If* $(T_i)_{i\geq 1}$ *is a sequence in* $B(X)$ *such that* $r(T_i)>1$ *for all* $i\geq 1$, *then there is a dense set* $D\subset X$ *such that* $\mathrm{Orb}(T_i,x)$ *tends to infinity for every* $x\in D$ *and* $i\geq 1$. *If, in addition, the operators are mutually commuting and have at least one of the properties (P.1) and (P.2) in Lemma 2.1, then for every* $n\geq 1$ *and every positive integers* $i_1<i_2<\ldots<i_n$ *the n-tuple orbit* $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^{n},x)$ *tends to infinity for every* $x\in D$.

Proof. Let $i\in\mathbb{N}$. Since $r(T_i)>1$ there is $\lambda_i\in\sigma(T_i)$ so that $|\lambda_i|>1$. By the Spectral Mapping Theorem, for every $n\in\mathbb{N}$, $\lambda_i^n\in\sigma(T_i^n)$ and hence,

$$|\lambda_i|^n\leq r(T_i^n)\leq\left\|T_i^n\right\|.$$

This would imply that

$$\sum_{n=1}^{\infty}\left\|T_i^n\right\|^{-1}\leq\sum_{n=1}^{\infty}|\lambda_i|^{-n}<\infty.$$

Now the conclusion follows from Theorem 2.5. □

Having in mind that every invertible operator is bounded bellow, we have the following corollary.

Corollary 2.7. *If* $(T_i)_{i\geq 1}$ *is a sequence of invertible, mutually commuting operators in* $B(X)$ *such that* $r(T_i)>1$, *for all* $i\geq 1$, *then there is a dense set* $D\subset X$ *so that* $\mathrm{Orb}(T_i,x)$ *such that for every* $n\geq 1$ *and every positive integers* $i_1<i_2<\ldots<i_n$ *the n-tuple orbit* $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^{n},x)$ *will tend to infinity for every* $x\in D$.

COMPETING INTERESTS

The authors declare that no competing interests exist.

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COMMON FIXED POINTS OF TWO T_f CHATTERJEA TYPE CONTRACTIONS IN A COMPLETE METRIC SPACE

UDC: 517.988.5:515.126.4

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Abstract. The focus in this paper are theorems about common fixed points for two T_f Chatterjea type contractions in a complete metric space (X,d). In doing so we defined T as continuous, injection and subsequentially convergent mapping, and f as a function of the class Θ, class of continuous monotony non-decreasing functions $f:[0,+\infty)\to[0,+\infty)$ such that $f^{-1}(0)=\{0\}$, and furthermore f is sub additive

1. INTRODUCTION

The Banach fixed-point theorem, as well as its generalizations presented by R. Kannan ([4]), S. K. Chatterjea ([7]) and P. V. Koparde, B. B. Waghmode ([3]), are well known. S. Moradi and D. Alimohammadi [9] generalized R. Kannan results using the sequentially convergent mappings. Using the sequentially convergent mappings, some generalizations of R. Kannan, S. K. Chatterjea and P. V. Koparde, B. B. Waghmode are proved [1]. The sequentially convergent mappings are defined as the following:

Definition 1 ([8]). Let (X,d) be a metric space. A mapping $T:X\to X$ is said sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ also is convergence. A mapping T is said sub sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ has a convergent subsequence.

Further, using sequentially convergent mappings are also proved several results about sharing fixed point for two R. Kannan, S. K. Chatterjea and P. V. Koparde, B. B. Waghmode types of mapping [5], 2006.

S. Moradi and A. Beiranvand [8], 2010, introduced the concept of T_f contractive mapping, applying the Θ class of continuous monotony non-decreasing functions $f:[0,+\infty)\to[0,+\infty)$ such that $f^{-1}(0)=\{0\}$, defined as the following:

2010 *Mathematics Subject Classification.* Primary: 47H10 Secondary: 54H25.
Key words and phrases. subsequentially convergent mapping, fixed point

Definition 2 ([8]). Let (X,d) be a metric space, $S,T:X\to X$ and $f\in\Theta$. A mapping S is said T_f – contraction if there exist $\alpha\in(0,1)$ such that
$$f(d(TSx,TSy))\leq\alpha f(d(Tx,Ty)),$$
for all $x,y\in X$.

Let us notice that if $f\in\Theta$ then $f^{-1}(0)=\{0\}$ implies that $f(t)>0$, for all $t>0$. S. Moradi and A. Beiranvand proved that if S is T_f contractive mapping, and then S has a unique fixed point. M. Kir и H. Kiziltunc, [2], 2014 generalized the S. Moradi and A. Beiranvand result about R. Kannan and S. K. Chatterjea types of mapping.

In our further observations we will present several results about sharing fixed points of two T_f Chatterjea type contractions in a complete metric space, such that the function f (f is a function of Θ class) we will additionally suppose that it is subadditive, i.e. $f(a+b)\leq f(a)+f(b)$, for all $a,b\in[0,+\infty)$.

2. MAINS RESULTS

Theorem 1. Let (X,d) be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ is such that $f(a+b)\leq f(a)+f(b)$, for all $a,b\in[0,+\infty)$ and the mapping $T:X\to X$ be continuous, injection and subsequentially convergent. If there exist $\alpha>0,\beta\geq0$ such that $2\alpha+\beta\in(0,1)$ and
$$f(d(TS_1x,TS_2y))\leq\alpha(f(d(Tx,TS_2y))+f(d(Ty,TS_1x)))+\beta f(d(Tx,Ty)) \quad (1)$$
for all $x,y\in X$, then S_1 and S_2 have a unique sharing fixed point.

Proof. Let x_0 be any point of X and let the sequence $\{x_n\}$ be defined as $x_{2n+1}=S_1x_{2n}$, $x_{2n+2}=S_2x_{2n+1}$, $n=0,1,2,3,...$.
If there exists $n\geq0$, such that $x_n=x_{n+1}=x_{n+2}$, then it is easy to be proven that $u=x_n$ is a sharing fixed point for S_1 and S_2. Therefore, let us assume that there no exist three consecutive equal terms of the sequence $\{x_n\}$. Then by applying the inequality (1), it is easy to prove the following inequalities:
$$f(d(Tx_{2n+1},Tx_{2n}))\leq\alpha(f(d(Tx_{2n-1},Tx_{2n}))+f(d(Tx_{2n},Tx_{2n+1})))+$$
$$+\beta f(d(Tx_{2n},Tx_{2n-1}))$$
and the above stated implies that for all $n=0,1,2,...$ and $\lambda=\frac{\alpha+\beta}{1-\alpha}\in(0,1)$ it holds that:
$$f(d(Tx_{n+1},Tx_n))\leq\lambda f(d(Tx_n,Tx_{n-1})). \quad (2)$$
Further, the inequality (2) implies
$$f(d(Tx_{n+1},Tx_n))\leq\lambda^n f(d(Tx_1,Tx_0)), \quad (3)$$

holds for all $n=0,1,2,\ldots$. Now, the metrics properties, the properties of the function f and the inequality (3) imply that for all $m,n\in\mathbb{N}$, $n>m$ holds the following

$$f(d(Tx_n,Tx_m))\le\tfrac{\lambda^m}{1-\lambda}f(d(Tx_1,Tx_0)).$$

According to that, the sequence $\{Tx_n\}$ is Caushy, and since (X,d) is a complete metric space, the sequence is convergent. Further, the mapping $T:X\to X$ is a subsequentially convergent, therefore the sequence $\{x_n\}$ consists a convergent subsequence $\{x_{n(k)}\}$, i.e. it exists $u\in X$ such that $\lim_{k\to\infty}x_{n(k)}=u$. Now, the continuity of T implies $\lim_{k\to\infty}Tx_{n(k)}=Tu$. But, $\{Tx_{n(k)}\}$ is a subsequence of the convergent sequence $\{Tx_n\}$, therefore

$$\lim_{n\to\infty}Tx_n=\lim_{k\to\infty}Tx_{n(k)}=Tu.$$

We will prove that $u\in X$ is a fixed point for the mapping S_1. So:

$$\begin{aligned}f(d(Tu,TS_1u))&\le f(d(Tu,Tx_{2n+2}))+f(d(Tx_{2n+2},TS_1u))\\&=f(d(Tu,Tx_{2n+2}))+f(d(TS_2x_{2n+1},TS_1u))\\&\le f(d(Tu,Tx_{2n+2}))+\alpha(f(Tx_{2n+1},TS_1u)+f(d(Tu,TS_2x_{2n+1})))+\\&\quad+\beta f(d(Tu,Tx_{2n+1}))\\&=f(d(Tu,Tx_{2n+2}))+\alpha(f(Tx_{2n+1},TS_1u)+f(d(Tu,Tx_{2n+2})))+\\&\quad+\beta f(d(Tu,Tx_{2n+1}))\end{aligned}$$

The mappings f and T are continuous, and therefore the metric space properties imply that for $n\to\infty$, the last inequality is transformed as

$$f(d(Tu,TS_1u))\le\alpha f(Tu,TS_1u)+(1+\alpha+\beta)f(0)$$

But, $1-\alpha>0$ and $f^{-1}(0)=\{0\}$, therefore the last inequality implies that $d(Tu,TS_1u)=0$, that is $TS_1u=Tu$. Finally, T is injection, and therefore $S_1u=u$, that u is fixed point for the mapping S_1. Analogously, u is also fixed point for the mapping S_2, i.e. u is common fixed point for the mapping S_1 and S_2.

Next, we will prove that S_1 and S_2 have a unique common fixed point. Let $v\in X$ be a fixed point for S_2, i.e. $S_2v=v$. Then

$$\begin{aligned}f(d(Tu,Tv))&=f(d(TS_1u,TS_2v)\,|)\\&\le\alpha(f(d(Tu,TS_2v))+f(d(Tv,TS_1u)))+\beta f(d(Tu,Tv)\\&=(2\alpha+\beta)f(d(Tu,Tv)).\end{aligned}$$

Now, $2\alpha+\beta<1$, therefore the last inequality implies $d(Tu,Tv)=0$, i.e. it holds $Tu=Tv$. But, T is injection, and therefore $u=v$, that is S_1 and S_2 have a unique common fixed point. ■

Corollary 1. Let (X,d) be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ is such that $f(a+b)\le f(a)+f(b)$, for all $a,b\in[0,+\infty)$ and the mapping $T:X\to X$ be continuous, injection and subsequentially convergent. If it exists $\lambda\in(0,1)$ such that

$$f(d(TS_1x,TS_2y))\le\lambda\sqrt[3]{f(d(Tx,TS_2y))\cdot f(d(Ty,TS_1x))\cdot f(d(Tx,Ty)}$$

for all $x,y\in X$, then S_1 and S_2 have a unique sharing fixed point.

Proof. The arithmetic-geometric mean inequality implies that

$$f(d(TS_1x,TS_2y))\le\tfrac{\lambda}{3}(f(d(Tx,TS_1x))+f(d(Ty,TS_2y))+f(d(Tx,Ty)))\,,$$

for all $x,y\in X$. Applying the Theorem 1 for $\alpha=\beta=\frac{\lambda}{3}$ we get the above corollary. ■

Corollary 2. Let (X,d) be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ be such that $f(a+b)\le f(a)+f(b)$,for all $a,b\in[0,+\infty)$ and the mapping $T:X\to X$ be continuous, injection and subsequentially convergent. If there exist $\alpha>0,\beta\ge0$ such that $2\alpha+\beta\in(0,1)$ and

$$f(d(TS_1x,TS_2y))\le\alpha\frac{f^2(d(Tx,TS_2y))+f^2(d(Ty,TS_1x))}{f(d(Tx,TS_2y))+f(d(Ty,TS_1x))}+\beta f(d(Tx,Ty)\,,$$

for all $x,y\in X$, then S_1 and S_2 have a unique sharing fixed point.

Proof. The inequality (1) is a direct implication of the given inequality. ■

Corollary 3. Let (X,d) be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ be such that $f(a+b)\le f(a)+f(b)$,for all $a,b\in[0,+\infty)$ and the mapping $T:X\to X$ be continuous, injection and subsequentially convergent. If it exists $\alpha\in(0,\frac{1}{2})$ such that

$$f(d(TS_1x,TS_2y))\le\alpha(f(d(Tx,TS_2y))+f(d(Ty,TS_1x)))\,,$$

for all $x,y\in X$, then S_1 and S_2 have a unique sharing fixed point.

Proof. The proof is directly implied by Theorem 1, for $\beta=0$. ■

Corollary 4. Let (X,d) be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ be such that $f(a+b)\le f(a)+f(b)$,for all $a,b\in[0,+\infty)$. If there exist $\alpha>0,\beta\ge0$ so that $2\alpha+\beta\in(0,1)$ and

$$f(d(S_1x,S_2y))\le\alpha(f(d(x,S_2y))+f(d(y,S_1x)))+\beta f(d(x,y))$$

for all $x,y\in X$, then S_1 and S_2 have a unique sharing fixed point.

Proof. The mapping $T:X\to X$ determined as $Tx=x$ is continuous, injection and subsequentially convergent.

Therefore, the proof is directly implied by Theorem 1, for $Tx=x$. ■

Corollary 5. Let (X,d) be a complete metric space $S_1,S_2 : X \to X$, $f \in \Theta$ be such that $f(a+b) \le f(a) + f(b)$, for all $a,b \in [0,+\infty)$. If it exists $\alpha \in (0,\frac{1}{2})$ so that

$$f(d(S_1x,S_2y)) \le \alpha(f(d(x,S_2y)) + f(d(y,S_1x))),$$

for all $x,y \in X$ it holds, then S_1 and S_2 have a unique sharing fixed point.

Proof. Direct implication from the Corollary 3 for $Tx = x$ or the Corollary 4 for $\beta = 0$. ■

Corollary 6. Let (X,d) be a complete metric space $S_1,S_2 : X \to X$, $f \in \Theta$ be such that $f(a+b) \le f(a) + f(b)$, for all $a,b \in [0,+\infty)$ and the mapping $T : X \to X$ be continuous, injection and subsequentially convergent. If there exist $p,q \in \mathbb{N}$ and $\alpha > 0, \beta \ge 0$ such that $2\alpha + \beta \in (0,1)$ and

$$f(d(TS_1^p x,TS_2^q y)) \le \alpha(f(d(Tx,TS_2^q y)) + f(d(Ty,TS_1^p x))) + \beta f(d(Tx,Ty))$$

for all $x,y \in X$, then S_1 and S_2 have a unique sharing fixed point.

Proof. The Theorem1 implies that the mappings S_1^p and S_2^q have a unique common fixed point $u \in X$. So, $S_1^p u = u$ and therefore

$$S_1u = S_1(S_1^p u) = S_1^p(S_1u),$$

that is S_1u is a fixed point for S_1^p . Analogously, $S_2^q u = u$ implies that

$$S_2u = S_2(S_2^q u) = S_2^q(S_2u),$$

that is S_2u a fixed point for S_2^q . But, the proof of the Theorem1 implies that S_1^p and S_2^q have unique fixed points. Therefore $S_1u = u$ and $S_2u = u$. According to this, $u \in X$ is a common fixed point for S_1 and S_2 .

For $v \in X$ is an arbitrary fixed point for S_1 and S_2 , we get that it is also a common fixed point for S_1^p and S_2^q . But the mappings S_1^p and S_2^q have a unique common fixed point, and therefore $v = u$. ■

Remark. The function $f : [0,+\infty) \to [0,+\infty)$ defined as $f(t) = t$, for each $t \in [0,1)$, it is a function of Θ class and it is a sub-additive. Moreover, each sequentially convergent mapping $T : X \to X$ is sub-sequentially convergent mapping. Therefore the Theorem2 and the Corollaries 7-12 [5], are directly implied by the above proved the Theorem1 and the Corollaries 1-6, respectively.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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CLASSIFICATIONS OF SYSTEMS OF LINEAR EQUATIONS BASED ON ITS GEOMETRICAL INTERPETATIONS

UDC: 512.644:514.124

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Abstract. In this paper, a classification of systems of $m\in\{1,2,3\}$ linear equations is given. Also, an idea for generalization of the classification for arbitrary number of equations is presented. These classes are described trough the geometric interpretations of the equations. It is proven that the classes are subclasses of the three classes determined by the generalized Cramer's rule.

1 CRAMER'S AND GEOMETRIC CLASSES

System of m linear equations with n unknowns ($m\times n$) over the set of the real numbers is the system of equations:

$$\begin{cases} a_{11}x_1+a_{12}x_2+...+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+...+a_{2n}x_n=b_2 \\ \quad ... \\ a_{m1}x_1+a_{m2}x_2+...+a_{mn}x_n=b_m \end{cases},$$

where m and n are positive integers, and a_{ij} and b_i are real numbers for $1\le i\le m$, $1\le j\le n$.

Let A be the matrix and $\overline{A}$ be the augmented matrix of the system $m\times n$, where $m\le n$. Let $D_{i_1i_2...i_m}$, where $i_1,i_2,...,i_m$ is a permutation without repetition of a class m of the set $I_{n+1}=\{1,2,...,n+1\}$, be a minor of $\overline{A}$ whose columns are matching with $i_1,i_2,...,i_m$ column of $\overline{A}$, respectively. Clearly, $D_{i_1i_2...i_m}\ne 0$ if and only if $D_{j_1j_2...j_m}\ne 0$, where $j_1,j_2,...,j_m$ is a permutation of $\{i_1,i_2,...,i_m\}$ such that $j_1<j_2<...<j_m$.

The generalized Cramer's rule [1], divides the systems $m\times n$, $m\le n$, into the following three classes:

1. There exists $D_{i_1i_2...i_m}\ne 0$ for some $i_1,i_2,...,i_m\in I_n$. Then the system has a unique solution for $m=n$, and infinite solutions for $m<n$, expressed through $n-m$ parameters.

Parameters can be taken to be x_k, $k\in I_n\setminus\{i_1,i_2,...,i_m\}$ and the solutions can be expressed explicitly by using the Cramer's formulas:

$$x_{i_1}=\frac{D_{x_{i_1}}}{D_{i_1i_2...i_m}},\ x_{i_2}=\frac{D_{x_{i_2}}}{D_{i_1i_2...i_m}},\ ...,\ x_{i_n}=\frac{D_{x_{i_n}}}{D_{i_1i_2...i_m}}.$$

2. $D_{i_1 i_2 \ldots i_m} = 0$ for every $i_1, i_2, ..., i_m \in I_n$ and $D_{i_1 i_2 \ldots i_{m-1}(n+1)} \neq 0$ for some $i_1, i_2, ..., i_{m-1} \in I_n$. In this case the system does not have solution.

3. $D_{i_1 i_2 \ldots i_m} = 0$ for every $i_1, i_2, ..., i_m \in I_{n+1}$. Then the system does not have solution or has infinitely many solutions, expressed through $n - r(A)$ parameters, where $r(A)$ is the rank of A and $n - r(A) \in \{n - m + 1, ..., n - 1, n\}$.

These classes will be called **Cramer classes**.

If $m > n$ the system is equivalent with the system in which the coefficients before the unknowns of $n+1, n+2, ..., m$-th column are zeros. In that system the first class is empty, and for $m - 1 > n$ the second class is again empty. In that cases there exist two or one Cramer (nonempty) class.

Linear combinations of m linear equations with n unknowns,

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1, \; a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2, \; ...,$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m,$$

with respect to the coefficients $\lambda_1, \lambda_2, ..., \lambda_n$, is the linear equation:

$$\lambda_1 (a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n - b_1) + \lambda_2 (a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n - b_2) + ...$$
$$+\lambda_n (a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n - b_m) = 0.$$

Definition 1.1. Zero linear equation is the equation in which all coefficients are zero. **Contradictory** linear equation is the equation in which all coefficients of the system (before unknowns) are zero, while the constant term is different from zero. Zero and contradictory linear equations will be called **singular**. **Regular** (nonsingular) linear equation is an equation in which at least one of the coefficients of the system is different than zero.

Thus, the zero linear equation is

$$0 \cdot x_1 + 0 \cdot x_2 + ... + 0 \cdot x_n = 0 \text{ i.e. } 0 = 0,$$

and contradictory linear equations are

$$0 \cdot x_1 + 0 \cdot x_2 + ... + 0 \cdot x_n = a, \; a \neq 0, \text{ i.e. } 0 = a, \; a \neq 0.$$

We will say that two equations are **contradictory to each other** (one is contradictory to the other) if they are regular and do not have a common solution. Two equations are equivalent if they have same solutions.

Definitions 1.2. One linear equation has rank 1 or **is in a general position** if it is a regular. m **linear equations are in a general position** if they consist a $m-1$ linear equations in a general position and the m-th equation is not equivalent or contradictory to any linear combination of the $m-1$ equations.

The system of m linear equations has a rank k if contains k linear equation in a general position, and do not contain $k+1$ linear equations in a general position. The system of m linear equations is **nonsingular** if it has a rank m.

Clearly $k \le \min\{m,n\}$, so for $m>n$ there are no m linear equations in a general position.

Theorem 1.3. Let be given m linear equations with n unknowns, $m \le n$. Then:

1. There exists $D_{i_1^0 i_2^0 \dots i_m^0} \ne 0$, for some $i_1^0, i_2^0, \dots, i_m^0 \in I_n$ if and only if the m-th equations are in a general position.

2. $D_{i_1 i_2 \dots i_m} = 0$ for every $i_1, i_2, \dots, i_m \in I_n$ and there exists $D_{i_1^0 i_2^0 \dots i_{m-1}^0 (n+1)} \ne 0$ for some $i_1^0, i_2^0, \dots, i_{m-1}^0 \in I_n$ if and only if there exist a $m-1$ equations in a general position, and the m-th equation is a contradictory to some linear combination of the other equations.

3. $D_{i_1 i_2 \dots i_m} = 0$ for every $i_1, i_2, \dots, i_m \in I_{n+1}$ if and only if there exist a $m-1$ equations in a general position, and the m-th equation is an equivalent to some linear combination of the other equations, or do not exist a $m-1$ equations in a general position.

Proof. One linear equation is a regular if and only if there exists $D_{i^0} \ne 0$, for some $i^0 \in I_n$. Let $m-1$ linear equations are in a general position if and only if $D_{i_1^0 i_2^0 \dots i_{m-1}^0} \ne 0$, for some $i_1^0, i_2^0, \dots, i_m^0 \in I_n$.

Let be given m linear equations with n unknowns and let the first $m-1$ linear equations are in a general positions. It follows that $D_{i_1^0 i_2^0 \dots i_{m-1}^0} \ne 0$, for some $i_1^0, i_2^0, \dots, i_m^0 \in I_n$.

Since $D_{i_1^0 i_2^0 \dots i_{m-1}^0} \ne 0$, the system

$$\begin{cases} \alpha_1 a_{1 i_1^0} + \alpha_2 a_{2 i_1^0} + \dots + \alpha_{m-1} a_{m-1, i_1^0} = a_{m i_1^0} \\ \alpha_1 a_{1 i_2^0} + \alpha_2 a_{2 i_2^0} + \dots + \alpha_{m-1} a_{m-1, i_2^0} = a_{m i_2^0} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha_1 a_{1 i_{m-1}^0} + \alpha_2 a_{2 i_{m-1}^0} + \dots + \alpha_{m-1} a_{m-1, i_{m-1}^0} = a_{m i_m^0} \end{cases}$$

has a unique solution $\left(\alpha_1^0, \alpha_2^0, \dots, \alpha_{m-1}^0\right)$.

If

$$a_{mk} = \alpha_1^0 a_{1k} + \alpha_2^0 a_{2k} + \dots + \alpha_{m-1}^0 a_{m-1,k} + \beta_k^0, \ k \in I_{n+1} \setminus \left\{ i_1^0, i_2^0, \dots, i_{m-1}^0 \right\},$$

then

$$D_{i_1^0 i_2^0 \dots i_{m-1}^0 k} = \begin{vmatrix} a_{1i_1^0} & a_{1i_2^0} & \dots & a_{1i_{m-1}^0} & a_{1k} \\ a_{2i_1^0} & a_{2i_2^0} & \dots & a_{2i_{m-1}^0} & a_{2k} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m-1,i_1^0} & a_{m-1,i_2^0} & \dots & a_{m-1,i_{m-1}^0} & a_{m-1,k} \\ \sum_{i=1}^{m-1} \alpha_i^0 a_{ii_1^0} & \sum_{i=1}^{m-1} \alpha_i^0 a_{ii_2^0} & \dots & \sum_{i=1}^{m-1} \alpha_i^0 a_{ii_{m-1}^0} & \sum_{i=1}^{m-1} \alpha_i^0 a_{ik} + \beta_k^0 \end{vmatrix} .$$

$$= \begin{vmatrix} a_{1i_1^0} & a_{1i_2^0} & \dots & a_{1i_{m-1}^0} & a_{1k} \\ a_{2i_1^0} & a_{2i_2^0} & \dots & a_{2i_{m-1}^0} & a_{2k} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m-1,i_1^0} & a_{m-1,i_2^0} & \dots & a_{m-1,i_{m-1}^0} & a_{m-1,k} \\ 0 & 0 & \dots & 0 & \beta_k^0 \end{vmatrix} = \beta_k^0 D_{i_1^0 i_2^0 \dots i_{m-1}^0}$$

It follows that

$$D_{i_1^0 i_2^0 \dots i_{m-1}^0 k} = 0 \text{ if and only if } \beta_k^0 = 0 \text{, for every } k \in I_{n+1} \setminus \{ i_1^0, i_2^0, \dots, i_{m-1}^0 \} .$$

From the last conclusion it follows that 1, 2 and the first statement of 3, are true. The second statement of 3 is also true, since we consider a cofactor expansion along the m-th row.■

According to the rank of the systems $m \times n$, $m \le n$, they form $m+1$ (nonempty) class, systems with rank $0,1,2,\dots,m$. The systems with rank m form one class matching with the first Cramer class (case 1), systems with rank $m-1$ are in the second and third class (case 2 and 3), and all other systems with rank less then $m-1$ are in the third class. For $m > n$ the systems form $n+1$ (nonempty) class, systems with rank $0,1,2,\dots,n$. If $m = n-1$ they belong to the second or the third Cramer class (the first class is empty), and if $m > n-1$ they belong only to the the third class. We will describe below a finer classification.

The systems with a rank 0, are divided into $m+1$ class, determined of the number of the zero equations l, $l \in \{0,1,\dots,m\}$. Then the number of the contradictory equations is $m-l$.

The systems with a rank 1, contain an equation with rank 1 (the other equations are regular and equivalent to it, contradictory to it, zero or contradictory), and are divided into classes determined by $k_1, k_2, \dots, k_s, l$ where $k_1, k_2, \dots, k_s$, $k_1 \ge k_2 \ge \dots \ge k_s$ are the numbers of the groups of maximally equivalent regular equations, and l is the number of zero equations (the number of contradictory equations is $m - k_1 - k_2 - \dots - k_s - l$).

Example 1.4. 1) The systems with 3 equations and rank 0, belong to 4 classes: $\underline{0}$, $\underline{1}$, $\underline{2}$ и $\underline{3}$.

The systems with 3 equations and rank 1, belong to

$$3+2\cdot 2+1\cdot 3=10 \text{ classes:}$$

$$1,1,1;\ 2,1;\ 3;\ 1,1,\underline{0};\ 1,1,\underline{1};\ 2,\underline{0};\ 2,\underline{1};\ 1,\underline{0};\ 1,\underline{1} \text{ and } 1,\underline{2}$$

(we write 1,1,1 for 1,1,1,$\underline{0}$ i.e. when the number of zero equations is 0).

2) The systems with 7 equations and rank 0, belong to 8 classes, and from rank 1 on:

$$11+11\cdot 2+7\cdot 3+5\cdot 4+3\cdot 5+2\cdot 6+1\cdot 7=108$$

classes.

The systems with rank 2 contain two linear equations in a general position Σ_1 and Σ_2. The other equations are: equivalent or contradictory to some linear combination Λ of the two equations that has 0, 1 or 2 nonzero coefficient i.e. the other equations are: zero or contradictory; equivalent or contradictory to one of the two equations; and equivalent or contradictory to a equations of arbitrary equation of a linear combination different of the two equations. If the system contains at least 4 regular equations, the other equations can be equivalent or contradictory to a same or different equation of: the both equations and an arbitrary equation of the linear combinations different of the two equations (Σ, Σ_1 and Σ_2, Σ and Λ_1, Λ_1 and Λ_2 ($\Sigma \in \{\Sigma_1,\Sigma_2\}$, $\Lambda_1,\Lambda_2 \in lk(\Sigma_1,\Sigma_2)\setminus \{\Sigma_1,\Sigma_2\}$, $\Lambda_1 \neq \Lambda_2$, where $lk(\Sigma_1,\Sigma_2)$ is the set of all linear combinations of Σ_1 and Σ_2). By combining the cases, a finer classification can be defined.

Then we can classified the systems with rank 3 and etc.

According to the previous criteria, the systems with rank $m-1$ are divided into $2m$ classes:

-The systems that belong to the case 2 are divided into m classes, depending on m-th equation, which is contradictory to the linear combinations with $0,1,...,m-1$ nonzero coefficients.

- The systems that belong to the first subcase of 3 are divided into m classes, depending on m-th equation, which is equivalent to the linear combinations with $0,1,...,m-1$ nonzero coefficients.

From the above discussion, it follows that these classes contain all the systems. In next section, using symbols, we will describe the classes of the systems with one, two or three linear equations.

The classification is based on geometric interpretations on the equations, and therefore the classes are called **geometric classes**.

2 GEOMETRIC INTERPRETATIONS OF SYSTEMS WITH ONE, TWO AND THREE LINEAR EQUATIONS

2.1. Geometric interpretation of the linear equations. The contradictory linear equation is interpreted by an empty set. For $n=1$, the zero equation geometricaly is interpreted by a line, while a regular $ax=b$, $a\neq 0$, by a point in line. For $n=2$, the zero equation geometrically is interpreted by a plane, while a regular $ax+by=c$, where $a\neq 0$ or $b\neq 0$ by a line in a plane. For $n=3$, the zero equation is interpreted by a space, and a regular $ax+by+cz=d$, where $a\neq 0$, $b\neq 0$ or $c\neq 0$, by a plane in a space.

The symbols for an empty set, a point, a line, a plane and a space are, respectively:

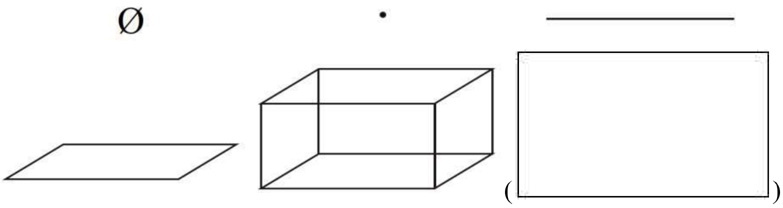
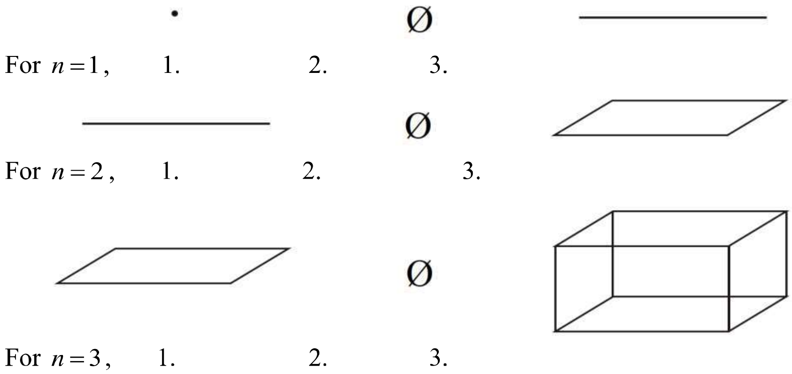


Figure 1

In Figure 2, geometric interpretations of the geometric classes are given.



1. there exists $D_{i_0}\neq 0$ for some $i_0\in I_n$;

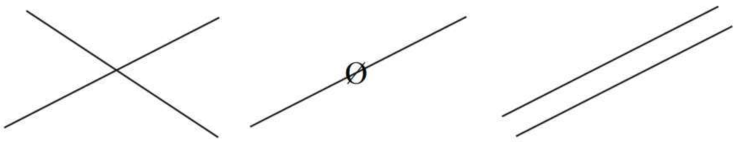
2. $D_i=0$ for every $i\in I_n$ and $D_{n+1}\neq 0$; 3. $D_i=0$ for every $i\in I_{n+1}$.

Figure 2

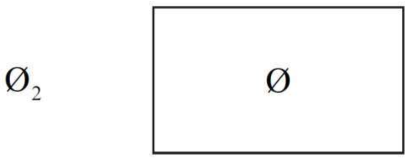
For $n>3$. The zero equation with n unknowns determines the n-dimensional space ($\mathbb{R}^n$), contradictory equation the empty set, while regular an $n-1$ dimensional plane. So, we can use the previous interpretation for $n=3$.

2.2. Geometric interpretation of systems of 2 linear equations. For the systems $m \times n$, $m,n \in \{2,3\}$, $m \le n$ the classification is given in the textbook [2]. In this paper we will generalize the classification for $m \in \{1,2,3\}$ and arbitrary $n \in \mathbb{N}$.

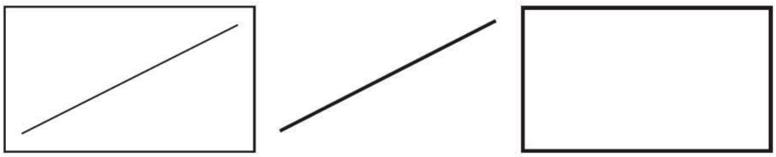
2×2. Two regular equations can be: in a general position, contradictory to each other, or equivalent. Their interpretations respectively are: two intersecting lines, two parallel lines or two matching lines. Therefore, there exist 8 class of the interpretations of the equations, given in Figure 3. Namely if the system has rank 0 then the two equations are zero (8), zero and contradictory (5), or contradictory (4). If the system has rank 1 and first of the equations is regular then the second equation is: equivalent to a linear combination of the first equation i.e. is zero (6) or regular and equivalent to the first equation (7); contradictory to linear combination of the first equation i.e. is contradictory (2) or regular and contradictory to the first equation (3). The last case is when the system has rank 2 or is nonsingular. In this case the equations are regular and have a unique common solution (1).



1. $D \ne 0$; 2-3. $D = 0 \wedge \left(D_x \ne 0 \vee D_y \ne 0\right)$;



4-5. do not have solution;

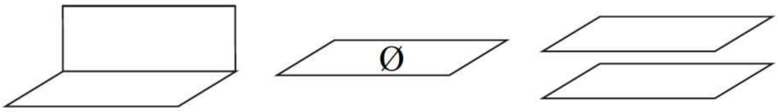


6-8. have infinite solutions expressed by: 6-7. one and 8. two parameters;

4-8. $D = D_x = D_y = 0$.

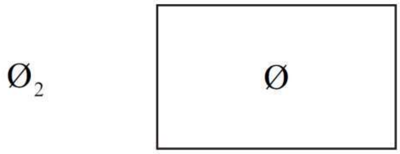
Figure 3

2×3. Two regular linear equations in general position, a contradictory to each other, or an equivalent are interpreted with two intersecting, parallel, or coincident planes, respectively. The classification is a restriction of the geometrical classification of the systems 2×3 that have 8 classes that are presented in Figure 4. For more visibility, the zero equation instead by a cuboid is presented by rectangular, a plane is presented by a parallelogram, and the lines are thickened in matching planes and spaces.

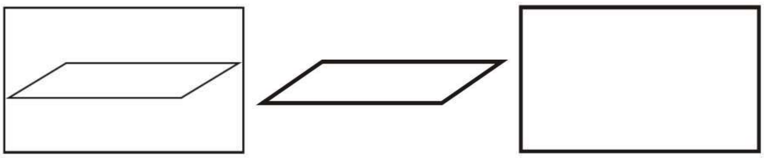


1. $D_{i_0 j_0} \neq 0$ for some $i_0, j_0 \in I_3$;

2-3. $D_{ij} = 0$ for every $i, j \in I_3$ and $D_{i_0 4} \neq 0$ for some $i_0 \in I_3$;



4-5. do not have solution;



6-8. have infinite solutions expressed by: 6-7 one and 8 three parameters;

4-8. $D_{ij} = 0$ for every $i, j \in I_4$, $i \neq j$.

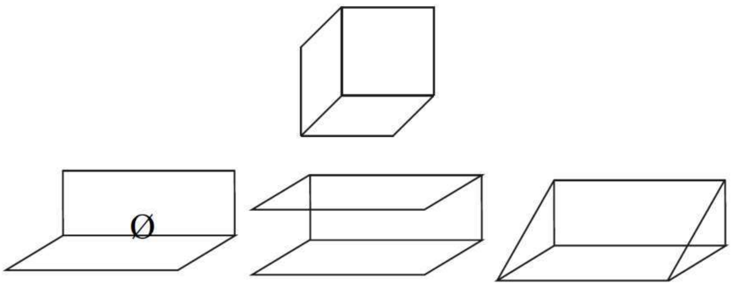
Figure 4

$2\times n$, $n > 3$. Two regular linear equations that are in a general position, contradictory to each other, or equivalent are interpreted with two intersecting $n-1$-planes, parallel planes or coincident planes, respectively. There exist 8 mutual positions of the interpretations of the equations, which can be sketched by using one of the previous interpretations, for example for $n = 3$.

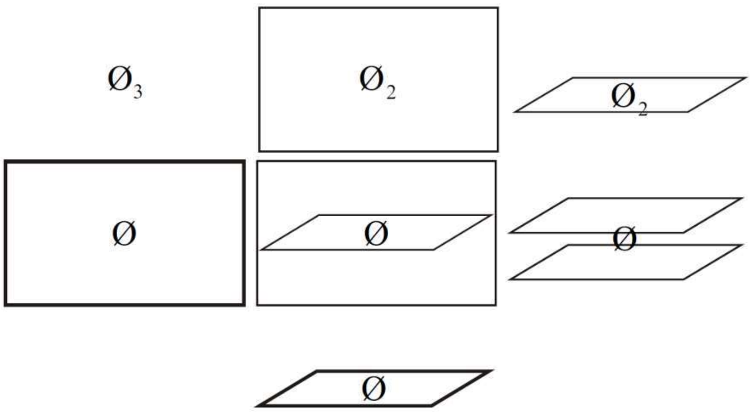
2×1. Two regular equations are contradictory to each other or equivalent are interpreted with two different points or two coinciding points. There exists 7 mutual positions of the interpretations of the equations (do not exist two equations in a general position, so the case 1 is not possible)

<u>2.3 Geometric interpretation of the system of 3 linear equations</u>. 3×3 Three equations in a general position define three planes intersecting at a point (Figure 5, 1). Two equations in a general position define two intersecting, in a line, planes. Their linear combination, depending on the number of non-zero coefficients, determine: a space (15), a plane that coincides with one of the planes (17) or a plane different from the planes passing trough its common line (16). The contradictory equation to they linear combinations determines an empty set (2), a plane parallel to one of the planes (3) or a plane that intersects the planes in a lines parallel to its common line (4). Let the rank of the system be 1. If the three equations are regular, then the three equations are mutually equivalent (20), only one pair (14) none (13). If exactly two equations are regular, then they are equivalent (11,19) or contradictory to each other (10,12). If exactly one equation is regular, depending of the number of zero equations, 3 case are possible (7,9,18). Let the rank of the system is 0. Then depending of the number of zero equations, 4 cases are possible (5,6,8,21).

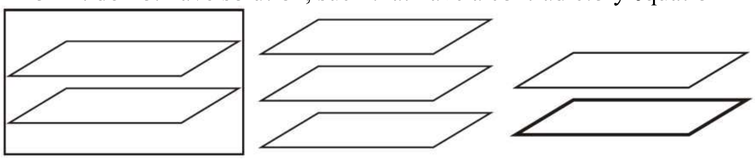
On Figure 5, the interpretations of all classes (a total 21), through symbols, sorted by Cramer's classes, and by the number of solutions of the systems are given. Of these, 8 classes are consisting of regular equations, determining 8 mutual positions of three planes in a space.



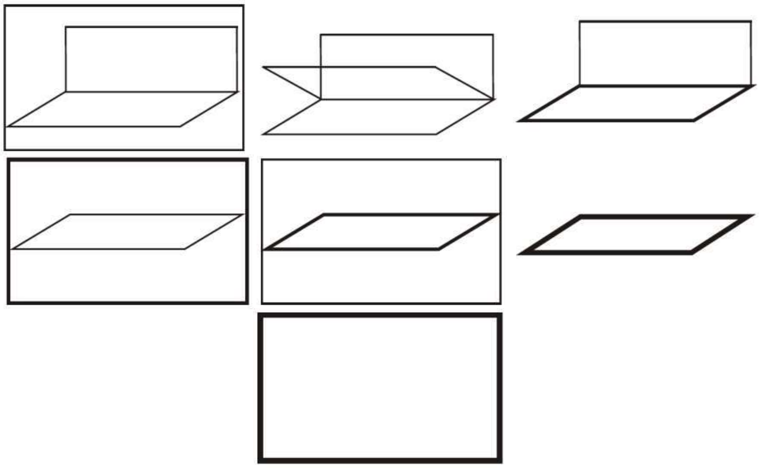
1. $D \neq 0$; 2-4. $D = 0 \wedge \left(D_x \neq 0 \vee D_y \neq 0 \vee D_z \neq 0 \right)$



5-11. do not have solution, such that have a contradictory equation



5-14. do not have solution



15-21. have infinite solutions expressed by: 15-17 one, 18-20 two, and 21 three parameters. 5-21. $D = D_x = D_y = D_z = 0$.

Figure 5

$3 \times n$, $n > 3$ For the systems $3 \times n$, $n > 3$ we can use interpretations for $n = 3$.

3×1, 3×2. If $n = 2$ there exist 20, while for $n = 1$, 14 geometric interpretations of the geometric classes, which also can be described by words or can be sketched. Compared to $n = 3$, for $n = 2$ there are not three equations in a general position (falls out in the case 1), and for $n = 1$ does not exist even two equations in a general position (falls out in the cases 1-4 and 15-17).

References

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