ARMAGANKA-Library Olympiads

Mathematical Olympiads

Macedonian Mathematical Olympiad 2016
Balkan Mathematical Olympiad 2016
European Girl's Mathematical Olympiad 2016
European Mathematical Cup 2015
Junior Macedonian Mathematical Olympiad 2016
Junior Balkan Mathematical Olympiad 2016
Mediterranean Mathematical Olympiad 2016

Aleksa Malcheski, Ph.D.
Slagjana Brsakoska, Ph.D.
Risto Malcheski, Ph.D.
Bojan Prangoski, Ph.D.
Daniel Velinov, Ph.D.
Sanja Atanasova, Ph. D
Pavel Dimovski, Ph.D.
Tomi Dimovski, M.Sc.
Vesna Andova, Ph.D
Samoil Malcheski, Ph. D
Methodi Glavche
Dimitar Treneski

ARMAGANKA-Union of mathematicians of Macedonia
Macedonia, Skopje 2016

President: Aleksa Malcheski
Publisher: Union of mathematicians of Macedonia-Armaganka
Adress: **ul. 2 br. 107A**
Vizbegovo, Butel, Skopje, Republic of Macedonia

CIP - Каталогизација во публикација
Национална и универзитетска библиотека "Св. Климент Охридски", Скопје

51(079.1)

MATHEMATICAL olympiads : Macedonian mathematical olympiads :
Balkan mathematical olympiads : European girl's mathematical olympiad
: European mathematical cup / Aleksa Malchevski ... [and others.]. -
Skopje : Macedonian mathematical society-Armaganka, 2016. - 64 стр. : ilustration.
; 24 см. - (#Библиотека #Олимпијади)

Authors: Aleksa Malcheski, Slagjana Brsakoska, Risto Malcheski, Bojan Prangoski, Daniel Velinov, Pavel Dimovski,Tomi Dimovski, Vesna Andova, Samoil Malcheski, Dimitar Treneski, Methodi Glavche

ISBN 978-9989-646-74-4 (кн.)
ISBN 978-9989-646-18-8 (библ.)

1. Malchevski, Aleksa [автор] [уредник]
а) Математика - Задачи од натпревари
COBISS.MK-ID 101387274

Foreword

This year in the Republic of Macedonia were held various competitions in the field of mathematics on all levels in primary and secondary school: school, municipality, regional, state competitions and Olympiads, as it is a tradition for many years in the past. Also, Macedonia was one of the participating countries on some world famous math competitions abroad.

On December 11, 2015, the European Mathematical Cup was held in the Faculty of Mechanical Engineering and the Faculty of Electrical Engineering and Information Technologies, Skopje. Students from all over the country were competing in two categories Junior and Senior.

In March 2016, a team of four girls was selected regarding their success on the previous competitions, who participated later in April 10-16, 2016 to the 5-th European Girls Mathematical Olympiad which was held in Bushteni, Romania.

On April 9, 2016 the 23-rd Macedonian Mathematical Olympiad, MMO 2016, was held in the Faculty of Electrical Engineering and Information Technologies, Skopje, for the students from secondary school. After all rigorous selection processes which raised from the complete system of the competitions in the Macedonia, the BMO team was formed, to participate on the 33-rd Balkan Mathematical Olympiad, BMO 2016, which was held in Tirana, Albania in May 5-10, 2016.

On May 6, 2016, the Mediterranean Mathematical Olympiad, MMC 2016, was held in the Faculty of Electrical Engineering and Information Technologies, Skopje. 50 students from all over the country, best in their categories, were competing.

On May 28, 2016 the 20-th Junior Macedonian Mathematical Olympiad, JMMO 2016, was held at FON University, Skopje on which the Macedonian team of the best 6 contestants under 15,5 years, was elected. They were participants in the 20-th Junior Balkan Mathematical Olympiad, JBMO 2016, which was held in June 24-29, 2016 in Slatina, Republic of Romania.

Then after the IMO team selection test on May 15, 2016, the IMO team was formed. This year the International Mathematical Olympiad, IMO 2016, will take place in Hong Kong, in July 06-16, 2016.

The content of this book is consisted of the mathematical competitions that already took place in Macedonia, the Balkan region and wider abroad, as well as their solutions.

57th IM

HONG KO

4^{th} EUROPEAN MATHEMATICAL CUP,

5^{th} December 2015 – 13^{th} December2015

Senior Category

Problem 1. $A=\{a,b,c\}$ is a set containing three positive integers. Prove that we can sind a set $B\subset A$, $B=\{x,y\}$ such that for all odd positive integers m,n we have

$$10\,|\,x^m y^n - x^n y^m$$

Problem 2. Let a,b,c be positive real numbers such that $abc=1$. Prove that

$$\frac{a+b+c+3}{4}\geq \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}.$$

Problem 3. Circles k_1 and k_2 intersect in points A and B, such that k_1 passes through the center O of the circle k_2. The line p intersects k_1 in points K and O and k_2 in points L and M, such that the point L is between K and O. The point P is orthogonal projection of the point L to the line AB. Prove that the line KP is parallel to the M-median of the triangle ABM.

Problem 4. A group of mathematicians is attending a conference. We say that a mathematician is k-content if he is in room with at least k people he admires or if he is admired by at least k other people in the room. It is known that when all participants are in a same room then they are all at least $3k+1$-content. Prove that you can assign everyone into one of 2 rooms in a way that everyone is at least k-content in his room and neither room is empty. *Admiration is not necessarily mutual and no one admires himself.*

Time alowed: 240 minutes **Each problem is worth** 10 points **Calculators are not** alowed

Solutions

Problem 1. $A=\{a,b,c\}$ is a set containing three positive integers. Prove that we can sind a set $B\subset A$, $B=\{x,y\}$ such that for all odd positive integers m,n we have

$$10\,|\,x^m y^n - x^n y^m$$

Solution. Let $f(x,y)=x^m y^n - x^n y^m$. If $n=m$, the problem statement will be fulfilled no matter how we choose B so from now on, without loss of generality, we consider $n>m$.

Since m and n are both odd, we have that $n-m$ is even and we get

$$f(x,y)=x^m y^m(y^{n-m}-x^{n-m})$$

$$f(x,y)=x^m y^m(y^2-x^2)Q(x,y)$$

$$f(x,y)=x^m y^m(y-x)(y+x)Q(x,y)$$

where $Q(x,y)=y^{n-m-2}+y^{n-m-4}x+....+x^{n-m-2}$.

Now if one of x,y is even, $f(x,y)$ is even. If both are odd, then $f(x,y)$ is again even since $x+y$ and $x-y$ are even in that case. This shows that we only need to consider divisibility by 5.

If A contains at least one element divisible by 5, we can put it in B and that will give us the solution easily.

Now we consider the case when none of the elements in A is divisible by 5. If some two numbers in A give the same remainder modulo 5, we can choose them and then $x-y$ will be divisible by 5 which solves the problem.

Now we consider the case when all remainders modulo 5 in A are different. Take a look at the pairs $(1,4)$ and $(2,3)$.

Since we have three different reamainders modulo 5, by pigeonhole principle one f these pairs has to be completely in A (when elements are considered modulo 5). Then if we pick the numbers from A that correspond to those two remaindes we get that $x+y$ is divisible by 5 so the problem statement is fulfilled again. This completes the proof.

Problem 2. Let a,b,c be positive real numbers such that $abc=1$. Prove that

$$\frac{a+b+c+3}{4}\geq\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}.$$

Solution 1. Rewrite the left hand side of inequality in following way:

$$\frac{a+b+c+3}{4}=\frac{a+b+c+3}{4\sqrt{abc}}=\frac{a+1}{4\sqrt{abc}}+\frac{b+1}{4\sqrt{abc}}+\frac{c+1}{4\sqrt{abc}}$$

Rewrite denominators:

$$\frac{a+1}{4\sqrt{abc}}+\frac{b+1}{4\sqrt{abc}}+\frac{c+1}{4\sqrt{abc}}=\frac{a+1}{2\sqrt{ab\cdot c}+2\sqrt{ac\cdot b}}+\frac{b+1}{2\sqrt{ba\cdot c}+2\sqrt{bc\cdot a}}+\frac{c+1}{2\sqrt{ca\cdot b}+2\sqrt{cb\cdot a}}=$$

and then by arithmetic mean-geometric mean inequality, we have

$$=\frac{a+1}{ab+c+ac+b}+\frac{b+1}{bc+a+ba+c}+\frac{c+1}{ca+b+cb+a}=\frac{a+1}{(a+1)(b+c)}+\frac{b+1}{(b+1)(a+c)}+\frac{c+1}{(c+1)(b+a)}=$$

$$=\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{b+a}=\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}.$$

Solution 2. We introduce change of variables: $x=a^3,b=y^3,c=z^3$. We now have the condition $xyz=1$.

We apply Schur inequality(with exponent $r=1$) to the numerator of the left hand side:

$$x^3+y^3+z^3+3xyz\geq x^2y+x^2z+y^2x+y^2z+z^2x+z^2y$$

to obtain inequality

$$\frac{x^2y+x^2z+y^2x+y^2z+z^2x+z^2y}{4}\geq\frac{1}{x^3+y^3}+\frac{1}{y^3+z^3}+\frac{1}{z^3+x^3}.$$

We apply arithmetic mean-geometric mean inequality for the denominators of the right hand side:

$$x^3+y^3\geq 2x^{3/2}y^{3/2}\qquad\Rightarrow\qquad\frac{1}{x^3+y^3}\leq\frac{1}{2x^{3/2}y^{3/2}}=\frac{1}{2}z^2\sqrt{xy}$$

and similarly to the other terms. We now have to prove

$$\frac{x^2y+x^2z+y^2x+y^2z+z^2x+z^2y}{4}\geq\frac{1}{2}x^2\sqrt{yz}+\frac{1}{2}y^2\sqrt{xz}+\frac{1}{2}z^2\sqrt{xy}$$

$$\frac{x^2y+x^2z+y^2x+y^2z+z^2x+z^2y}{2}\geq x^2\sqrt{yz}+y^2\sqrt{xz}+z^2\sqrt{xy}.$$

We apply arithmetic mean-geometric mean inequality in pairs on the left hand side

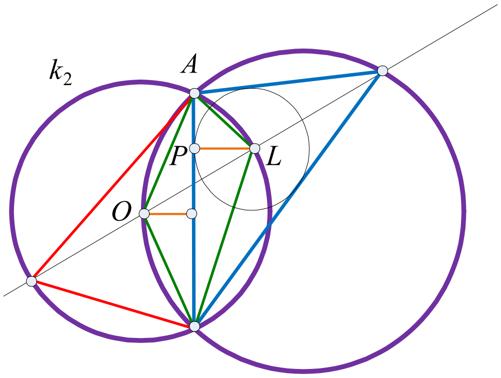
$$\frac{x^2y+x^2z}{2}\geq x^2\sqrt{yz}$$

$$\frac{y^2z+y^2x}{2}\geq y^2\sqrt{xz}$$

$$\frac{z^2x+z^2y}{2}\geq z^2\sqrt{xy}\,.$$

Summing up ineqailities from above finishes the proof.

Problem 3. Circles k_1 and k_2 intersect in points A and B, such that k_1 passes through the center O of the circle k_2. The line p intersects k_1 in points K and O and k_2 in points L and M, such that the point L is between K and O. The point P is orthogonal projection of the point L to the line AB. Prove that the line KP is parallel to the M-median of the triangle ABM.



Solution. Let the point C be the midpoint of the line segment AB. We have to prove $MC \| KP$.

Let us introduc angle $\alpha=\angle BKA$. Notice that

$$\angle BLA=180^\circ-\angle BMA=180^\circ-\frac{1}{2}\angle BOA=$$
$$=180^\circ-\frac{1}{2}(180^\circ-\angle BKA)=90^\circ+\frac{1}{2}\alpha\,.$$

Also, notice that the point O is midpoint of the arc $\widehat{AB}$. Thus the line KO is bisector of the angle $\angle BKA$. From the two calims above, we deduce that L is incenter of the triangle ABK.

Moreover, notice that ML is diameter of the circle k_2, thus $\angle ABM=90^\circ$. Since BL is angle bisector of the angle $\angle ABK$ we deduce that BM is exterior angle bisector of the same angle.

Thus, since M lies on angle bisector KM and exterior angle bisector BM, M is the center of the excircle for the triangle ABK.

Thus, we have to prove that the line passing though the incenter L of the triangle ABK and point of the tangency of incircle of the same triangle is parallel to the line passing through the center of the excircle M and the midpoint C of the line segment AB. This is a well known lemma, which completes the proof.

Problem 4. A group of mathematicians is attending a conference. We say that a mathematician is k-content if he is in room with at least k people he admires or if he is admired by at least k other people in the room. It is known that when all participants are in a same room then they are all at least $3k+1$-content. Prove that you can assign everyone into one of 2 rooms in a way that everyone is at least k-content in his room and neither room is empty. *Admiration is not necessarily mutual and no one admires himself.*

Solution. We will for simplicity and clarity of presentation use some basic graph theoretic terms, this is in no way essential.

We represent the situation by a directed graph (abbr. digraph) $G(V,E)$ where each vertex $v\in V(G)$ represents a mathematician and each edge $e\in E(G)$ represents an admiration relation. Given $v\in V(G)$ we define out-degree of v denoted $o(v)$ as the number of edges starting in v (so the number of mathematicians v admires) and in-degree $i(v)$ as the number of edges ending in v (so the number of mathematicians who admire v). Given $X\subseteq V$ by $G(X)$ we denote the induced subgraph (a graph with vertex set X and edges inherited from G). We say that a digraph is a k-digraph if for every $v\in V(G)$ we have $i(v)\geq k$ or $o(v)\geq k$.

So the question can be reformulated as: Given G is a $3k+1$-digraph we can split its vertices into 2 vertex disjoint classes such that each induced subgraph on class is a k digraph.

We call a subset X of vertices of G $k-tight$ if for any $Y \subseteq X$ we have a vertex $v \in Y$ such that $i_{G(Y)}(v) \leq k$ and $o_{G(Y)}(v) \leq k$. A partition of V, (A_1, A_2) is feasible if A_1 is k-tight and A_2 is k-tight.

We first assume there are no feasible partitions.

In this case consider a minimal size subset $A_1 \subseteq V(G)$ subject to $G(A_1)$ being a k-digraph, we define $A_2 \equiv V(G) - A_1$. Given a subset $X \subset A_1$, $G(X)$ is not a k-digraph so there is a vertex $v \in X$ such that $o_{G(X)}(v) < k$ and $i_{G(X)}(v) < k$ which shows that any proper subset of A_1 satisfies the condition of k-tightness. For the case of $X \equiv A_1$ by removing any vertex $v \in A_1$ the graph $G' \equiv G(A_1 - \{v\})$, by minimality assumption on A_1, must contain a vertex w such that $o_{G'}(w) < k$ and $i_{G'}(w) < k$ so as there is only one extra vertex in $G(A_1)$, namely v $o_{G(A_1)}(w) \leq k$, $i_{G(A_1)}(w) \leq k$. In particular this shows A_1 is k-tight.

This implies A_2 is not k-tight by our assumption so there exists an $A_2' \subseteq A_2$ such that A_2' is a $k+1$ digraph. Now applying the following proposition to extend the pair (A_1, A_2') to a full partition which satisfies the condition of the problem.

Given disjoint subsets $A, B \subseteq V(G)$ we say (A, B) is a solution pair if both $G(A)$ and $G(B)$ are k-digraphs.

Proposition. If a $2k+1$ digraph G admits a solution pair it admits a partition with both induced graphs of both classes being k-digraphs.

Proof. Take a maximal solution pair (A, B), the condition in the lemma guaranteeing it exists. Let $C = V(G) - (A \cup B)$, if C is empty we are done so assume $|C| > 0$. By our assumption $(A, B \cup C)$ is not a solution pair so there is some $x \in C$ such that $o_{G(B \cup C)}(x), i_{G(B \cup C)}(x) < k$ so as G is $2k+1$ digraph $i_G(x) \geq 2k+1$ or $o_G(x) \geq 2k+1$ so either $o_{G(A \cup \{x\})}(x) > k+1$ or $i_{G(A \cup \{x\})}(x) > k+1$ so in particular $(A \cup \{x\}, B)$ is a solution pair contradicting maximality and completing our argument.

Hence we are left with the case in which we have at least one feasible partition. We pick the feasible partition (A, B) maximizing $w(A < B) = |E(G(A))| + |E(G(B))|$. The fact that A is k-tight implies there is an x with $o_{G(A)}(x) \leq k$, $i_{G(A)}(x) \leq k$ so x needs to have at least $k+1$ edges in or out of B so $|B| \geq k+1$ and by simmetry $|A| \geq k+1$.

We now prove that there exist an $X \subseteq A$ such that $G(X)$ is a k-digraph, by contradiction. Assuming the opposite we notice that for any $x \in B$, $B - \{x\}$ is still k-tight while B being k-tight implies there is an $x \in B$ such that $o_{G(B)}(x) \leq k$, $o_{G(B)}(x) \leq k$ so for this x we have $A \cup \{x\}$ is also k-tight. Hence, for $A' = A \cup \{x\}$ and $B' = B - \{x\}$, (A', B') is a feasible partition. We considering the change in edges which moving x causes we have $w(A', B') - w(A, B) \geq 3k+1-k-k-k = 1$ as we know $i_{G(X)} \geq 3k+1$ or $o_{G(X)} \geq 3k+1$ so moving x from B to A increases number of edges in A by at least $3k+1-k$ while the choice of x in B means we lose at most $k+k$ edges in B. This is a contradiction to maximality of (A, B).

Analogously we can find $Y \subseteq B$ with $G(Y)$ a k-digraph. Now applying the above proposition yet again we are done.

Remark. The same argument slightly modified weight function can be used to show the result for non symmetric rooms, in particular if the graph is a $k+l+\max(k,l)+1$ digraph it can be partitioned into k-digraph and l digraph parts.

4^{th} EUROPEAN MATHEMATICAL CUP,

5^{th} December 2015 – 13^{th} December2015

MLADI NADARENI MATEMATIČARI Marin Getaldić

Junior Category

Problem 1. We are given an $n\times n$ board. Rows are labeled with numbers 1 to n downwards and columns are labeled with numbers 1 to n from left to right. On each field we write the number x^2+y^2 where (x,y) are its coordinates. We are given a figure and can initally place it on any field. In every step we can move the figure from one field to another if the other field has not already been visited and if at least one of the following conditions is satisfied:

- the numbers in those 2 fields give the same remainders when divided by n
- those fields are point reflected with respect to the center of the board

Can all the fields be visited in case:

a) $n=4$

b) $n=5$

Problem 2. Let m,n,p be fixed positive real numbers which satisfy $mnp=8$. Depending on these constants, find the minimum of

$$x^2+y^2+z^2+mxy+nxz+pyz$$

where x,y,z are arbitrary positive real numbers satisfying $xyz=8$. When is the equality attained? Solve the problem for:

a) $m=n=p=2$

b) arbitrary (but fixed) positive real numbers m,n,p.

Problem 3. Let $d(n)$ denote the number of positive divisors of n. For positive integer n we define $f(n)$ as

$$f(n)=d(k_1)+d(k_2)+d(k_3)+...+d(k_m),$$

where $1=k_1<k_2<...<k_m=n$ are all divisors of the number n. We call an integer $n>1$ almost perfect if $f(n)=n$. Find all almost perfect numbers.

Problem 4. Let ABC be an acute angled triangle. Let B',A' be points on the perpendicular bisectors of AC,BC respectively such that $B'A\perp AB$ and $AB'\perp AB$. Let P be a point on the segment AB and O the circumcenter of the triangle ABC. Let D,E be points on BC,AC respectively such that $DP\perp BO$ and $EP\perp AO$. Let O' be the circumcenter of the triangle CDE. Prove that B',A' and O' are collinear.

Time alowed: 240 minutes, **Each problem is worth** 10 points **Calculators are not** alowed

Solutions

Problem 1. We are given an $n\times n$ board. Rows are labeled with numbers 1 to n downwards and columns are labeled with numbers 1 to n from left to right. On each field we write the number x^2+y^2 where (x,y) are its coordinates. We are given a figure and can initally place it on any field. In every step we can move the figure from one field to another if the other field has not already been visited and if at least one of the following conditions is satisfied:

- the numbers in those 2 fields give the same remainders when divided by n
- those fields are point reflected with respect to the center of the board

Can all the fields be visited in case:

a) $n=4$

b) $n=5$

Solution.a) The answer is NO.

	1	2	3	4
1	2	5	10	17
2	5	8	13	20
3	10	13	18	25
4	17	20	25	36

	1	2	3	4
1	2	1	2	1
2	1	0	1	0
3	2	1	2	1
4	1	0	1	0

On the left we have the board from the problem, on the right we have the same board, but with remainders of the values from the board instead of the values themselves.

We will denote *field* i for a field with number i written on it in the right table. Let's assume that we can visit all of the fields. That means that at some point we will visit a *field* i. Obviously, when using the first type of move, we can visit any other *field* 1 which hasn't yet been visited. Also, it easy to notice, that for *field* 1, the reflection of that field is also a *field* 1. That means that both types of moves lead to another *field* 1. Also, in same fashion we conclude thatfor the each step, if the figure is on the *field* 1, then in the step after(if that wasn't the last one) and in the step before(if that wasn't the first one) should be *field* 1.

Now we conclude that the first visited field 1 must be the field visited in the first step. Same way we conclude that the last visited field 1 must be the field visited in the last step. But, we know that all of fields 1 are visited consecutively, in exactly 8 moves(because there are 8 fields 1), while there are exactly 16 moves that we have to make. This leads to contradiction.

b) The answer is YES.

	1	2	3	4	5
1	2	5	10	17	26
2	5	8	13	20	29
3	10	13	18	25	34
4	17	20	25	36	41
5	26	29	34	41	50

	1	2	3	4	5
1	2	0	0	0	1
2	0	3	3	0	4
3	0	3	3	0	4
4	2	0	0	2	1
5	1	4	4	1	0

Again, on the left we have the board from the problem, on the right we have the same board, but with remainders of the values from the board instead of the values themselves.

We can move from any field to another with the same number written n the field in the right table by using the second move.

One idea to visit all the fields is the following:

● find the 4 pairs of the fields of types field i and field j, such that all 8 fields are different, in each pair $i\neq j$, those two field in one pair are symmetric, and the second member of the n-th pair has the same value on the right board as the first member of the $(n+1)$-th pair. Also, we want that all the values of the right table are mentoined through members of those pairs. For example:

$((2,2),(4,4)),((1,4),(5,2)),((3,5),(3,1)),((2,1),(4,5))$

● Now, the algorithm is: after second member of n-th pair and before the first member of the $(n+1)$-th pair visit all fields by using the first step. Of course, before first pair and after fourth pair move in similar way. Jump from the first member of the pair to the second member of the pair by using second step.

This is one of the ways to do it: We start with the field $(3,3)$. Then we visit all of the *field* 3, using the first move, in any way as long as the last visited field is $(2,2)$. Then, using the second move, we wisit the field $(4,4)$. Again, using the first move we wisit all *fields* 2 in any way as long as the last visited field is $(1,4)$. Using the second move we visit the field $(5,2)$. Then, using the first move we visit all *fields* 4 in any way as long as the last visited field $(3,5)$. In same fashion, using the second move we visit the field $(4,5)$ using the second move. We conclude by visiting all ***fields*** 1 in any way.

Problem 2. Let m,n,p be fixed positive real numbers which satisfy $mnp=8$. Depending on these constants, find the minimum of $x^2+y^2+z^2+mxy+nxz+pyz$ where x,y,z are arbitrary positive real numbers satisfying $xyz=8$. When is the equality attained?

Solve the problem for:

c) $m=n=p=2$

d) arbitrary (but fixed) positive real numbers m,n,p.

Solution 1.a) Use AM-GM and $xyz=8$ to get

$$x^2+y^2+z^2+xy+xy+xz+xz+yz+yz\geq 9\sqrt[9]{x^6y^6z^6}=36\,.$$

We have equality for $x=y=z=2$.

b) Using $xyz=8$, we can transform the given expression:

$$x^2+y^2+z^2+mxy+nxz+pyz=x^2+\frac{8p}{x}+y^2+\frac{8n}{y}+z^2+\frac{8m}{z}\,.$$

Since all numbers are positive reals, we can apply AM-GM inequality to get:

$$x^2+\frac{8p}{x}=x^2+\frac{4p}{x}+\frac{4p}{x}\geq 6\sqrt[3]{p^2}\,.$$

When we apply the same procedure for x,y,z and sum the inequalities, we get:

$$x^2+y^2+z^2+mxy+nxz+pyz=x^2+\frac{8p}{x}+y^2+\frac{8n}{y}+z^2+\frac{8m}{z}\geq 6\sqrt[3]{2}(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2})\,.$$

In order to get equality, we must have equality in all above inequalities and that happens for

$$x=\sqrt[3]{4p},\qquad y=\sqrt[3]{4n},\qquad z=\sqrt[3]{4m}\,.$$

Desired minimum is therefore $6\sqrt[3]{2}(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2})$.

Solution 2. We only present solution for b) part here, marking sheme for a) part is the same as in first solution. We use weighted AM-GM:

$$x^2+y^2+z^2+mxy+nxz+pyz=\sqrt[3]{p^2}\frac{x^2}{\sqrt[3]{p^2}}+\sqrt[3]{n^2}\frac{y^2}{\sqrt[3]{n^2}}+\sqrt[3]{m^2}\frac{z^2}{\sqrt[3]{m^2}}+2\sqrt[3]{m^2}\frac{mxy}{2\sqrt[3]{m^2}}+2\sqrt[3]{n^2}\frac{nxz}{2\sqrt[3]{m^2}}+2\sqrt[3]{p^2}\frac{pyz}{2\sqrt[3]{p^2}}\geq$$

$$=3(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2})\cdot\sqrt[3(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2})]{\left(\frac{x^2}{\sqrt[3]{p^2}}\right)^{\sqrt[3]{p^2}}\left(\frac{y^2}{\sqrt[3]{n^2}}\right)^{\sqrt[3]{n^2}}\left(\frac{z^2}{\sqrt[3]{m^2}}\right)^{\sqrt[3]{m^2}}}\,.$$

$$\sqrt[3(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2})]{\left(\frac{\sqrt[3]{mxy}}{2}\right)^{2\sqrt[3]{m^2}}\left(\frac{\sqrt[3]{nxz}}{2}\right)^{2\sqrt[3]{n^2}}\left(\frac{\sqrt[3]{pyz}}{2}\right)^{2\sqrt[3]{p^2}}}=$$

$$=3(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2})\sqrt[3(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2})]{\left(\frac{xyz}{2}\right)^{2(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2})}}=$$

$$=3(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2})\sqrt[3]{\left(\frac{xyz}{2}\right)^2}=$$

$$=3(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2})\sqrt[3]{4^2}=6\sqrt[3]{2}(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2})$$

We have shown that the minimum value the expression can take is $6\sqrt[3]{2}(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2})$. Equality can only be achieved when $x=\sqrt[3]{4p}, y=\sqrt[3]{4n}, z=\sqrt[3]{4m}$.

Problem 3. Let $d(n)$ denote the number of positive divisors of n. For positive integer n we define $f(n)$ as

$$f(n)=d(k_1)+d(k_2)+d(k_3)+...+d(k_m),$$

where $1=k_1<k_2<...<k_m=n$ are all divisors of the number n. We call an integer $n>1$ almost perfect if $f(n)=n$. Find all almost perfect numbers.

Solution 1. Alternative way to define $f(n)$ is

$$f(n)=\sum_{k|n,k\geq 1} d(k).$$

Let $n=p_1^{\alpha_1}p_2^{\alpha_2}...p_r^{\alpha_r}$ be the prime factorization of n. We have $d(n)=\prod_{i=1}^{r}(\alpha_i+1)$.

We prove the function f is multiplicative, inparticular, given coprime n,m we have $f(mn)=f(m)f(n)$.

Using n,m are coprime for the second ineqiality and the fact that function d is multiplicative we get:

$$f(mn)=\sum_{k|mn} d(k)=\sum_{k_1|n,k_2|m} d(k_1k_2)=\sum_{k_1|n,k_2|m} d(k_1)d(k_2)=\left(\sum_{k_1|n} d(k_1)\right)\left(\sum_{k_2|m} d(k_2)\right)=f(n)f(m).$$

If $r=1$ we have $n=p_1^{\alpha_1}$. We note that divisors of n are $1,p_1,p_1^2,....,p_1^{\alpha_1}$, so

$$f(n)=\sum_{i=0}^{\alpha_1}(i+1)=\frac{(\alpha_1+1)(\alpha_1+2)}{2}.$$

Combining this with the multiplicativity result for f we deduce $f(n)=\prod_{i=1}^{n}\frac{(\alpha_i+1)(\alpha_i+2)}{2}$.

We now prove that for primes $p\geq 5$ and $p=3$ provided $a\geq 3$ we have $f(p^a)=\frac{(a+1)(a+2)}{2}<\frac{2}{3}p^a$ by induction on a. As a basis $3<\frac{2p}{3}$ for $p\geq 5$ and $6<\frac{2}{3}\cdot 3^3$. For the step it is enough to notice that $\frac{a+3}{a+1}\leq 2<p$ in both cases.

Similarly we can prove for $p=2$ that $f(p^a)<p^a$ provided $a\geq 4$. By explicitly checking the remainig cases $p=2$ and $a=1,2,3$ and $p=3,a=1,2$ we conclude $f(p^a)\leq\frac{2}{3}p^a$ for all p,a and $f(p^a)\leq p^a$ for all $p\geq 3$ and $p=2,a\geq 4$.

Assuming $f(n)=n$ we would have $\prod_{i=1}^{k}\frac{f(p_i^{\alpha_i})}{p^{\alpha_i}}=1$ so the above considerations imply that only possible prime divisors are $2,3$. If $k=1$ the only possible solution is $n=3$. If $k=2$ we have $p_1=2,p_2=3$ and $1\leq a_1\leq 2$ and $1\leq a_2\leq 2$ which give 4 cases to check giving the other 2 solutions $n=18,36$.

So, all almost perfect numbers are $3,18,36$.

Solution 2. We hereby present one similar but different solution which does not use a lot of properties of the function f.

Firstly, we will prove the following lemma:

Lemma. For any positive integer $n>1$ and prime p we have

$f(pn) \leq 3f(n)$.

The equality holds if and only if $GCD(p,n)=1$.

Proof. For every integer m we have that the set of divisors of the number pm is the union of the following two sets:

- set of divisors of m
- set of divisors of m multiplied by p.

Also those two mentioned sets are disjoint if and only if $GCD(p,m)=1$ (if we have that p,m are disjoint, then it is obvious that none of the divisors of pm are in both sets; if they are not coprime, then the number p belongs to both sets).

This is why we have $d(pm) \leq 2d(m)$ and

$$f(pn)=\sum_{k|pn} d(k) \leq \sum_{k|n} d(k)+\sum_{k|n} d(pk) \leq f(n)+\sum_{k|n} 2d(k)=3f(n).$$

In both inequalities equality holds if and only if sets from before are disjoint, i.e. when $GCD(p,n)=1$.

Also, we simply see that $f(2^k)=d(1)+d(2)+...+d(2^k)=1+2+3+...+(k+1)=\frac{(k+1)(k+2)}{2}$.

Notice that if for some positive integer n we have $f(n)<n$, then for every $p \geq 3$ we have $f(pn) \leq 3f(n) \leq pf(n)<pn$. Consequently, if $f(n)<n$, then for every odd m we have $f(mn)<mn$.

Because of this, we will introduce new terms. Number n is *nice multiple* of m if $m|n$ and $\frac{m}{n}$ is odd number. Analogously, we define *nice divisor*. Our statement from above is: if for some n we have $f(n)<n$, then neither of its nice multiplies is almost perfect number.

Our strategy will be the following: check the cases of the small numbers and see ratio of numbers n and $f(n)$. When we have that $n>f(n)$, conclude that there are not almost perfect numbers among their nice multiplies. With formula for $f(2^k)$ conclude that for sufficiently big k (when $f(2^k)<2^k$ this is enough to conclude that there are no more almost perfect numbers.

By induction, it is simple to prove that $f(2^k)<2^k$ for $k \geq 4$. Thus, there are no almost perfect numbers of the form $2^k \cdot m$, where $k \geq 4$ and m is odd, since they all have 2^k as their nice divisor. We only have to check the numbers of the form $2^k \cdot m$, where $k \leq 3$ and m is odd.

First case: $k=0$

For any odd prime p we have $f(p)=d(1)+d(p)=3 \leq p$. From that we see that $n=3$ is solution. Moreover, we do not have any more solutions: if some odd number has a prime divisor different from 3, since $f(p)<p$ this number can not be almost perfect number; if it is a power of 3 bigger than 3, since $f(9)<3f(3)=9$, there are no more solutions as well (9 is nice divisor of every power of 3 bigger that 3).

Second case: $k=1$.

For any odd prime we have $f(2p)=3f(2)=9$. If $p>5$ thenwe have $2p>f(2p)$, so for all almost perfect numbers of the form $2^1 m$ number m has to have prime divisors 3 and/or 5.

We directly see that neither 6 or 10 is almost perfect. SO, in this case, almost perfect numberhas to a nice divisor of the form $2 \cdot 9, 2 \cdot 15$ or $2 \cdot 25$. For $n=18$ we have another solution, in other two cases we have inequality $f(n)<n$. If we want to seek new solution in this case, since they cannot be nice multiplies of 30 and 50, the only possibility is that almost perfect number has nice divisor $2 \cdot 27$. But we have(equality case in lemma) that $f(2 \cdot 27)<3f(2 \cdot 9)=2 \cdot 27$. So, there are no more solutions in this case.

Third case: $k=2$

For any odd prime we have $f(4p)=3f(4)=18$. If $p>5$ then we have $4p>f(4p)$, so for all almost perfect numbers of the form $2^1 \cdot m$ number m has to have prime divisors 3 and/or 5.

We directly see that neither 12 or 20 is almost perfect. So, in this case, almost perfect number has to have a nice divisor of the form $4\cdot 9, 4\cdot 15$ or $4\cdot 25$. For $n=36$ we have another solution, in other two cases we have inequality $f(n)<n$. If we want to seek new solution in this case, since they cannot be nice multiplies of 60 and 100, the only possibility is that almost perfect number has nice divisor $4\cdot 27$. But we have (equality case in lemma(, that $f(4\cdot 27)<3f(4\cdot 9)=4\cdot 27$.

So, there are no more solutions in this case.

Fourts case: $k=3$

For any odd prime we have $f(8p)=3f(8)=30$. Similarly to other cases, we only observe candidates of the form $8\cdot 3^l$. Number $8\cdot 3$ is not almost perfect, all other candidates have nice divisor $8\cdot 9$. But, we have $f(72)=60<72$. As we always concluded, we do not have any new solutions.

So allalmost perfect numbers are $3,18,36$.

Problem 4. Let ABC be an acute angled triangle. Let B',A' be points on the perpendicular bisectors of AC,BC respectively such that $B'A\perp AB$ and $AB'\perp AB$. Let P be a point on the segment AB and O the circumcenter of the triangle ABC. Let D,E be points on BC,AC respectively such that $DP\perp BO$ and $EP\perp AO$. Let O' be the circumcenter of the triangle CDE. Prove that B',A' and O' are collinear.

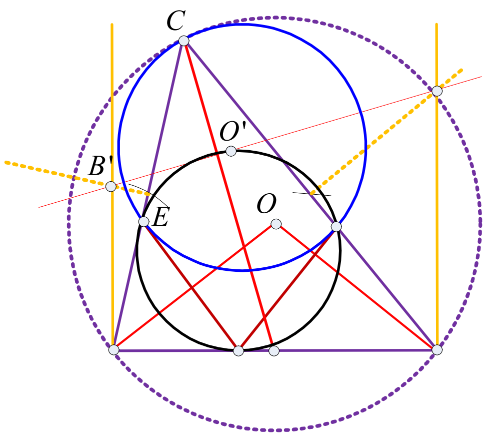
Solution. Remark. We first start by giving some intuition on how the problem canbe approached. We won't go into detail here but do give partial marks for correct ideas. We believe that any essential correct solution should have them in the background so we don't require them to be written down explicitly.

We notice that if $P\equiv A$ then $O'\equiv B'$ while if $P\equiv B$ we have $O'\equiv A'$. So the problem is equivalent to showing that as P varies on the segment AB respective O' map to a segment and we are now interested in identifying this segment.

It is hence natural to draw a picture not containing anything dependent on P and try to identify the line $A'B'$. Which turns out to be perpendicular to CM where M is the midpoint of AB.

Furthermore we note that $B'M^2-B'C^2=AM^2=A'M^2-A'C^2$ and this defines the line uniquely (and shows $A'B'\perp CM$).

The following sketch represents the problem setting when we do include the elements depending on P.



We now start with the formal proof.

It is enough to show that $O'M^2-O'C^2=AM^2$ for all P, including $P=A,B$ which allows us to draw the following sketch omitting B',C'.

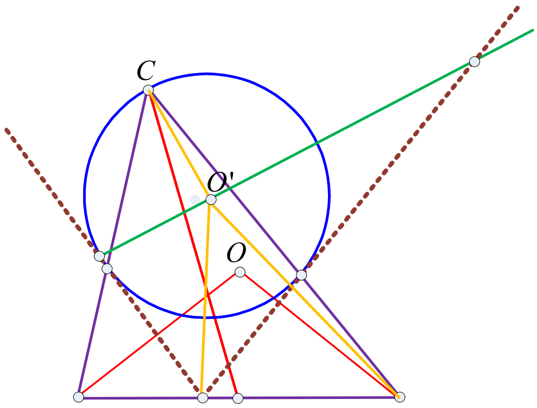
We first prove that $O'EPD$ is a cyclic quadrilateral. This follows as $EO'D=2ABC=APE+BPD=\pi-EPD$ as $ABC=APE=BPD$. This in turn implies PO' is an angle bisector of the angle EPD and $PO'\perp AB$.

We now have all the ingredients to show $O'M^2-O'C^2=AM^2$. The following sketch illustrates the last part of the proof.

We introduce the point D' as the second intersection of the line PE and the circumcircle of CDE so that $O'P^2-O'C^2=PE\cdot PD'$.

Now as PO' is the angle bisector of EPD we have $PD=PD'$ by the extended $S-S-K$ congruency theorem and the following observation. There is some care needed here, mainly the options we get by $S-S-K$ are $PD=PD'$ or $PD=PE$ but if $PD=PE$ triangles $P'EO'$ and $P'DO'$ are congruent by

$S-S-S$ congruency theorem so in particular $EO'P = DO'P = CAB$ while $EPO' = DPO' = \frac{\pi}{2} - CAB$ so PD and PE are tangents os in fact $D' \equiv E$ so the above claim is still true.



Now noticing triangles APE and BPD are similar we get $\frac{PE}{AP} = \frac{PB}{PD}$ implying $AP \cdot BP = PE \cdot PD = PE \cdot PD'$.

As $PO' \perp AB$ by using Pythagoras theorem we get

$$O'M^2 - O'C^2 - AM^2 =$$
$$= O'P^2 - O'C^2 + PM^2 - AM^2 = .$$
$$= PD' \cdot PE - AP \cdot PB = 0$$

Where we used $O'P^2 - -O'C^2 = PE \cdot PD'$ by the power of the point P to the circumcircle of CDE and

$$AM^2 - PM^2 = (BM - PM)(AM - PM) = AP \cdot PB.$$

This completes the proof.

European girl's Mathematical Olympiad
Bushteni, Romania, April 10.04-16.04.2016

Day 1
Tuesday, April 12 , 2016

Problem 1. Let n be an odd positive integer, and let $x_1,x_2,...,x_n$ be non-negative real numbers. Show that

$$\min_{i=1,2,...,n}(x_i^2+x_{i+1}^2)\leq \max_{j=1,2,...,n}\left(2x_jx_{j+1}\right),$$

where $x_{n+1}=x_1$.

Problem 2. Let $ABCD$ be a cyclic quadrilateral, and let diagonals AC and DC intersect at X. Let C_1,D_1 and M be the midpoint of segments CX,DX and CD rspectively. Lines AD_1 and BC_1 intersect at Y, and line MY intersect diagonals AC and BD at different points E and F, respectively. Prove that line XY is tangent to the circle through E,F and X.

Problem 3. Let m be a positive integer. Consider a $4m\times 4m$ array of square unit cells. Two different cells are *related* to each other if they are in either the same row or in the same column. No cell is related to itself. Some cells are coloured blue, such that every cell is related to at least two blue cells. Determine the minimum number of blue cells.

Solutions

Problem 1. Let n be an odd positive integer, and let $x_1,x_2,...,x_n$ be non-negative real numbers. Show that

$$\min_{i=1,2,...,n}(x_i^2+x_{i+1}^2)\leq \max_{k=1,2,...,n}\left(2x_kx_{k+1}\right),$$

where $x_{n+1}=x_1$.

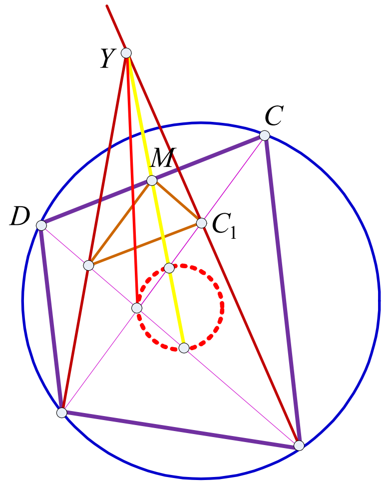
Solution. In what follows, indices are reduced modulo n. Consider the n differences $x_{k+1}-x_k$, $k=1,2,...,n$. Since n is odd, there exists an index j such that $(x_{j+1}-x_j)(x_{j+2}-x_{j+1})\geq 0$. Without loss of generality, we may and will assume both factors non-negative, so $x_j\leq x_{j+1}\leq x_{j+2}$. Consequently,

$$\min_{i=1,2,3,...,n}(x_i^2+x_{i+1}^2)\leq x_j^2+x_{j+1}^2\leq 2x_{j+1}^2\leq 2x_{j+1}x_{j+2}\leq \min_{k=1,2,3,...,n}2x_kx_{k+1}.$$

Remark. If $n\geq 3$ is odd, and one of the x_k is negative, then the conclusion may no longer hold. This is the case if, for instance, $x_1=-b$, and $x_{2k}=a$, $x_{2k+1}=b$, $k=1,2,...,\frac{n-1}{2}$, where $0\leq a<b$, so the string of numbers is

$$-b,a,b,a,b,...,b,a.$$

If n is even, the conclusion may again no longer hold, as shown by any string of alternate real numbers: $a,b,a,b,...,a,b$, where $a\neq b$.



Problem 2. Let $ABCD$ be a cyclic quadrilateral, and let diagonals AC and DC intersect at X. Let C_1,D_1 and M be the midpoint of segments CX,DX and CD rspectively. Lines AD_1 and BC_1 intersect at Y, and line MY intersect diagonals AC and BD at different points E and F, respectively. Prove that line XY is tangent to the circle through E,F and X.

Solution. We are to prove that $\angle EXY = \angle EFX$; alternatively, but equivalently,

$$\angle AYX + \angle XAY = \angle BYF + \angle XBY .$$

Since the quadangle $ABCD$ is cyclic, the triangles XAD and XBC are similar, and since AD_1 and BC_1 are corresponding medians in these triangles, it follows that $\angle XAY = \angle XAD_1 = \angle XBC_1 = \angle XBY$.

Finally, $\angle AYX = \angle BYF$, since X and M are corresponding points in the similar triangles ABY and C_1D_1Y : indeed, $\angle XAB = \angle XDC = \angle MC_1D_1$, and $\angle XBA = \angle XCD = \angle MD_1C_1$.

Problem 3. Let m be a positive integer. Consider a $4m \times 4m$ array of square unit cells. Two different cells are *related* to each other if they are in either the same row or in the same column. No cell is related to itself. Some cells are coloured blue, such that every cell is related to at least two blue cells. Determine the minimum number of blue cells.

Solution 1.(Israel) The required minimum is $6m$ and is achieved by a diagonal string of m 4×4 blocks of the form below*bullets mark centers of blue cells):

●

●

●

In particular, this configuration shows that the reqired minimum does not exceed $6m$.

We now show that any configuration of blue cells satisfying the condition in the statement has cardinality at least $6m$.

Fix such a configuration and let m_1^r be the number of blue cells in rows containing exactly one such, let m_2^r be the number of blue cells in rows containing exactly two such, and let m_3^r be the number of blue cells in rows containing at least three such; the numbers m_1^c, m_2^c and m_3^c are defined similarly.

Begin by noticing that $m_3^c \ge m_1^r$ and similarly, $m_3^r \ge m_1^c$. Indeed, if a blue cell is alone in its row, respectively column, then there eare at least two other blue cells in its column, respectively row, and the calim follows.

Suppose now, if possible, the total number of blue cells is less than $6m$. We will show that $m_1^r > m_3^r$ and $m_1^c > m_3^c$ and reach a contradiction by the preceding: $m_1^r > m_3^r \ge m_1^c > m_3^c \ge m_1^r$.

We prove the first inequality; the other one is dealt with similarly. TO this edn, notice that there are no empty rows-otherwise, each column would contain at least two blue cells, whence a total of at least $8m > 6m$ blue cells, which is contradiction. Next, count rows to get $m_1^r + \frac{m_2^r}{2} + \frac{m_3^r}{3} \ge 4m$, and count blue cells to get

$m_1^r+m_2^r+m_3^r<6m$. Subtraction of the latter from the former multiplied by $\frac{3}{2}$ yields $m_1^r-m_3^r>\frac{m_2^r}{2}\geq 0$, and the conclusion follows.

Solution 2. To prove that a minimal configuration of blue cells satisfying the condition in the statement has cardinality at least $6m$, consider a bipartite graph whose vertex parts are the rows and the columns of the array, respectively, a row and a column being joined by an edge if and only if the two cross at a blue cell. Clearly, the number of blue cells is equal to the number of edges of this graph, and the relationship condition in the statementreads: for every row r and every column c, $\deg r+\deg c-\varepsilon(r,c)\geq 2$, where $\varepsilon(r,c)=2$ if r and c joined by an edge, and $\varepsilon(r,c)=0$ otherwise.

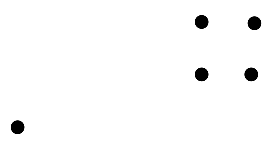
Notice that there are no empty rows/columns, so the graph has no isolated vertices. By the preceding, the cardinalty of every connected component of the graph is at least 4, so there are at most $2\cdot\frac{4m}{4}=2m$ such and consequently, the graph has at least $8m-2m=6m$ edges. This completes the proof.

Remarks. The argument in the first solution shows that equality to $6m$ is possible only if $m_1^r=m_3^r=m_1^c=m_3^c=3m$, $m_2^r=m_2^c=0$, and there are no rows, respectively columns, containing four blue cells or more.

Consider the same problem for an $n\times n$ array. The argument in the second solution shows that the corresponding minimum is $\frac{3n}{2}$ if n is divisible by 4, and $\frac{3n}{2}+\frac{1}{2}$ if n is odd; if $n\equiv 2\,(\mathrm{mod}\,4)$, the minimum in question is $\frac{3n}{2}+1$. To describe corresponding minimal configurations C_n, refer to the minimal configuration C_2,C_3,C_4,C_5 below:

The case $n\equiv 0(\mathrm{mod}\,4)$ was dealt with above: a C_n consists of a diagonal string of $\frac{n}{4}$ blocks C_4. If $n\equiv r\,(\mathrm{mod}\,4)$, $r=2,3$, a C_n consists of a diagonal string of $\left\lfloor\frac{n}{4}\right\rfloor$ blocks C_4 followed by a C_r, and if $n\equiv 1\,(\mathrm{mod}\,4)$, a C_n consists of a diagonal string of $\left\lfloor\frac{n}{4}\right\rfloor-1$ blocks C_4 followed by a C_5.

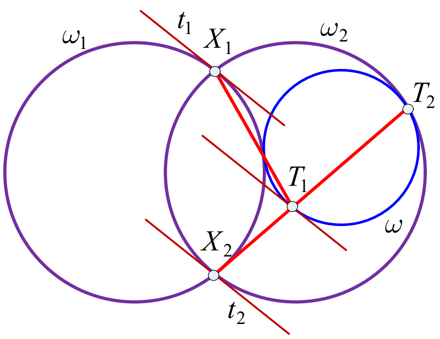
Minimal configuration are not necessary unique(two configurations being equivalent if one is obtained from the other by permuting the rows and/or the columns). For instance, if $n=6$, the configurations below are both minimal:



European girl's Mathematical Olympiad
Bushteni, Romania, April 10.04-16.04.2016

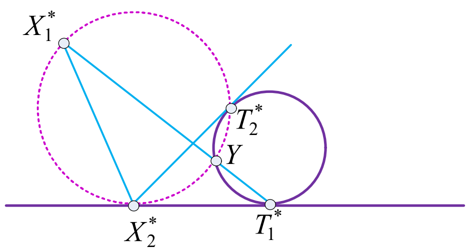
Day 2
Wensday, April 13 , 2016

Problem 4. Two circles, ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at a point T_1, and internaly tangent to ω_2 at a point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω.



Solution 1. Let the line X_kT_k and ω meet again at X_k', $k=1,2$, and notice that the tangent t_k to ω_k at X_k and the tangent t_k' to ω at X_k' are parallel. Since the ω_k have equal radii, the t_k are parallel, so the t_k' are parallel, and consequently the points X_1' and X_2' coincide(they are not antipodal, since they both lie on the same side of the line T_1T_2. The conclusion follows.

Solution 2. The circle ω is the image of ω_k under a homothety h_k centred at T_k, $k=1,2$. The tangent to ω at $X_k'=h_k(X_k)$ is therefore parallel to the tangent t_k to ω_k at X_k. Since the ω_k have equal radii, the t_k are parallel, so $X_1'=X_2'$ and since the points X_k, T_k and X_k' are collinear, the conclusion follows.



Solution 3. Invert from X_1 and use an asterisk to denote images under this inversion. Notice that ω_k^* is the tangent from X_2^* to ω^* at T_k^*, and the pole X_1 lies on the bisektrix of the angle formed by the ω_k^*, not containing ω^*. Letting $X_1T_1^*$ and ω^* meet again at Y, standard angle chase shows that Y lies on the circle $X_1X_2^*T_2^*$ and the conclusion follows.

Remarks. The product h_1h_2 of the two homotheties in the first solution is reflexion across the midpoint of the segment X_1X_2, which lies on the line T_1T_2.

Various arguments, involving similarities, radical axes, and the like, work equally well to prove the required result.

Problem 5. Let k and n be integers such that $k \geq 2$ and $k \leq n \leq 2k-1$. Place rectangular tiles, aech of size $1 \times k$ or $k \times 1$, on an $n \times n$ chessboard so that each tile covers exactly k cells, and no two tiles overlap. Do this until no further tile be placed in this way. For each such k and n, determine the minimum number of tiles such an arrangement may contain.

Solution. The required minimum is n if $n=k$, and it is $\min\{n, 2n-2k+2\}$ if $k<n<2k$.

The case $n=k$ being clear, assume hence forth $k<n<2k$. Begin by describing maximal arrangement of the board $[0,n]\times[0,n]$, having the above mentioned cardinalities.

If $k<n<2k-1$, then $\min\{n,2n-2k+2\}=2n-2k+2$. To obtain a maximal arrangement of this cardinality, place four tiles, $[0,k]\times[0,1]$, $[0,1]\times[0,k+1]$, $[1,k+1]\times[k,k+1]$ and $[k,k+1]\times[0,k]$ in the square $[0,k]\times[0,k]$, stack $n-k-1$ horizontal tiles in the rectangle $[1,k+1]\times[k+1,n]$, and erect $n-k-1$ vertical tiles in the rectangle $[k+1,n]\times[1,k+1]$.

If $n=2k-1$, then $\min\{n,2n-2k+2\}=n=2k-1$. A maximal arrangement of $2k-1$ tiles is obtained by stacking $k-1$ horizontal tiles in the rectangle $[0,k]\times[0,k-1]$, another $k-1$ horizontal tiles in rectangle $[0,k]\times[k,2k-1]$, and ading the horizontal tile $[k-1,2k-1]\times[k-1,k]$.

The above examples show show that the required minimum does not exceed the mentoined values.

To prove the reverse inequality, consider a maximal arrangement and let r, respectively c, be the number of rows, respectively columns, not containing a tile.

If $r=0$ or $c=0$, the arrangement clearly contains at least n tiles.

If r and c are both positive, we show that the arrangement contains at least $2n-2k+2$ tiles. To this end, we will prove that the rows, respectively columns, not containing a tile are consecutive. Assume this for the moment, to notice that these r rows and c columns cross to form an $r\times c$ rectangular array containing no tile at all, so $r<k$ and $c<k$ by maximality. Consequently, there are $n-r\geq n-k+1$ rows containing at least one horizontal tile each, and $n-c\geq n-k+1$ columns containing at least one vertical tile each, whence a total of at least $2n-2k+2$ tiles.

We now show that the rows not containing a tile are consecutive; columns are dealt with similarly. Consider a horizontal tile T. Since $n<2k$, the nearest horizontal side of the board is at most $k-1$ rows away from the row containing T. Thee rows, if any, cross the k columns T crosses to form a rectangular array no vertical tile fits in. Maximality forces each of these rows to contain a horizontal tile and the claim follows.

Consequently, the cardinality of every maximal arrangement is at least $\min\{n,2n-2k+2\}$, and the conclusion follows.

Remarks.(1) If $k\geq 3$ and $n=2k$ the minimum is $n+1=2k+1$ and is achieved, for instance, by the maximal arrangement consisting of the vertical tile $[0,1]\times[1,k+1]$ along with $k-1$ horizontal tiles stacked in $[1,k+1]\times[0,k-1]$, another $k-1$ horizontal tiles stacked in $[1,k+1]\times[k+1,2k]$, and two horizontal tiles stacked in $[k,2k]\times[k-1,k+1]$. This example shows that the corresponding minimum does not exceed $n+1<2n-2k+2$. The argument in the solution also applies to the case $n=2k$ to infer that for a maximal arrangement of minimal cardinality either $r=0$ or $c=0$, and the cardinality is at least n. Clearly, we may and will assume $r=0$. Suppose, if possible, such an arrangement contains exactly n tiles. Since there is no room left for an additional tile, some tile T must cover a cell of the leftmost column, so it covers the k leftmost cells along its row, and there is then room for another tile along that row-a contradiction.

(2) For every pair (r,c) of integers in the range $2k-n,...,k-1$, at least one of which is positive, say $c>0$, there exists a maximal arrangement of cardinality $2n-r-c$.

Use again the board $[0,n]\times[0,n]$ to stack $k-r$ horizontal tiles in each of the rectangles $[0,k]\times[0,k-r]$ and $[k-c,2k-c]$, erect $k-c$ vertical tiles in each of the rectangles $[0,k-c]\times[k-r,2k-r]$ and $[k,2k-c]\times[0,k]$, then stack $n-2k+r$ horizontal tiles in the rectangle $[k-c,2k-c]\times[2k-r,n]$, and erect $n-2k+c$ vertical tiles in the rectangle $[2k-c,n]\times[1,k+1]$.

Problem 6. Let S be the set of all positive integers n such that n^4 has a divisor in the range $n^2+1, n^2+2, ..., n^2+2n$. Prove that there are infinitely many elements of S of the forms $7m, 7m+1, 7m+2, 7m+5, 7m+6$ and no elements of S of the form $7m+3$ or $7m+4$, where m is an integer.

Solution. The conclusion is a consequence of the lemma below which actually provides a recursive description of S. The proof of the lema is at the end of the solution.

Lemma. *The forth power of a positive integer n has a divisor in the range in the range $n^2+1, n^2+2, ..., n^2+2n$ if and only if at least one of the numbers $2n^2+1$ and $12n^2+9$ is a perfect square.*

Consequently, a positive integer n is a member of S if and only if $m^2-2n^2=1$ or $m^2-12n^2=9$ for some positive integer m.

The former is a Pell equation whose solutions are $(m_1, n_1)=(3,2)$ and

$$(m_{k+1}, n_{k+1})=(3m_k+4n_k, 2m_k+3n_k),\ k=1,2,3,....$$

In what follows, all congruences are modulo 7. Iteration shows that $(m_{k+3}, n_{k+3}) \equiv (m_k, n_k)$. Since $(m_1, n_1) \equiv (3,2)$, $(m_2, n_2) \equiv (3,-2)$ and $(m_3, n_3) \equiv (1,0)$ it follows that S contains infinitely many integrs from each of the residue classes 0 and ± 2 modulo 7.

The other equations is easily transformed into a Pell equation, $m'^2-12n'^2=1$, by noticing that m and n are both divisible by 3, say $m=3m'$ and $n=3n'$. In this case, the solutions are $(m_1, n_1)=(21,6)$ and

$$(m_{k+1}, n_{k+1})=(7m_k+24n_k, 2m_k+7n_k),\ k=1,2,3,....$$

This time iteration shows that $(m_{k+4}, n_{k+4}) \equiv (m_k, n_k)$. Since $(m_1, n_1) \equiv (0,-1)$, $(m_2, n_2) \equiv (-3,0)$, $(m_3, n_3) \equiv (0,1)$ and $(m_4, n_4) \equiv (3,0)$, it follows that S contains infinitely many integers from each of the residue classes 0 and ± 1 modulo 7.

Finally, since the n_k from the two sets of formulae exhaust S, by the preceding no integer in the residue classes ± 3 modulo 7 is a member of S.

We now turn to the lemma. Let n be a member of S, and let $d=n^2+m$ be a divisor of n^4 in the range $n^2+1, n^2+2, ..., n^2+2n$, so $1 \le m \le 2n$. Consideration of the square of $n^2=d-m$ shows m^2 divisible by d, so m^2/d is a positive integer. Since $n^2<d<(n+1)^2$, it follows that d is not a square; in particular, $m^2/d \ne 1$, so $m^2/d \ge 2$. On the other hand, $1 \le m \le 2n$, so $\frac{m^2}{d}=\frac{m^2}{n^2+m} \le \frac{4n^2}{n^2+1}<4$. Consequently, $\frac{m^2}{d}=2$ or $\frac{m^2}{d}=3$; that is, $\frac{m^2}{n^2+m}=2$ or $\frac{m^2}{n^2+m}=3$. In the former case, $2n^2+1=(m-1)^2$, and in the latter, $12n^2+9=(2m-3)^2$.

Conversely, if $2n^2+1=m^2$ for some positive integer m, then $1<m^2<4n^2$, so $1<m<2n$, and $n^4=(n^2+m+1)(n^2-m+1)$, so the first factor is the desired divisor.

Similarly, if $12n^2+9=m^2$ for some positive integer m, then m is odd, $n \ge 6$, and $n^4=\left(n^2+\frac{m}{2}+\frac{3}{2}\right)\left(n^2-\frac{m}{2}+\frac{3}{2}\right)$, and again the first factor is the desired divisor.

23-ta Makedonska
matemati~ka olimpijada

23-rd Macedonian mathematical olympiad

Faculty of Electrical Engineering and Information Technologies-Skopje
09.04.2016

1. Solve the equation $1+x^z+y^z=\mathrm{lcm}(x^z,y^z)$ in the set of natural numbers.

2. A magical square of dimensions 3×3 is a square with side 3, consisting of 9 unit squares, so that the real numbers written in the unit squares (one number in each unit square) satisfy the property: the sum of the numbers in the unit squares in any row is equal to the sum of the numbers in the unit squares in any column and is equal to the sum of the numbers in the unit squares in the two diagonals.

A rectangle of dimensions $m\times n, m\geq 3, n\geq 3$ is given, which consists of mn unit squares. If in each unit square one number is written in such a way that each square of dimensions 3×3 is magical, then how many different numbers can be used at most to fill the rectangle?

3. Solve the equation $xyz+yzt+xzt+xyt=xyzt+3$ in the set of natural numbers.

4. A segment AB and its midpoint K are given. An arbitrary point C , different from K is chosen on the perpendicular to AB through K. Let N be the intersection of AC and the line passing through B and the midpoint of the segment CK. Let U be the intersection of AB with the line that passes through C and the midpoint L of the segment BN. Prove that the ratio of the areas of the triangles CNL and BUL doesn't depend on the choice of point C.

5. Let $n\geq 3$ and $a_1,a_2,...,a_n$ be positive real numbers for which $\frac{1}{1+a_1^4}+\frac{1}{1+a_2^4}+...+\frac{1}{1+a_n^4}=1$ holds. Prove the inequality $a_1a_2\cdot...\cdot a_n\geq (n-1)^{n/4}$.

Solutions

1. Solve the equation $1+x^z+y^z=\mathrm{lcm}(x^z,y^z)$ in the set of natural numbers.

Solution. Let $d=\gcd(x,y)$. Then $d\,|\,lcm(x^z,y^z)$, $d\,|\,x^z$ and $d\,|\,y^z$, from where we get $d=1$. The equation is transformed into $1+x^z+y^z=x^zy^z$, or, equivalently $(x^z-1)(y^z-1)=2$. We get $x^z-1=1, y^z-1=2$ or $x^z-1=2, y^z-1=1$, from where it follows that $x=2,y=3,z=1$ or $x=3,y=2,z=1$.

2. A magical square of dimensions 3×3 is a square with side 3, consisting of 9 unit squares, so that the real numbers written in the unit squares (one number in each unit square) satisfy the property: the sum of the numbers in the unit squares in any row is equal to the sum of the numbers in the unit squares in any column and is equal to the sum of the numbers in the unit squares in the two diagonals.

A rectangle of dimensions $m\times n$, $m\ge3, n\ge3$ is given, which consists of mn unit squares. If in each unit square one number is written in such a way that each square of dimensions 3×3 is magical, then how many different numbers can be used at most to fill the rectangle?

Solution. We consider the magical square:

A_1	A_2	A_3
B_1	B_2	B_3
C_1	C_2	C_3

Then

$$\begin{aligned}A_1+A_2+A_3&=B_1+B_2+B_3=C_1+C_2+C_3=A_1+B_1+C_1\\&=A_2+B_2+C_2=A_3+B_3+C_3=A_1+B_2+C_3=C_1+B_2+A_3=S,\end{aligned}$$

or, equivalently

$$\begin{aligned}4S&=(B_1+B_2+B_3)+(A_2+B_2+C_2)+(A_1+B_2+C_3)+(C_1+B_2+A_3)\\&=(A_1+A_2+A_3)+(B_1+B_2+B_3)+(C_1+C_2+C_3)+3B_2=3S+3B_2.\end{aligned}$$

We get $S=3B_2$. In what follows we will denote the central element B_2 by x.

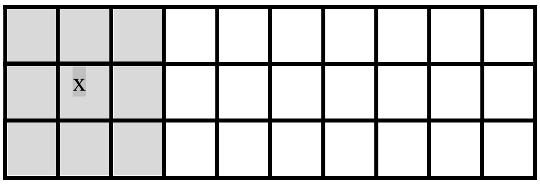
We have proven that if the central element in a magical square is x, then $S=3x$ …………..….(1)

If the rectangle is of dimensions 3×3, then it is a magical square and we can fill it with 9 different numbers, for example

.

1	10	4
8	5	2
6	0	9

We will show that a rectangle of dimensions $n=3, m>3$ has to be filled with a single number. Let $n=3, m>3$ and let x be the number in the first central unit square (Picture 1).



Picture 1.

From (1) we get that if the unit square from the rectangle is filled as in Picture 1, then S of the designated square is $3x$. We consider the square designated in Picture 2.

	x	x							

Picture 2.

Then its central unit square has to be x again, because the second column has sum equal to $3x$. Analogously, by moving the square to the right we get a rectangle that has to be filled in the following way:

	x	x	x	...	...	x	x	x	

From the colored squares, it follows that the entire second row it filled with x.
Let's assume that the rectangle is filled in the following way:

a	c								
x	x	x	x	...	...	x	x	x	x
b	d								

Since the sum of the numbers in the first row of the colored square is equal to the sum of the numbers in the diagonals, the rectangle has to be filled in the following way

a	c	a							
x	x	x	x	...	...	x	x	x	x
b	d	b							

Picture 3.

Next we consider the colored square in Picture 4. Because $2a+c=3x$ and $2b+d=3x$ we get that the rectangle is filled in the following way:

a	c	a	a						
x	x	x	x	...	...	x	x	x	x
b	d	b	b						

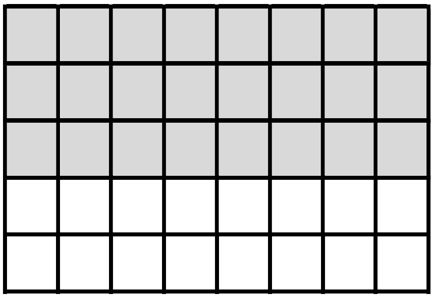
Picture 4

Analogously to the way the colored square was filled in Picture 3, we get that $c=a, b=d$. But then

a	a	a	a						
x	x	x	x	...	...	x	x	x	x
b	b	b	b						

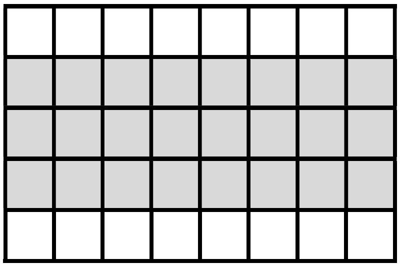
, from where $a=b=c=d=x$ i.e. all elements of the rectangle have to be equal.

Let $n>3, m>3$. Then, because of the previous discussion, the rectangle of width 3 and length m has to be filled with one number (Picture 5).



Picture 5.

For the same reasons, the same holds for the colored rectangle and every rectangle obtained by vertical translation.



Finally, if $n=m=3$, then the rectangle can be filled with 9 different numbers. If $n>3$ or $m>3$, then the rectangle can be filled only with a single number.

3. Solve the equation $xyz+yzt+xzt+xyt=xyzt+3$ in the set of natural numbers.

Solution. After dividing the equation by $xyzt$ we get $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}=1+\frac{3}{xyzt}$. Because of symmetry, without loss of generality, we can assume that

$$x\leq y\leq z\leq t \quad ... \qquad (1)$$

from where it follows that $\frac{1}{x}\geq\frac{1}{y}\geq\frac{1}{z}\geq\frac{1}{t}$. We get $\frac{4}{x}\geq\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}=1+\frac{3}{xyzt}>1$, from where we have $x<4$.

Case 1. Let $x=3$. Then the equation is of the form $3yz+yzt+3zt+3yt=3yzt+3$, or, equivalently $3(yz+zt+yt)=2yzt+3$. After dividing this equation by yzt we get

$$3\left(\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)=2+\frac{3}{yzt}>2, \frac{9}{y}>2, \text{ from where we have } y\leq 4.$$

The possible values for y are 3 and 4.

a) For $y=4$ we get

$$3(4z+zt+4t)=8zt+3, 12(z+t)=5zt+3, 12\left(\frac{1}{z}+\frac{1}{t}\right)=5+\frac{3}{zt}>5, \frac{24}{z}>5,$$

from where we have $z\leq 4$. From (1) it follows that $z=4$ and the equation gets the form $12(4+t)=20t+3$, or, equivalently $8t=45$, which implies that t is not a natural number.

б) For $y=3$, we get

$$3(3z+zt+3t)=6zt+3, 3(z+t)=zt+1, 3\left(\frac{1}{z}+\frac{1}{t}\right)=1+\frac{1}{zt}>1, \frac{6}{z}>1, z<6.$$

The possible values for z are $3,4,5$.

-Let $z=3$. Then $3(3+t)=3t+1$ which is impossible.

-If $z=4$, then $3(4+t)=4t+1,\ t=11$.

-If $z=5$, then $3(5+t)=5t+1, t=7$.

We get that the quadriples $(3,3,4,11)$, $(3,3,5,7)$ are solutions.

Case 2. Let $x=2$.

Then the equation is if the form

$$2yz+yzt+2zt+2yt=2yzt+3,$$

or, equivalently,

$$2(yz+zt+yt)=yzt+3 \text{ ....} \qquad (2).$$

Then each of the numbers y,z,t is odd. After dividing this equation by yzt we get $2\left(\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)=1+\frac{3}{yzt}>1$ from where we have $\frac{6}{y}>1$, or, equivalently $y<6$.

a) If $y=5$ then (2) is of the form $2(5z+zt+5t)=5zt+3$, or, equivalently $10(z+t)=3zt+3$. Hence $10\left(\frac{1}{z}+\frac{1}{t}\right)=3+\frac{3}{zt}>3$, therefore $\frac{1}{z}>\frac{3}{20}$, or equivalently $z\leq 6$. The only possibility is $z=5$. We get $10(5+t)=15t+3$, or, equivalently $5t=47$ which implies that t is not a natural number.

б) If $y=3$, (2) is of the form $2(3z+zt+3t)=3zt+3$, or equivalently $6(z+t)=zt+3$. Then $6\left(\frac{1}{z}+\frac{1}{t}\right)=1+\frac{3}{zt}>1$, from where $\frac{12}{z}>1$, or, equivalently $z<12$. The possibilities for z are $3,5,7,9,11$.

-If $z=3$, then $6(3+t)=3t+3$, from where we have $3t=-15$, or equivalently $t=-5\notin\mathbb{N}$.

- If $z=5$, then $6(5+t)=5t+3$, $t=-27\notin\mathbb{N}$.

- If $z=7$, then $6(7+t)=7t+3$, $t=39$.

- If $z=9$, then $6(9+t)=9t+3$, from where we have $3t=51$, or, equivalently $t=17$.

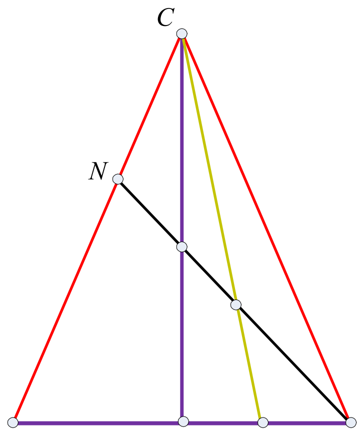
Therefore in this case the solutions are the quadriples $(2,3,7,39)$, $(2,3,9,17)$.

Case 3. The case remains when $x=1$. Then the equation is of the form $yz+yzt+zt+yt=yzt+3$, or, equivalently $yz+zt+yt=3$. From (1) we get $3yz\leq 3$, or equivalently $yz\leq 1$, from where $y=1$ and $z=1$. Then $1+2t=3$, or equivalently $t=1$. The quiadriple $(1,1,1,1)$ is a solution.

Finally, the solutions to the initial equation are all permutations of $(3,3,4,11)$, $(3,3,5,7)$, $(2,3,7,39)$, $(2,3,9,17)$, $(1,1,1,1)$.

4. A segment AB and its midpoint K are given. An arbitrary point C , different from K is chosen on the perpendicular to AB through K. Let N be the intersection of AC and the line passing through B and the midpoint of the segment CK. Let U be the intersection of AB with the line that passes through C and the midpoint L of the segment

BN. Prove that the ratio of the areas of the triangles CNL and BUL doesn't depend on the choice of point C.



Solution. Let M be the midpoint of the segment CK. From Menelaus' theorem for the triangle AKC and the line BN we have

$$\frac{\overline{CN}}{\overline{NA}} \cdot \frac{\overline{AB}}{\overline{BK}} \cdot \frac{\overline{KM}}{\overline{MC}} = 1.$$

From this we get $\overline{NA} = 2\overline{NC}$, from which it follows that $\overline{AC} = 3\overline{NC}$. Hence $P_{BNC} = \frac{1}{3}P_{ABC}$. From Menelaus' theorem for the triangle ABN and the line CU we have

$$\frac{\overline{AU}}{\overline{UB}} \cdot \frac{\overline{BL}}{\overline{LN}} \cdot \frac{\overline{NC}}{\overline{CA}} = 1.$$

Therefore we get $\overline{AU} = 3\overline{UB}$. Therefore U is the midpoint of the segment BK. It follows that $P_{BUC} = \frac{1}{4}P_{ABC}$. Let $x = P_{CNL}$ and $y = P_{BLU}$. Since L is the midpoint of BN, we have $P_{BLC} = x$. Now

$$x + y = P_{BLC} + P_{BLU} = P_{BUC} = \frac{1}{4}P_{ABC},$$

on the other hand we have

$$2x = P_{CNL} + P_{BLC} = P_{BNC} = \frac{1}{3}P_{ABC}.$$

If we divide these two equalities we get

$$\frac{1}{2} + \frac{y}{2x} = \frac{3}{4}, \text{ hence } \frac{y}{x} = \frac{1}{2},$$

from where we get the required statement.

5. Let $n \geq 3$ and $a_1, a_2, ..., a_n$ be positive real numbers for which $\frac{1}{1+a_1^4} + \frac{1}{1+a_2^4} + ... + \frac{1}{1+a_n^4} = 1$ holds. Prove the inequality $a_1 a_2 \cdot ... \cdot a_n \geq (n-1)^{n/4}$.

Solution. Let $a_i^2 = \tan x_i, x_i \in \left[0, \frac{\pi}{2}\right), i = 1, 2, ..., n.$ Then $\sum_{i=1}^{n} \cos^2 x_i = 1$.

From the inequality between the arithmetical and the geometrical mean it follows that

$$\sin^2 x_i = 1 - \cos^2 x_i \geq (n-1)\left(\prod_{j=1, j\neq i}^{n} \cos x_j\right)^{2/(n-1)}, i = 1, 2, ..., n.$$

By multiplying the n inequalities above, we get $\prod_{i=1}^{n} \sin^2 x_i \geq (n-1)^n \prod_{i=1}^{n} \cos^2 x_i$. The last inequality is equivalent to the inequality

$$\prod_{i=1}^{n} \tan x_i \geq (n-1)^{n/2}.$$

Finally, $\prod_{i=1}^{n} a_i = \left(\prod_{i=1}^{n} \tan x_i\right)^{1/2} \geq (n-1)^{n/4}$, which was to be proven.

33-th Balkan mathematical Olympiad
05.05.-10.05.2016, Tirana Albania

Problem 1

Find all injective functions $f:\mathbb{R}\to\mathbb{R}$ such that for every real number x and every positive integer n,

$$\left|\sum_{i=1}^{n} i(f(x+i+1)-f(f(x+i))\right|<2016$$

Problem 2

Let $ABCD$ be a cyclic quadrilateral with $AB<CD$. The diagonals intersect at the point F and lines AD and BC intersect at the point E. Let K and L be the orthogonal projections of F onto lines AD and BC respectively, and let M,S and T be the midpoints of EF,CF and DF respectively. Prove that the second intersection point of the circumcircles of triangles MKT and MLS lies on the segment CD.

Problem 3

Find all monic polynomials f with integer coefficients satisfying the following condition: there exists a positive integer N such that p divides $2(f(p)!)+1$ for every prime $p>N$ for which $f(p)$ is a positive integer.

Note. *A monic polynomial has leading coefficient equal to 1.*

Problem 4

The plane is divided into unit squares by two sets of parallel lines, forming an infinite gird. Each unit squares is coloured with one of 1201 colours so that no rectangle with perimeter 100 contains two squares of the same colour. Show that no rectangle of size 1×1201 or 1201×1 contains two squares of the same colour.

Note. *Any rectangular is assumed here to have sides contained in the lines of the gird.*

Solutions

Problem 1

Find all injective functions $f:\mathbb{R}\to\mathbb{R}$ such that for every real number x and every positive integer n,

$$\left|\sum_{i=1}^{n} i(f(x+i+1)-f(f(x+i))\right|<2016$$

Solution. From the condition of the problem we get

$$\left|\sum_{i=1}^{n-1} i(f(x+i+1)-f(f(x+i))\right|<2016.$$

Then

$$|n(f(x+n+1)-f(f(x+n))|=$$

$$=\left|\sum_{i=1}^{n} i(f(x+i+1)-f(f(x+i))-\sum_{i=1}^{n-1} i(f(x+i+1)-f(f(x+i))\right|<2\cdot 2016=4032$$

implaying

$$|f(x+n+1)-f(f(x+n))|<\frac{4032}{n}$$

for every real number x and every positive integer n.

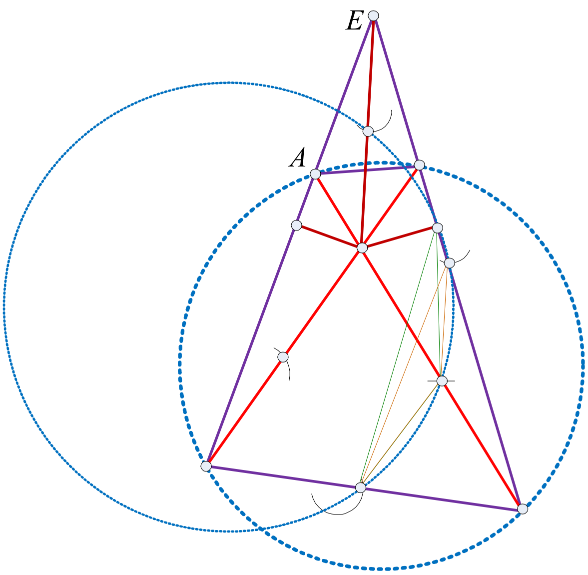
Let $y\in\mathbb{R}$ be arbitrary. Then there exists x such that $y=x+n$. Wwe obtain

$$|f(y+1)-f((f(y))|<\frac{4032}{n}$$

for every real number y and every positive integer n. The last inequality holds for every positive integer n from where $f(y+1)=f(f(y))$ for every $y\in\mathbb{R}$ and since the function f is an injection, then $f(y)=y+1$. The function $f(y)=y+1$ satisfies the required condition.

Problem 2

Let $ABCD$ be a cyclic quadrilateral with $AB<CD$. The diagonals intersect at the point F and lines AD and BC intersect at the point E. Let K and L be the orthogonal projections of F onto lines AD and BC respectively, and let M,S and T be the midpoints of EF,CF and DF respectively. Prove that the second intersection point of the circumcircles of triangles MKT and MLS lies on the segment CD.



Solution. Let N be the midpoint of CD. We will prove that the circumcircles of the triangles MKT and MLS pass through N.(1)

First will prove that the circumcircle of MLS passes through N. (2)

Let Q be the midpoint of EC. Note that the circumcircle of MLS is the Euler circle(2) of the triangle EFC, so it passes also through Q (*)(3)

We will prove that

$$\angle SLQ=\angle QNS \quad \text{or} \quad \angle SLQ+\angle QNS=180^\circ. \tag{4}$$

Indeed, since FLC is right-angled and LS is its median, we have that $SL=SC$ and

$$\angle SLC=\angle SCL=\angle ACB. \tag{5}$$

In addition, since N and S are the midpoint of DC and FC we have that $SN\|FD$ and similarly, since Q and N are the midpoints of EC and CD, so $QN\|ED$.

It follows that the angles $\angle EDB$ and $\angle QNS$ have parallel sides, and since $AB<CD$ they are acute, and as a result we have that

$$\angle EDB=\angle QNS \quad \text{or} \quad \angle EDB+\angle QNS=180^\circ. \tag{6}$$

But, from the cyclic quadrilateral $ABCD$, we get that

$$\angle EDB=\angle ACB. \tag{7}$$

Now, from (2), (3) and (4) we obtain immediately (1), so the quadrilateral $LNSQ$ is cyclic. Since from (*), its circumcircle passes also through M, we get that the points M,L,Q,S,N are cocyclic and this means that the circumcircle of MLS passes through N.

Similarly, the circumcircle of MKT passes also through N and we have the desired.

Problem 3

Find all monic polynomials f with integer coefficients satisfying the following condition: there exists a positive integer N such that p divides $2(f(p)!)+1$ for every prime $p>N$ for which $f(p)$ is a positive integer.

Note. *A monic polynomial has leading coefficient equal to 1.*

Solution. If f is a constant polynomaila then it's obvious that the condition cannot hold for

$p\geq 5$ since $f(p)=1$. (1)

From the divisibility relation $p|2(f(p))!+1$ we conclude that:

$f(p)<p$ for all primes $p>N$ (*) (2)

In fact, if for some prime number p we have $f(p)\geq p$, then $p|(f(p))!$ and then $p|1$ which is absurd.

Now suppose that $\deg f=m>1$. Then $f(x)=x^m+Q(x)$, $\deg Q\leq m-1$ and so $f(p)=p^m+Q(p)$. Hence for some large enough prime number p holds that $f(p)>p$, which contradicts (*). Therefore we must have $\deg f(x)=1$ and $f(x)=x-a$ for some positive integer a. (3)

Thus the given condition becomes:

$p|2(p-a)!+1$. (4)

But we have(using Wilson theorem)

$2(p-3)!\equiv -(p-3)!(p-2)\equiv -(p-2)!\equiv -1\,(\mathrm{mod}\ p)$

$p|2(p-3)!+1$. (5)

From (1) and (2) we get

$2(p-3)!\equiv -(p-3)!(p-2)\equiv -(p-2)!\equiv -1\,(\mathrm{mod}\ p)$

$(p-3)!(-1)^a(a-1)!\equiv (p-1)!(-1)^a(a-1)!(\mathrm{mod}\ p)$

$(p-3)!(-1)^a(a-1)!\equiv 1\,(\mathrm{mod}\ p)$.

Since $-2(p-3)!\equiv 1\,(\mathrm{mod}\ p)$, it follows that

$(-1)^a(a-1)!\equiv -2\,(\mathrm{mod}\ p)$. (6)

Taking $p>(a-1)!$, we conclude that $a=3$, we conclude that $a=3$ and so $f(x)=x-3$, for all x.

The function $f(x)=x-3$ satisfies the required condition.

Problem 4

The plane is divided into unit squares by two sets of parallel lines, forming an infinite gird. Each unit squares is coloured with one of 1201 colours so that no rectangle with perimeter 100 contains two squares of the same colour. Show that no rectangle of size 1×1201 or 1201×1 contains two squares of the same colour.

Note. *Any rectangular is assumed here to have sides contained in the lines of the gird.*

Solution. Let the centers of the unit squares be the integer points in the plane, and denote each unit square by the coordinates of the center.

Consider the set D of all unit squares (x,y) such that $|x|+|y|\leq 24$. Any integer translate of D is called a diamond.

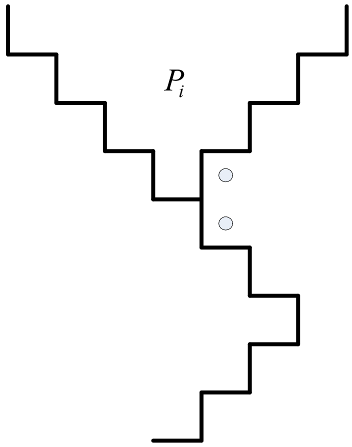
Since any two unit squares that belong to the same diamond also belong to some rectangle of perimeter 100, a diamond cannot contain two squares of the same colour. Since a diamond contains exactly $24^2+25^2=1201$ unit squares, a diamond must contain every colur exactly once.

Choose one colur, say, green, and let $a_1,a_2,...$ be all green unit squares. Let P_i be the the diamond of center a_i. We will show that no unit square is covered by two P's and that every unit square is covered by some P_i.

Indeed, suppose first that P_i and P_j contain the same unit square b. Then their centers lie within the same rectangle of perimeter 100, a contradiction.

Let, on the other hand, b be an arbitrary unit square. The diamond of center b must contain some green unit square a_i. The diamond P_i of center a_i will then contain b.

Therefore, $P_1, P_2, ...$ from a covering of the plane in exactly one layer. It is easy to see, through, that, up to translation and reflection, there exists a unique such covering.(indeed, consider two



neighbouring diamonds.Unless they fit neatly, uncoverable spaces of two unit squares are created near the corners: see Fig.1.)

Figure 1

Without loss of generality, then, this covering is given by the diamonds of centers (x, y) such that $24x + 25y$ is divisble by 1201. (See fig.2 for an analogous covering with smaller diamonds.) It follows from this that no rectangle of size 1×1201 can contain two green unit squares, and analogous reasoning works for the remaining colours.

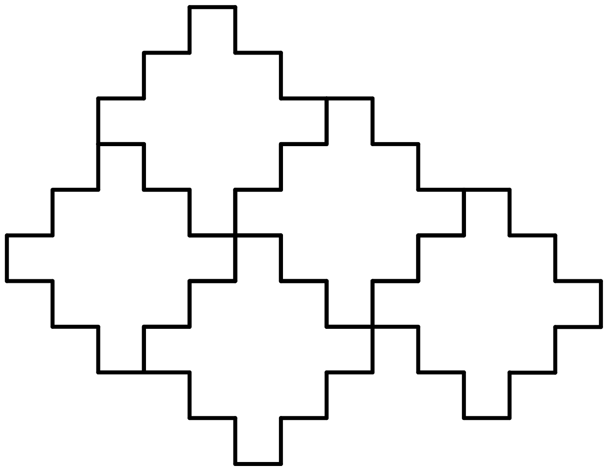


Figure 2

Селекционен тест за учество на ИМО 2016
Факултет за електротехника и информациски технологии-Скопје
15.05.2016 година

Problems and solutions

Задача 1. Нека ABC е остроаголен триаголник и нека H е неговиот ортоцентар. Точката G припаѓа на рамнината на триаголникот при што $ABGH$ е паралелограм. Точката I припаѓа на правата правата GH така што правата AC ја полови отсечката HI. Правата AC ја сече опишаната кружница околу триаголникот GCI по вторпат во точката J. Докажи дека $IJ = AH$.

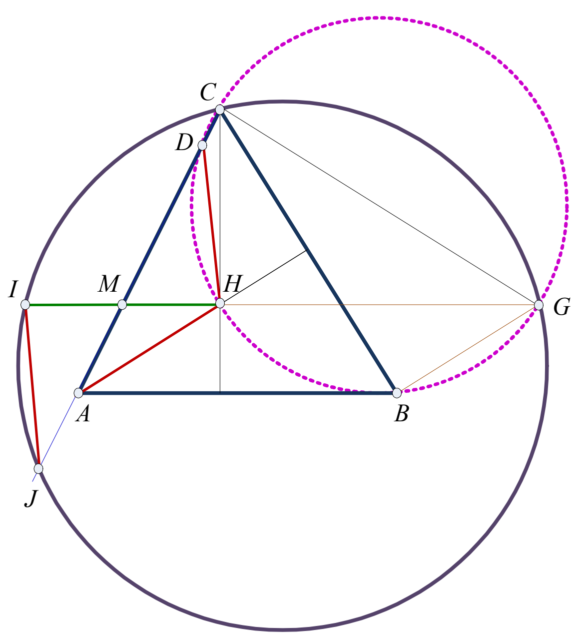
Решение 1. Бидејќи $HG \| AB$ и $BG \| AH$, добиваме дека $BG \perp BC$ и $CH \perp GH$. Според тоа, четириаголникот $BGCH$ е тетивен. Бидејќи H е ортоцентар на триаголникот ABC, добиваме дека $\angle HAC = 90° - \angle ACB = \angle CBH$. Бидејќи $BGCH$ и $CGJI$ се тетивни четириаголници, добиваме дека

$$\angle CJI = \angle CGH = \angle CBH = \angle HAC.$$

Нека M е пресечна точка на AC и GH, и нека $D \neq A$ е точка од правата AC така што $AH = HD$. Тогаш $\angle MJI = \angle HAC = \angle MDH$.

Бидејќи $\angle MJI = \angle MDH$, $\angle IMJ = \angle HMD$ и $IM = MH$, добиваме дека триаголниците IMJ и HMD се складни, па според тоа $IJ = HD = AH$, што требаше да се докаже.

Решение 2. Равенството $\angle CGH = \angle CGB$ го добиваме на потполно ист начин како и во претходното решение. Во паралелограмот $ABGH$ имаме $\angle BAH = \angle HGB$. Од таму добиваме дека



$$\angle HMC = \angle BAC = \angle BAH + \angle HAC = \angle HGB + \angle CGB = \angle CGB.$$

Според тоа правоаголните триаголници CMH и CGB се слични. Исто така од опишаната круница околу триаголникот GCI лесно се добива дека триаголниците MIJ и MCG се слични. Но, тогаш

$$\frac{IJ}{CG} = \frac{MI}{MC} = \frac{MH}{MC} = \frac{GB}{GC} = \frac{AH}{CG},$$

од каде го добиваме равенството $IJ = AH$.

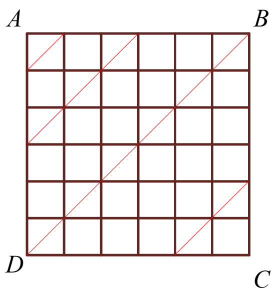
Problem2. Let a square scheme $2n\times 2n$, made of unit white squares be given. Allowed move is to change the color of three consecutive unit squares in a particular row or three consecutive unit squares in a particular column - unit square with white color goes to unit square with black color and vice versa.

Find all nonnegative integers, $n\geq 2$, for which with allowed moves the given square scheme can be colored like chess table. (Belorusian mathematitical olympiad 2016)

Solution. We will call black unit squares which one when the square scheme is colored like a chess table are black and white unite squares those which will not change their color. It is not difficult to see when the square scheme is colored like chess table, we will have $2n^2$ black and $2n^2$ white unit squares, i.e. we will have even number of black and white unit squares.

Let the square scheme is colored like a chess table with finite number of moves. Every black unit square it must be recolored odd number times and every white unit square must be recolored even number times (some of the white unit squares can be not colored at all, i.e. to be colored zero times). According to that, the number of recoloring of the unit squares must be even number, hence the number of the moves need for the recoloring is even number, since in each allowed move three unit squares are recolored.

We will show that if $n\not\equiv 0\,(\text{mod}\,3)$, the number of moves with which we can make recoloring is an odd number.



Such a contradiction for us will show that for all such nonnegative integers it is not possible to made such recoloring, i.e. the square scheme to be colored like a chess table. The vertices of the square scheme, starting from the left upper vertex and moving in clockwise direction we will denote with A,B,C,D (see the image).

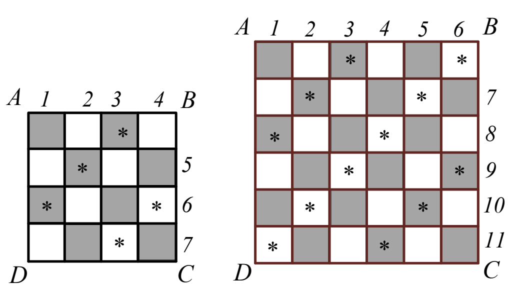
Let we consider the unit square which has a side which is a part of the sides AB and BC on the given square scheme. We will say the diagonal of the square scheme which starts from such a unit square and all the unit squares in which one can pass the chess bishop, starting from up going down, or from left to right which is same as previous (square scheme with dimensions 6×6 has 11 diagonals, on the given image are denoted only four of them). It is obvious that the square $ABCD$ has $4n-1$ diagonals and each diagonal is consisting only of white unite squares or only of black unit squares. Diagonal consisting only of white unite squares we will call white diagonal and diagonal consisting only of black unit squares we will call black diagonal.

Without loss of generality we can assume that the unit square containing the vertex A as its own vertex is a black square. The unit squares which are on the sides AB and BC, starting with the vertex A, we will denote with the numbers from 1 to $4n-1$ continuously (the case $n=2$ and $n=3$ is given on the following image). Next we will consider the diagonals of the square scheme starting with unit square which has an ordinal number divisible with 3.
In every unit square of such diagonal we will write * (on the image bellow, the cases $n=2$ and $n=3$ are given).
On that way we will obtain even or odd number of black unit squares in which one is written * in general case.

It is obvious that a diagonal which has ordinal number divisible with 3 will be black diagonal and if his ordinal number is not divisible with 2. It will be white in every other case.
It is obvious that the diagonal starting with odd ordinal number will has an odd number of unit squares.
Hence the parity of the black unit squares in which ones we have * is the same as the parity of the number of all odd numbers which are divisible with 3, between 1 and $4n-1$. We will find that number.

a) $n\equiv 0\,(\text{mod}\,3)$

In this case $n=3k, k\in\mathbb{N}$, so $4n-1=12k-1$ and between the numbers from 1 to $4n-1$ which are odd and are divisible with 3, are the numbers $3\cdot 1, 3\cdot 3, ..., 3\cdot(4k-3), 3\cdot(4k-1)$. The number of such numbers is an even number, i.e. that number is $2k$.



b) $n\equiv 1\,(\text{mod}\,3)$

In this case $n=3k+1, k\in\mathbb{N}$, so $4n-1=12k+3$ and between the numbers from 1 to $4n+1$ which are odd and divisible with 3 are the numbers $3\cdot 1, 3\cdot 3,..., 3\cdot(4k-1), 3\cdot(4k+1)$. The number of such numbers is an odd number, i.e. that number is $2k+1$.

c) $n\equiv 2 \pmod 3$

In this case $n=3k+2, k\in\mathbb{N}$, so $4n-1=12k+7$ and between the numbers from 1 to $4n+1$ which are odd and divisible with 3 are the numbers $3\cdot 1, 3\cdot 3,..., 3\cdot(4k-1), 3\cdot(4k+1)$. Hence the number of such numbers is an odd number, i.e. that number is $2k+1$.

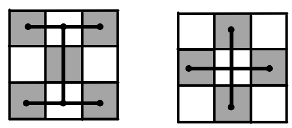
Hence, * will be written in odd number odd black unit squares if $n\not\equiv 0 \pmod 3$, and if $n\equiv 0 \pmod 3$, * will be written in even number of black unit squares.

Now, let the square scheme is colored like a chess table. Let we note that when we recolor in one allowed move we recolor only one unit square in which one is written *. So all the allowed moves are divided in two cases:

1) Allowed moves in which one we recolor white unit square with written symbol *
2) Allowed moves in which one we recolor black unit square with written symbol *

Moves like in case 1) which have to be made is even number, since each white unit square in which one is written * must be recolored even number times. Moves like in case 2), in case when $n\not\equiv 0 \pmod 3$ is an odd number since the number of black unit squares is odd and each of them must be recolred odd number times. Hence, to have a coloring in these cases like a chess table it must be odd number of colorings. But, this is a contradiction with the fact that the recoloring will be made if are made only even number of recolorings, i.e. even number of allowed moves.
Hence, if $n\not\equiv 0 \pmod 3$, recoloring of the square scheme like a chess table with allowed moves is not possible.

If $n\equiv 0 \pmod 3$, such a coloring of the square scheme with allowed moves is possible. In that case $2n$ is divisible with 3 and the square scheme can be divided on squares 3×3 and each one can be recolored with allowed moves in one of the given cases on the image below.



3. Нека m и n се позитивни цели броеви такви што $m>n$. Дефинираме $x_k=\frac{m+k}{n+k}$ за $k=1,2,...,n+1$. Докажи дека ако $x_1,x_2,...,x_{n+1}$ се цели броеви, тогаш $x_1x_2..x_{n+1}-1$ е делив со барем еден прост непарен број.

Решение. Нека препоставиме дека $x_1,x_2,...,x_{n+1}$ се цели броеви. Ги дефинираме целите броеви

$$a_k=x_k-1=\frac{m+k}{n+k}-1=\frac{m-n}{n+k}>0,$$

за $k=1,2,...,n+1$.

Нека $P=x_1x_2..x_{n+1}-1$. Потребно е да докажеме дека P е делив со барем еден непарен прост број, или дека P не е степен на бројот 2. За таа цел, ќе ги испитаме степените на 2 кои ги делат броевите a_k.

Нека 2^d е најголем степен на 2 кој го дели $m-n$, а нека 2^c е најголем степен на 2 кој не го надминува $2n+1$. Тогаш $2n+1\le 2^{c+1}-1$, па $n+1\le 2^c$. Значи, добиваме дека 2^c е еден од броевите $n+1,n+2,...,2n+1$, и дека единствен степен на 2 е 2^c кој се наоѓа меѓу тие броеви. Нека l природен број таков што $n+l=2^c$. Бидејќи $\frac{m-n}{n+l}$ е цел број, добиваме дека $d\ge c$. Според тоа $2^{d-c+1}\nmid a_l=\frac{m-n}{n+l}$, додека $2^{d-c+1}|a_k$ за секој $k\in\{1,2,3,...,n+1\}\setminus\{l\}$.

Ќе пресметаме конгруенција по модуло 2^{d-c+1}, при што добиваме

$$P=(a_1+1)(a_2+1)...(a_{n+1}+1)-1\equiv(a_l+1)\cdot 1^n-1=a_l\not\equiv 0 \pmod{2^{d-c+1}}.$$

Според тоа $2^{d-c+1}\nmid P$.

Од друга страна, за секој $k\in\{1,2,...,n+1\}\setminus\{l\}$ имаме $2^{d-c+1}|a_k$. Според тоа $P>a_k\ge 2^{d-c+1}$, за некое k од каде следува дека P не е степен на бројот 2.

19th Meditarranean mathematical olympiad

Fakultet za elektrotehnika i informaciski tehnologii
06.05.2016, Skopje, Republic of Macedonia

Problems and solutions

Problem 1

Determine all integers $n \geq 1$ for which the number $n^8+n^6+n^4+4$ is prime.
Solution. We use factorization

$$n^8+n^6+n^4+4=(n^4-n^3+n^2-2n+2)(n^4+n^3+n^2+2n+2).$$

The first factor $f(n)$ satisfies

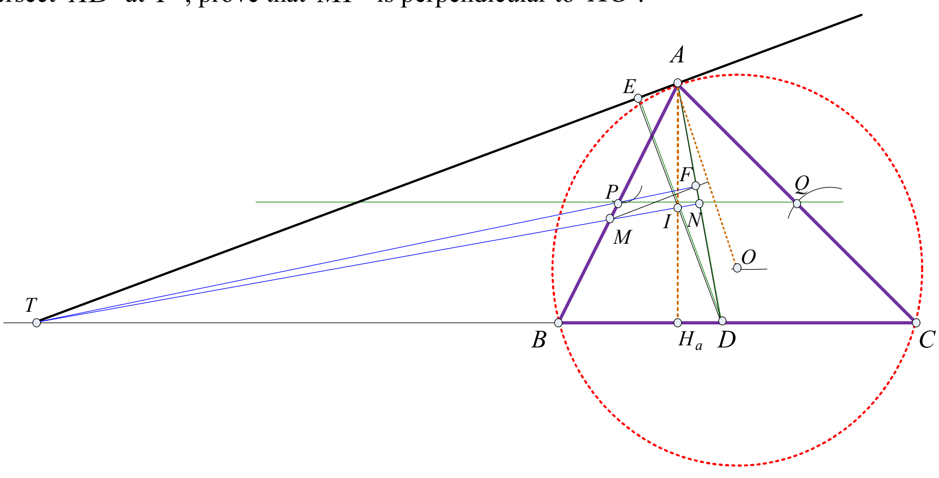
$$f(n)=n^4-n^3+n^2-2n+2=n^3(n-1)+(n-1)^2+1$$

and hence satisfies $f(n) \geq 2$ for all $n \geq 2$. The second factor $g(n)=n^4+n^3+n^2+2n+2$ is strictly greather than 2 for all $n \geq 2$. This only leaves the case $n=1$ as a potential candidate for a prime, and indeed $f(1)g(1)=1 \cdot 7=7$ is prime.

Problem 2

Let ABC be a triangle. D is the foot of the internal bisector of the angle A. The perpendicular from D to the tangent AT (T belong to BC) to the circumscribed circle of ABC intersect the altitude AH_a at the point I (H_a belong to BC).

If P is the midpoint of AB and O is the circumcircle, TI intersect AB at M and PT intersect AD at F, prove that MF is perpendicular to AO.



Solution. Let Q be the midpoint of AC and N the intersection of AD and PQ. Then N is the midpoint of AD. As DE is perpendicular to AT, being E the intersection point of DI

and AT, and as OA is perpendicular to AT, we get that DE is parallel to OA, and so the angles OAN and ADE are equal. As a consequence, triangles ADE and DAH_a are congruent.

In particular angle DAT equals to angle H_aAD, that is, ATD is isosceles and point I is the orthocenter of ABC.

So, TI is perpendicular to AD, and the intersection point of TI and AD is the midpoint of AD (N, say).

The four points M,N,I,T are collinear.

We will apply the Ceva theorem in the triangle APT with the cevians PN, AD and TM. We get

$$\frac{FP}{FT}\cdot 1\cdot\frac{MA}{PM}=1 \qquad \Leftrightarrow \qquad \frac{PF}{TF}=\frac{MP}{MA}.$$

(Observe that NP cut AT in its midpoint).

So, MF is parallel to AT, and form this MF is perpendicular to AO, as claimed.

Problem 3

Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that

$$\sqrt{\frac{b}{a^2+3}}+\sqrt{\frac{c}{b^2+3}}+\sqrt{\frac{a}{c^2+3}}\leq\frac{3}{2}\sqrt[4]{\frac{1}{abc}}.$$

Solution. Putting $\vec{u}=\left(\frac{1}{\sqrt{a^2+3}},\frac{1}{\sqrt{b^2+3}},\frac{1}{\sqrt{c^2+3}}\right)$ and $\vec{v}=(\sqrt{b},\sqrt{c},\sqrt{a})$ in CBS inequality, we get

$$\left(\sqrt{\frac{b}{a^2+3}}+\sqrt{\frac{c}{b^2+3}}+\sqrt{\frac{a}{c^2+3}}\right)^2\leq\left(\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}\right)(a+b+c)=$$

$$=3\left(\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}\right)$$

on account of the constain relation.

We have

$$a^2+3=a\ \ +1+1+1=\geq 4\sqrt[4]{a^2}=4\sqrt{a}.$$

Likewise, we get

$$b^2+3\geq 4\sqrt{b}$$

$$c^2+3\geq 4\sqrt{c}.$$

Therefore,

$$\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}\leq\frac{1}{4}\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)=\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{4\sqrt{abc}}\leq\frac{\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}}{4\sqrt{abc}}\leq$$

$$=\frac{a+b+c}{4\sqrt{abc}}$$

on account of AM-GM inequality.

Combining the proceeding results, we get

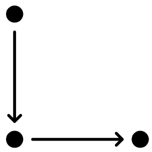
$$\left(\sqrt{\frac{b}{a^2+3}}+\sqrt{\frac{c}{b^2+3}}+\sqrt{\frac{a}{c^2+3}}\right)^2 \leq 3\frac{a+b+c}{4\sqrt{abc}}=\frac{9}{4\sqrt{abc}}$$

from which the statement follows. Equality holds when $a=b=c=1$ and we are done.

Problem 4

Consider a 25×25 chessboard with cells $C(i,j)$ for $1\leq i,j\leq 25$. Find the smallest possible number n of colors with these cells can be colored subject to the following condition: For $1\leq i<j\leq 25$ and for $1\leq s<t\leq 25$, the three cells $C(i,s),C(j,s),C(j,t)$ carry at least two different colors.

Solution. The forbiden is given by



For a 3×3 chessboard, the minimum number is given by 2. Indeed:

1	1	2
1	2	2
2	2	1

If we deal with a 5×5 chessboard, it is sufficient to consider 3 colours:

1	1	2	2	3
1	2	2	3	3
2	2	3	3	1
2	3	3	1	1
3	3	1	1	2

It seems that $m_n=\frac{n+1}{2}$ colurs is sufficient for an $n\times n$ chessboard for any odd n. So we will prove that 13 colours are sufficient for the 25×25 chessboard. We consider the colurs $\{1,2,3,...,11,12,0\}$ and the chessboard coloured as:

1	1	2	2	3	.....	11	11	12	12	0
1	2	2	3	3	.....	11	12	12	0	0
2	2	3	3	4	.....	12	12	0	0	1
2	3	3	4	4	.....	12	0	0	1	1
.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
11	12	12	0	0	.....	8	9	9	10	10
12	12	0	0	1	.....	9	9	10	10	11
12	0	0	1	1	.....	9	10	10	11	11
0	0	1	1	2	.....	10	10	11	11	12

which satisfies the condition at first sight. In fact, it is easy that $C[i,j]=\left[\frac{i+j}{2}\right](\bmod 13)$ for any $1\leq i,j\leq 25$. So if the condition fails, then $C[i,s]=C[j,s]=C[j,t]$ for some $1\leq i<j\leq 25$ and $1\leq s<t\leq 25$, which implies that

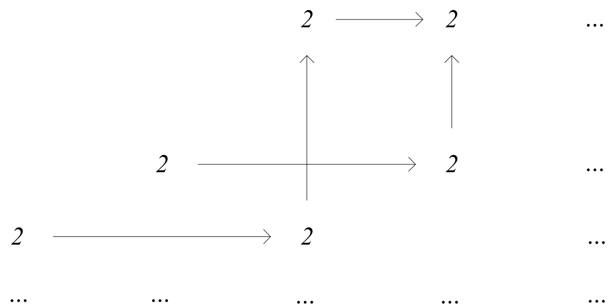
$$\left[\frac{i+s}{2}\right](\bmod 13)=\left[\frac{j+s}{2}\right](\bmod 13)=\left[\frac{j+t}{2}\right](\bmod 13).$$

From $C[i,s]=C[j,s]$ it is clear that $\left[\frac{j+s}{2}\right]=\left[\frac{i+s}{2}\right]$ since

$$0\leq\left[\frac{j+s}{2}\right]-\left[\frac{i+s}{2}\right]<\frac{j+s}{2}-\left(\frac{i+s}{2}-1\right)=\frac{j-i}{2}+1\leq\frac{24}{2}+1=13,$$

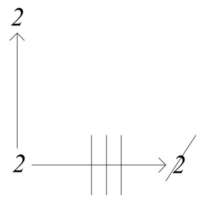
so $\left[\frac{i+s}{2}\right]-\left[\frac{j+s}{2}\right]\leq 12$ and the remainders must coincide. Analogously, from $C[j,s]=C[j,t]$ C we have that $\left[\frac{j+s}{2}\right]=\left[\frac{j+t}{2}\right]$ so we conclude that $\left[\frac{i+s}{2}\right]=\left[\frac{j+t}{2}\right]$, so we conclude that $\left[\frac{i+s}{2}\right]=\left[\frac{j+t}{2}\right]$, which is impossible since $\left[\frac{i+s}{2}\right]\leq\frac{i+s}{2}\leq\frac{j+t}{2}-1<\left[\frac{j+t}{2}\right]$.

Now we prove that 13 colours are necessary. Fix a 25×25 chessboard with a configuration sastifying the condition. We fix any color, for instance colour number 2. We will call 2-cells that



ones which are coloured with colour number 2. The total number of 2-cells will be denoted by c_2. We delete all the colours and only remain 2-cells. From any 2-cell, we draw horizontal arrows from left to right and vertical arrows from down to up joining consecutive 2-cells. These arrows will be called 2-arrows.

Any 2-cell cannot have two or more out-going 2-arrows since otherwise the forbidden configuration would occur:



Therefore, the total number of 2-arrows satisfies $c_2\geq a_2$. It is clear that in any row, if there are k 2-cells, then there are $k-1$ 2-arrows in that row, so the total number of horizontal 2-arrows equals to c_2-25 because there are 25-rows. Analogously, looking at the columns, the total number of vertical 2-arrows also equals to c_2-25. So the total number of 2-arrows is $a_2=2(c_2-25)$ and we obtain $c_2\geq a_2\geq 2c_2-50$, so $c_2\leq 50$. Since there are $25\times 25=625$ cells and $\frac{625}{50}>12$, we need at least 13 colours to get the configuration.

20-th Junior Macedonian Mathematical Olympiad
FON University - Skopje
28.05.2016

1. Solve the equation

$$x_1^4 + x_2^4 + \ldots + x_{14}^4 = 2016^3 - 1.$$

in the set of integers.

2. Let $ABCD$ be a parallelogram and let E, F, G and H be the midpoints of the sides AB, BC, CD and DA, respectively. If $BH \cap AC = I$, $BD \cap EC = J$, $AC \cap DF = K$ and $AG \cap BD = L$, then prove that the quadrilateral $IJKL$ is a parallelogram.

3. A square of dimension 4×4 is given, which consists of 16 squares of side 1. Non-negative integers are filled in each square of dimension 1×1 from the square, so that the sum of any five of them which can be covered with one of the figures in the picture (the figures can be translated and turned over) is 5. How many different numbers can be used to fill in the square?

4. Let x, y, z be positive real numbers. Prove that

$$\sqrt{\frac{xy}{x^2+y^2+2z^2}} + \sqrt{\frac{yz}{y^2+z^2+2x^2}} + \sqrt{\frac{zx}{z^2+x^2+2y^2}} \le \frac{3}{2}.$$

When does equality hold?

5. Solve the equation

$$x + y^2 + (\gcd(x,y))^2 = xy \cdot \gcd(x,y).$$

in the set of natural numbers.

Solutions

1. Solve the equation

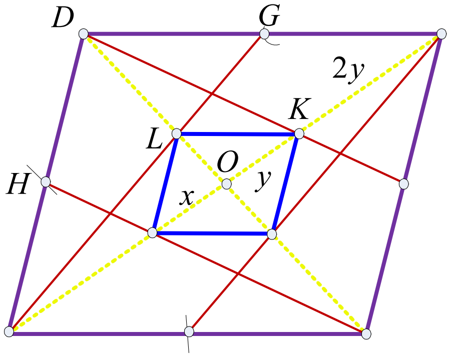
$$x_1^4 + x_2^4 + \ldots + x_{14}^4 = 2016^3 - 1.$$

in the set of integers.

Solution. For $x = 2k$, $x^4 = 16k^4 \equiv 0 \pmod{16}$.

For $x = 2k+1$, $x^4 - 1 = 8k(k+1)(2k^2+2k+1) \equiv 0 \pmod{16}$, i.e. $x^4 \equiv 1 \pmod{16}$. Since $2016^3 - 1 \equiv 15 \pmod{16}$, and the sum of the numbers on the left-hand side never gives a remainder 15 when divided by 16, it follows that the given equation has no solution in the integers.

2. Let $ABCD$ be a parallelogram and let E,F,G and H be the midpoints of the sides AB,BC,CD and DA, respectively. If $BH\cap AC=I$, $BD\cap EC=J$, $AC\cap DF=K$ and $AG\cap BD=L$, then prove that the quadrilateral $IJKL$ is a parallelogram.



Proof. Let $AC\cap BD=O$. Clearly, AO and BH are medians in the triangle ABD, hence I is the centroid of ABD. Similarly K is the centroid of BCD. If $\overline{IO}=x$, then $\overline{AI}=2x$. Similarly, if $\overline{KO}=y$, then $\overline{CK}=2y$. Therefore $3x=\overline{AO}=\overline{CO}=3y$, i.e. $x=y$. We analogously prove that $\overline{JO}=\overline{LO}$. It follows that $IJKL$ is a parallelogram.

3. A square of dimension 4×4 is given, which consists of 16 squares of side 1. Non-negative integers are filled in each square of dimension 1×1 from the square, so that the sum of any five of them which can be covered with one of the figures in the picture (the figures can be translated and turned over) is 5. How many different numbers can be used to fill in the square?

Solution. For each rectangle of dimension 3×4

a	b	c	d
e	f	g	h
i	j	k	l

it holds that $a+(e+f+g+h)=d+(e+f+g+h)=i+(e+f+g+h)=l+(e+f+g+h)$, i.e. $a=d=i=l$.

Let the square be filled as in the picture.

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

Then from the previous discussion it follows that $a=c=m=o$, $b=d=n=p$, $a=d=i=l$ and $e=h=m=p$. Therefore, $a=b=c=d=h=l=p=o=n=m=i=e=X$, the square is of the form

X	X	X	X
X	f	g	X
X	j	k	X
X	X	X	X

and $5X=5$, i.e. $X=1$.
On the other hand, $f+g+3X=j+k+3X=f+j+3X=g+k+3X=5X$, i.e. $f+g=j+k=f+j=g+k=2X$. From the first and fourth, and the second and fourth equation, respectively, it follows that $f=k=Y$ and $j=g=Z$. According to that, the square is of the form

1	1	1	1
1	Y	Z	1
1	Z	Y	1
1	1	1	1

and $Y+Z=2$.

The following cases are possible:

1) $Y=0, Z=2$,
2) $Y=1, Z=1$ and
3) $Y=2, Z=0$.

Therefore, at most 3 different numbers can be used to fill in the square (case 1 or case 3).

4. Let x,y,z be positive real numbers. Prove that

$$\sqrt{\frac{xy}{x^2+y^2+2z^2}}+\sqrt{\frac{yz}{y^2+z^2+2x^2}}+\sqrt{\frac{zx}{z^2+x^2+2y^2}}\leq\frac{3}{2}.$$

When does equality hold?

Solution: We have

$$\sqrt{\frac{xy}{x^2+y^2+2z^2}}+\sqrt{\frac{yz}{y^2+z^2+2x^2}}+\sqrt{\frac{zx}{z^2+x^2+2y^2}}\leq$$

$$\sqrt{\frac{xy}{xy+yz+zx+z^2}}+\sqrt{\frac{yz}{xy+yz+zx+x^2}}+\sqrt{\frac{zx}{xy+yz+zx+y^2}}=$$

$$\sqrt{\frac{xy}{(z+x)(y+z)}}+\sqrt{\frac{yz}{(x+y)(z+x)}}+\sqrt{\frac{zx}{(y+z)(x+y)}}\leq$$

$$\frac{\frac{x}{z+x}+\frac{y}{y+z}}{2}+\frac{\frac{y}{x+y}+\frac{z}{z+x}}{2}+\frac{\frac{z}{y+z}+\frac{x}{x+y}}{2}=$$

$$\frac{\frac{x+y}{x+y}+\frac{y+z}{y+z}+\frac{z+x}{z+x}}{2}=\frac{3}{2}.$$

Equality holds if and only if $x=y=z$.

5. Solve the equation

$$x+y^2+(\gcd(x,y))^2=xy\cdot\gcd(x,y).$$

in the set of natural numbers.

Solution. We introduce the substitution $z=НЗД(x,y)$ and we get the equation $x+y^2+z^2=xyz$. There exist natural numbers a and b such that $x=az$ and $y=bz$. Then the equation gets the form $az+b^2z^2+z^2=abz^3$, i.e. $a+b^2z+z=abz^2$. Since the right-hand side is divisible by z and two of the summands on the left-hand side are divisible by z, it follows that a is divisible by z.

Therefore, there exists a natural number c s.t. $a=cz$. By substituting in the equation, the equation gets the form $cz+b^2z+z=cbz^3$, or $c+b^2+1=cbz^2$. Hence we get $b^2+1=c(bz^2-1)$. It is clear that $bz^2\neq 1$, since if that was not the case we would get $b^2+1=0$, which is impossible. Then we have $c=\frac{b^2+1}{bz^2-1}$. By multiplying the equation by z^2, we get $cz^2=\frac{b^2z^2+z^2}{bz^2-1}=b+\frac{b+z^2}{bz^2-1}$. Since cz^2 is a natural number $\frac{b+z^2}{bz^2-1}$ is also a natural number. Therefore, $bz^2-1\leq b+z^2$, i.e.

$$\left(z^2-1\right)(b-1)\leq 2 \text{ ....(1)}$$

If $b=1$, then $c=\frac{2}{z^2-1}$ and hence $z^2=2$ or $z^2=3$, which is impossible. If $b=2$, then $c=\frac{5}{2z^2-1}$. If $2z^2-1=1$, then $z=1$. It follows that $c=5$, $a=5$, i.e. $x=5$ and $y=2$. If $2z^2-1=5$, then $z^2=3$, which is impossible. If $b=3$, then $c=\frac{10}{3z^2-1}$. The cases $3z^2-1=1$, $3z^2-1=5$ and $3z^2-1=10$ are impossible. If $3z^2-1=2$, then $z=1$. It follows that $c=5$, $a=5$, i.e. $x=5$ and $y=3$. If $b>3$ then from (1) it follows that $z=1$. Then $c=\frac{b^2+1}{b-1}=b+1+\frac{2}{b-1}$, from where we have $b=2$ or $b=3$, which contradicts the assumption that $b>3$. Therefore the solutions to the equation are $(x,y)=\{(5,2),(5,3)\}$.

20-th Junior Balkan mathematical Olympiad
24.06.-29.06.2016, Slatina, Romania

Problem 1

A trapezoid $ABCD$ ($AB \| CD$, $AB > CD$) is cicumscribed. The incircle of the triangle ABC touches the lines AB and AC at the points M and N , respectively. Prove that the incenter of the trapezoid $ABCD$ lies on the line MN .

Problem 2

Let a,b and c be positive real number. Prove that

$$\frac{8}{(a+b)^2+4abc}+\frac{8}{(b+c)^2+4abc}+\frac{8}{(c+a)^2+4abc}+a^2+b^2+c^2 \geq \frac{8}{a+3}+\frac{8}{b+3}+\frac{8}{c+3}.$$

Problem 3

Find all the triples of integers (a,b,c) such that the number

$$N=\frac{(a-b)(b-c)(c-a)}{2}+2$$

is a power of 2016 .

(A power of 2016 is an integer of the form 2016^n , where n is a non-negative integer).

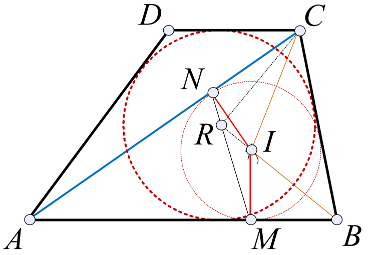
Problem 4

A 5×5 table is called *regular* if each of its cells contains one of four pairwise distinct real numbers, such that each of them occurs exactly once in every 2×2 subtable. The sum of all numbers of a *regular table* is called the *total sum* of the table. With any four numbers, one constructs all possible regular tables, computers their total sums and counts the distinct outcomes. Determine the maximum possible count.

Solutions

Problem 1

A trapezoid $ABCD$ ($AB \| CD$, $AB > CD$) is cicumscribed. The incircle of the triangle ABC touches the lines AB and AC at the points M and N , respectively. Prove that the incenter of the trapezoid $ABCD$ lies on the line MN .



Solution

Version 1. Let I be the incenter of triangle ABC and R be the common point of the lines BI and MN . Since

$m(\widehat{ANM})=90^\circ-\frac{1}{2}m(\widehat{MAN})$ and $m(\widehat{BIC})=90^\circ+\frac{1}{2}m(\widehat{MAN})$

the quadrilateral $IRNC$ is cyclic. (1)

It follows that $m(\widehat{BRC})=90^\circ$ and therefore

$$m(\widehat{BCR})=90^\circ-m(\widehat{CBR})=90^\circ-(180^\circ-m(\widehat{BCD})=\frac{1}{2}m(\widehat{BCD}) \quad (2)$$

So, (CR is the angle bisector of $\widehat{DCB}$ and R cis the incenter of the trapezoid. (3)

Versions 2. If R is the incentre of the trapezoid $ABCD$, then B,I and R are collinear, (1')

and $m(\widehat{BRC})=90^\circ$. (2')

The quadrilateral $IRNC$ is cyclic. (3')

Then $m(\widehat{MNC})=90^\circ+\frac{1}{2}\cdot m(\widehat{BAC})$ (4')

and $m(\widehat{RNC})=m(\widehat{BIC})=90^\circ+\frac{1}{2}\cdot m(\widehat{BAC})$, (5')

so that $m(\widehat{MNC})=m(\widehat{RNC})$ and the points M,R and N are collinear. (6')

Version 3. If R is the incentre of the trapezoid $ABCD$, let $M'\in(AB)$ and $N'\in(AC)$ be the unique points, such that $R\in M'N'$ and $(AM')\equiv(AN')$. (1")

Let S be the intersection point of CR and AB. Then $CR=RS$. (2")

Consider $K\in AC$ such that $SK\parallel M'N'$. Then N' is the midpoint of (CK). (3")

We deduce

$$AN'=\frac{AK+KC}{2}=\frac{AS+AC}{2}=\frac{AB-BS+AC}{2}=\frac{AB+AC-BC}{2}=AN. \quad (4")$$

We conclude that $N=N'$, hence $M=M'$, and R,M,N are collinear. (5")

Problem 2

Let a,b and c be positive real number. Prove that

$$\frac{8}{(a+b)^2+4abc}+\frac{8}{(b+c)^2+4abc}+\frac{8}{(c+a)^2+4abc}+a^2+b^2+c^2\geq\frac{8}{a+3}+\frac{8}{b+3}+\frac{8}{c+3}.$$

Solution. Since $2ab\leq a^2+b^2$, it follows that $(a+b)^2\leq 2(a^2+b^2)$ (1)

and $4abc\leq 2c(a^2+b^2)$, for any positive reals a,b,c. (2)

Adding these inequalities, we find

$$(a+b)^2+4abc\leq 2(a^2+b^2)(c+1), \quad (3)$$

so that

$$\frac{8}{(a+b)^2+4abc}\geq\frac{4}{(a^2+b^2)(c+1)}. \quad (4)$$

Using the AM-GM inequality, we have

$$\frac{4}{(a^2+b^2)(c+1)}+\frac{a^2+b^2}{2}\geq 2\sqrt{\frac{2}{c+1}}=\frac{4}{\sqrt{2(c+1)}} \quad (5)$$

respectively

$$\frac{c+3}{8}=\frac{(c+1)+2}{8}\geq\sqrt{\frac{2(c+1)}{4}}. \quad (6)$$

We conclude that

$$\frac{4}{(a^2+b^2)(c+1)}+\frac{a^2+b^2}{2}\geq\frac{8}{c+3}, \quad (7)$$

and finally

$$\frac{8}{(a+b)^2+4abc}+\frac{8}{(b+c)^2+4abc}+\frac{8}{(c+a)^2+4abc}+a^2+b^2+c^2\geq\frac{8}{a+3}+\frac{8}{b+3}+\frac{8}{c+3}. \quad (8)$$

Problem 3

Find all the triples of integers (a,b,c) such that the number

$$N=\frac{(a-b)(b-c)(c-a)}{2}+2$$

is a power of 2016.

(A power of 2016 is an integer of the form 2016^n, where n is a non-negative integer).

Solution. Let a,b,c be integers and n be a positive integer such that

$(a-b)(b-c)(c-a)+4=2\cdot 2016^n$.

We set $a-b=-x$, $b-c=-y$ and we rewrite the equation as

$$xy(x+y)+4=2\cdot 2016^n . \qquad (1)$$

If $n>0$, then the right hand side is divisible by 7 , so we have that

$$xy(x+y)+4\equiv 0 \pmod 7 \qquad (2)$$

or

$$3xy(x+y)\equiv 2 \pmod 7 \qquad (3)$$

or

$$(x+y)^3-x^3-y^3\equiv 2 \pmod 7 . \qquad (4)$$

Note that, by Fermat's Little Theorem, for any integer k the cubic residues are $k^3\equiv -1,0,1 \pmod 7$. (5)

It follows that in (1) some of $(x+y)^3, x^3$ and y^3 should be divisible by 7 .

But in this case, $xy(x+y)$ is divisible by 7 and this is a contradiction. (6)

So, the only possibility is to have $n=0$ and consequently, $xy(x+y)+4=2$, or, equivalently, $xy(x+y)+4=-2$. (7)

The solutions for this are $(x,y)\in\{(-1,-1),(2,-1),(-1,2)\}$, (8)

so the required triples are $(a,b,c)=(k+2,k+1,k)$, $k\in\mathbb{Z}$, an all their cyclic permutations. (9)

Alternative version: If $n>0$ then 9 divides $(a-b)(b-c)(c-a)+4$, that is, the equation $xy(x+y)+4\equiv 0 \pmod 9$ has the solution $x=b-a, y=c-b$. (1')

But then x and y have to be 1 modulo 3, implying $xy(x+y)\equiv 2 \pmod 9$, which is a contradiction.

We can continue now as in the first version.

Problem 4

A 5×5 table is called *regular* if each of its cells contains one of four pairwise distinct real numbers, such that each of them occurs exactly once in every 2×2 subtable. The sum of all numbers of a *regular table* is called the *total sum* of the table. With any four numbers, one constructs all possible regular tables, computers their total sums and counts the distinct outcomes. Determine the maximum possible count.

Solution. We will prove that the maximum number of total sums is 60 .

The proof is based on the following claim.

Claim. In a regular table either each row contains exactly two of the numbers, or each column contains exactly two of the numbers.

Proof of the Calim. Indeed, let R be the a row containing at least three of the numbers. Then, in row R we can find three of the numbers in consecutive position, let x,y,z be the numbers in consecutive positions (where $\{x,y,s,z\}=\{a,b,c,d\}$). Due to our hypothesis that in every 2×2 subarray each number is used exactly once, in the row, above R (if there is such a row), precisely above the numbers x,y,z will be the numbers z,t,x in this order. And above them will be the numbers x,y,z in this order. The same happens in the rows below R (see at the following figure).

$$\begin{pmatrix} \bullet & x & y & z & \bullet \\ \bullet & z & t & x & \bullet \\ \bullet & x & y & z & \bullet \\ \bullet & z & t & x & \bullet \\ \bullet & x & y & z & \bullet \end{pmatrix}$$

Completing all the array, it easily follows that each column contains exactly two of the numbers and our claim is proven. (1)

Rotating the matrix (if it is necessary), we may assume that each row contains exactly two of the numbers. If we forget the first row and column from the array, we obtain 4×4 array, that can be divided into four 2×2 subarrays, containing thus each number exactly four times, with a total sum of $4(a+b+c+d)$.

It suffices to find how many different ways are there to put the numbers in the first row R_1 and the first column C_1. (2)

Denoting by a_1,b_1,c_1,d_1 the number of appearances of a,b,c and respectively d in R_1 and C_1, the total sum of the numbers in the entire 5×5 array will be

$$S=4(a+b+c+d)+a_1\cdot a+b_1\cdot b+c_1\cdot c+d_1\cdot d\ . \qquad (3)$$

In the first, the third and the fifth row contain the numbers x,y with x denoting the number at the entry $(1,1)$, then the second and the fourth row will contain only the numbers z,t, with z denoting the number at the entry $(2,1)$. Then $x_1+y_1=7$ and $x_1\geq 3$, $y_1\geq 2$, $z_1+t_1=2$, and $z_1\geq t_1$. Then $\{x_1,y_1\}=\{5,2\}$ or $\{x_1,y_1\}=\{4,3\}$, respectively $\{z_1,t_1\}=\{2,0\}$ or $\{z_1,t_1\}=\{1,1\}$. (4)

Then $\{a_1,b_1,c_1,d_1\}$ is obtained by permuting one of the following quadriples:

$$(5,2,2,0)\ ,\ (5,2,1,1)\ ,\ (4,3,2,0)\ ,\ (4,3,1,1)\ . \qquad (5)$$

There are a total of $\frac{4!}{2!}=12$ permutations of $(5,2,2,0)$, also 12 permutations of $(5,2,1,1)$, 24 permuitations of $(4,3,2,0)$ and finally, there are 12 permutations of $(4,3,1,1)$. Hence, there are at most 60 different possible total sums. (6)

We can obtain indeed each of these 60 combinations: take three rows *ababa* alternating with two rows *cdcdc* to get $(5,2,2,0)$; take three rows *ababa* alternating with one row *cdcdc* and a row (*dcdcd*) to get $(5,2,1,1)$; take three rows *ababc* alternating with two rows *cdcda* to get $(4,3,2,0)$; take three rows *abcda* alternating with two rows *cdabc* to get $(4,3,1,1)$. (7)

By choosing for example $a=10^3, b=10^2, c=10, d=1$, we can make all these sums different. (8)

Hence, 60 is indeed the maximum possible number of different sums.

Alternative version. Consider a regular table containing the four distinct numbers a,b,c,d. The four 2×2 corners contain each all the four numbers, so that, if a_1,b_1,c_1,d_1 are the numbers of appearances of a,b,c and respectively d in the middle row and column, then

$$S=4(a+b+c+d)+a_1\cdot a+b_1\cdot b+c_1\cdot c+d_1\cdot d\ . \qquad (1')$$

Consider the numbers x in position $(3,3)$, y in position $(3,2)$, y' in position $(3,4)$, z in position $(2,3)$ and z' in position $(4,3)$.

If $z\neq z'=t$, then $y=y'$, and in position $(3,1)$ and $(3,5)$ there will be the number x. (2’)

The second and fourth row can only contain now the numbers z and t, respectively the forst and fifth row only x and y. (3’)

Then $x_1+y_1=7$ and $x_1\geq 3$, $y_1\geq 2$, $z_1+t_1=2$, and $z_1\geq t_1$. Then $\{x_1,y_1\}=\{5,2\}$ or $\{x_1,y_1\}=\{4,3\}$, respectively $\{z_1,t_1\}=\{2,0\}$ or $\{z_1,t_1\}=\{1,1\}$. (4’)

One can continue now as in the first version.

SEEMOUS REGULATIONS
http://www.massee-org.eu/images/seemous/SEEMOUS_2016_regulations.pdf

These regulations were approved by the MASSEE (Mathematical Society of South Eastern Europe) on April 1, 2006.

1. The aims of the SEEMOUS include:
a. The challenging, encouragement and development of mathematically gifted higher education students in all participating institutions and corresponding countries;
b. The fostering of friendly relationships among higher education students and educators of the participating institutions;
c. The creation of opportunities for the exchange of information on higher education syllabi and the development of partnerships and networks between the participating institutions;
d. The development of young researchers in mathematics and its applications.
2. The official language of the SEEMOUS is English.
3. The SEEMOUS is organized once every year within the first 15 days of the month of March.
4. Countries or universities interested to host SEEMOUS should apply to MASSEE at least 15 months before the date of organization.
5. The SEEMOUS Jury shall consist of all leaders of the participating teams representing an institution.
6. New participants have to be accepted by MASSEE at least three months before participation.
7. Teams represent institutions but results per country will be computed for teams of six students made up by the best six scores of students participating from each country, except if a National team is officially participating. The National teams have to be specified, in writing, by the national Mathematical Society or the Ministry of Education of the country.
8. All decisions by the Jury are based on simple majority unless it is otherwise specified. The Chairman may vote only when a tie break is needed.
9. The Jury could decide to suggest changes to the regulations. Suggestions are submitted to MASSEE by the chairman of the Jury for changes to be applicable from the next Olympiad. Changes in the regulations can only be approved by MASSEE.
10. Deputy leaders may participate in the Jury and they may also replace their leaders in his/her absence.
11. Each participating institution has one vote regardless of the size of their team.
12. National teams can participate with students not representing institutions or a mixture. If a national team participates with a leader then the leader becomes a member of the jury and has one vote.
13. The minutes of the Jury meeting a re approved at the last meeting of the jury and before the closing of SEEMOUS. The Chairman of the Jury of the SEEMOUS is obliged to give the minutes of the Jury meetings to all leaders and to send them to the MASSEE Council.
14. The Jury may consider and decide on any matter raised, which is not covered by any other regulation item, provided that such decision does not violate the constitution of MASSEE.
15. Additional regulations may be added by the Jury, in which case at least two thirds majority is needed. New regulations become effective beginning the next SEEMOUS, provided they are approved by the MASSEE Council.
16. The Chairman of the Jury may call as many day meetings as he/she deems necessary during an SEEMOUS or when at least one third of the participating in stitutions or national representations request an additional Jury meeting.
17. Proposals to host an SEEMOUS are discussed during a Jury meeting and recommended to MASSEE by the Jury in an order of preference. The MASSEE shall always approve the host countries/institutions of the next two SEEMOUS.