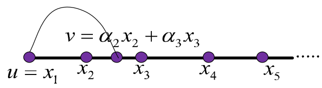
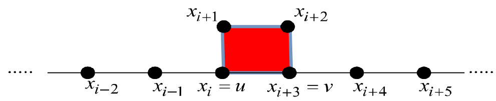
.Parallel to this we have vector $v = x_3$ for which we ask the question whether it can be a loop of a loop 2-subspace generated by the kernel 2-sub-space which is already generated by the element x_3. Of course the answer is yes, it is a loop of the kernel 2-subspace generated by x_1, x_2, x_3 and the 2-vector (x_3, x_4).



Sub case 6. $u = x_i$, $v = x_j$ where $j = i+3$.



In this sub case we will consider two situations, i.e.:

a) $i = 1$, and b) $i > 1$

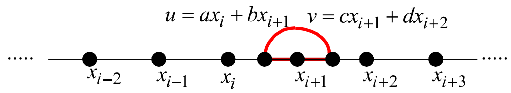
Situation a)

If $i = 1$, then we have a cyclic 2-subspace generated by the 2-vectors $(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_1)$. Then this 2-subspace is followed by loop-connecting 2-subspace with a loop in x_4. It is generated by the 2-vectors $(x_3, x_4), (x_1, x_4), (x_5, x_4)$. Then, from x_5 on, is followed a branch 2-subspace, which is one sided.

Situation b)

This situation is completely analogous to the situation a), and here the role of the vector x_1 takes the vector x_i. Additionally, we have that the vectors $x_1, x_2, ..., x_{i-1}$ form a branch 2-subspace, which is a subspace from M'.

Sub case 7. $u = ax_i + bx_{i+1}$, $v = cx_{i+1} + dx_{i+2}$ where $ab \neq 0$ and $cd \neq 0$



In this sub case we have that the 2-vectors (v,u) and (x_{i+1}, u) belong in the new 2-subspace M', so, in this 2-subspace belongs also the 2-vector

$$\begin{bmatrix} \frac{1}{d} & 0 \\ 0 & 1 \end{bmatrix} \left((v,u) + \begin{bmatrix} -c & 0 \\ 0 & 1 \end{bmatrix} (x_{i+1}, u) \right) = \begin{bmatrix} \frac{1}{d} & 0 \\ 0 & 1 \end{bmatrix} \left((cx_{i+1} + dx_{i+2}, u) + (-cx_{i+2}, u) \right) = \begin{bmatrix} \frac{1}{d} & 0 \\ 0 & 1 \end{bmatrix} (dx_{i+2}, u) = (x_{i+2}, u)$$

Now it is clear that we have 2-subspace which is fully analogous to the 2-subspace which is generated as in sub case 4 from this case, which is equivalent with the sub case 2, and this sub case is fully described. So, the 2-vectors $(x_{i+1}, x_{i+2}), (x_{i+2}, x_i), (x_i, x_{i+1})$ all belong in M', where from we get that the kernel 2-subspace generated by them is also a subspace from M'. So, the kernel 2-subspace generated by $(x_{i+1}, v), (v, u), (u, x_{i+1})$ is consisted both in the kernel 2-subspace generated by $(x_{i+1}, x_{i+2}), (x_{i+2}, x_i), (x_i, x_{i+1})$, and in M'. In any case, we have 2-subspace determined with

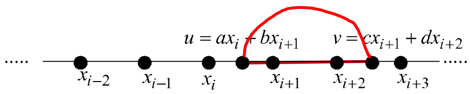
$M' = M \cup L^2(x_i, x_{i+1}, x_{i+2})$.

In some form we should consider the role of the vectors x_i and x_{i+2}. The vector x_{i+2} in the case $i=1$ is also a loop element of the 2-subspace generated from any two linearly independent elements from the kernel 2-subspace and the vector x_{i+3}.

In this case it is certain that $x_i = x_1$, then M' begins with kernel 2-subspace, as already described, and continues through the loop x_{i+2} in a branch 2-subspace. If $i=2$, then in M' we have two loops x_i and x_{i+2}, i.e. two loop 2-subspaces, one of them a kernel 2-subspace and the other one is one branch 2-subspace. If we have $i \geq 3$, then we have two branch 2-subspaces, one of them is a kernel 2-subspace and two loop 2-subspace.

Sub case 8. $u = ax_i + bx_{i+1}$ **,** $v = cx_{i+2} + dx_{i+3}$ where $ab \neq 0$**,** $cd \neq 0$ and $i > 1$**.**

In this situation the vectors u, x_{i+1}, x_{i+2}, v are four linearly independent elements.



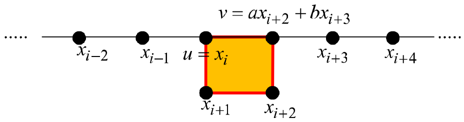
Here, all four pairs of elements $(u, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, v), (v, u)$ belong in the new 2-subspace M' and they for themselves form a cyclic 2-subspace, which at the same time is 2-subspace which is also in X^2 and is a part of M'. Now, it is not clear whether we can consider the elements u and v everyone of them separately for loops of two 2-subspces.

Sub case 9. $u = ax_1 + bx_2$, $v = cx_3 + dx_4$, where $ab \neq 0$, $cd \neq 0$

This sub case is fully described in sub case 7 from this case.

Sub case 10. $u = x_i$, $v = ax_{i+2} + bx_{i+3}$ **,** $i > 1$

In this situation we have a sub case which is analogous to the previous case. But, now since $i > 1$, we get that the vectors $x_1, x_2, ..., x_{i-1}$ are linearly independent and



the pairs of 2-vectors $(x_1, x_2), (x_2, x_3), ..., (x_{i-2}, x_{i-1})$ form finite branch 2-subspace. Additionally, in relation to the previous sub case, the vector $u = x_i$ will be a loop element of a loop 2-subspace.

Sub case 11. $u = x_i$, $v = \alpha_j x_j + \alpha_{j+1} x_{j+1} + \alpha_{j+2} x_{j+2} + \alpha_{j+3} x_{j+3}$, where i can be any positive integer, as well as j, and $\alpha_j \alpha_{j+3} \neq 0$.

Here, we are interested only the end cases, because the remaining case is the same as for the two-sided branch.

Situation 1. $u = x_1$, $v = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, where $\alpha_1 \alpha_4 \neq 0$.

The vector v is not a coordinate of any 2-vector from M. Enough reason for this is the fact that $\alpha_1, \alpha_4 \neq 0$. According to this, the branch generated by $x_1, x_2, x_3, ...$ is simply supplemented with a first element $v = x_\circ$, and that will be a new branch $v = x_\circ, x_1, x_2, x_3, ...$.

Situation 2. $u = x_2$, $v = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, where $\alpha_1 \alpha_4 \neq 0$.

From the condition $\alpha_1, \alpha_4 \neq 0$, we have that the vector v is not a coordinate of any 2-vector from M. According to this, the 2-vectors $(x_1, x_2), (v, x_2), (x_3, x_2)$ form a loop 2-subspace S' which has the form

$$S' = \bigcup_{w \in L(x_1, v, x_3)} L(w, x_2) \times L(w, x_2).$$

The vector x_4 is ending vector of an one-sided infinite branch, i.e. $x_4, x_5, \ldots$. We will denote that 2-subspace with S. So,

$$M' = S' \cup S.$$

Situation 3. $u = x_3$, $v = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, where $\alpha_1 \alpha_4 \neq 0$,

From the condition $\alpha_1, \alpha_4 \neq 0$, we have that the vector v is not a coordinate of any 2-vector from M. Now, on the vector $u = x_3$, as basic vector, one 2-subspace S' is generated and it is a loop 2-subspace. Its generator elements are $(u, x_2), (u, x_4), (u, v)$. The vector (x_1, x_2), which is attached to itself in this loop 2-subspace, which is trivial. That is the subspace $S'' = \{A(x_1, x_2) / A \in M_2(\Phi)\}$. From the other side the vectors $x_4, x_5, x_6, x_7, \ldots$ form one-sided branch 2-subspace S''' which is also attached to the loop 2-subspace which is previously described. According to this, in this case the extension is

$$M' = S' \cup S'' \cup S'''$$

Situation 4. $u = x_4$, $v = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, where $\alpha_1 \alpha_4 \neq 0$,

In this situation we have similar position as in the previous three cases. Because of the condition $\alpha_1 \alpha_4 \neq 0$, the vector v can not be a coordinate in any vector from M. Now, the vector $u = x_4$ will become loop centre of the three 2-vectors $(u, x_3), (u, x_5), (u, v)$ which form a loop 2-subspace S'. The previous 2-vectors $(x_1, x_2), (x_2, x_3)$ form finite branch 2-subspace which will be denoted with S'', and the vectors from the sequence $x_5, x_6, \ldots$ form branch 2-subspace which will be denoted with S'''. Finally, this 2-subspace will be

$$M' = S' \cup S'' \cup S'''.$$

Situation 5. $u = x_5$, $v = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, where $\alpha_1 \alpha_4 \neq 0$,

This situation is almost identical to the previous situation. Here, the starting branch 2-subspace S'' is generated by the 2-vectors $(x_1, x_2), (x_2, x_3), (x_{3,4}, x_4)$. The vector $u = x_5$ is a vector which is a loop element for the new 2-subspace M', which will be denoted as S'. It is a basis of the 2-vectors $(x_4, u), (x_6, u), (v, u)$. On the other hand, the vectors $x_6, x_7, \ldots$ form a branch 2-subspace S'''. So,

$$M' = S' \cup S'' \cup S'''.$$

Situation 6. $u = x_1$, $v = \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5$, where $\alpha_2 \alpha_5 \neq 0$,

The proof is trivial. It is the same as if to the vector x_1 is added one vector, and now the vectors $v, x_1, x_2, x_3, \ldots$ form a branch 2-subspace. Certainly, the vector v does not belong in any coordinate of the 2-vector from M'.

Situation 7. $u = x_1$, $v = \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5 + \alpha_6 x_6$, where $\alpha_3 \alpha_6 \neq 0$, The proof is trivial. It is the same as if to the vector x_1 is added one vector, and now the vectors $v, x_1, x_2, x_3, ...$ form a branch 2-subspace. Certainly, the vector v does not belong in any coordinate of the 2-vector from M'.

In this situation we have one vector which is a coordinate of a 2-vector from M and one vector which is not a coordinate of a 2-vector from M. The discussion here is same as in all other cases. The rest of the cases. i.e. when we have indexes i and $j, j+1, j+2, j+3$, where $i<j$ we have totally analogous situation as in the situation 7, which is true for every $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}+\alpha_{j+3}x_{j+3}$, $j>3$.

Sub case 11'. $u=x_i$, $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}+\alpha_{j+3}x_{j+3}$, $i>1$ where i can be any positive integer greater than 1, same as j, where $\alpha_j\alpha_{j+3}\neq 0$.

We should mention here that one vector u is always a coordinate of 2-vector from M and the other vector v is not a coordinate of none of the 2-vectors from M .

Here, we are interested only the ending cases, because the rest of the situations are same as the ones for the two-sided branch.

Situation 1. $u=x_i$, $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$, where $\alpha_{i-1}\alpha_{i+2}\neq 0$

Situation 2. $u=x_i$, $v=\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}+\alpha_{i+3}x_{i+3}$, where $\alpha_i\alpha_{i+3}\neq 0$

Situation 3. $u=x_i$, $v=\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}+\alpha_{i+3}x_{i+3}+\alpha_{i+4}x_{i+4}$, where $\alpha_{i+1}\alpha_{i+4}\neq 0$,

Situation 4. $u=x_i$, $v=\alpha_{i+2}x_{i+2}+\alpha_{i+3}x_{i+3}+\alpha_{i+4}x_{i+4}+\alpha_{i+5}x_{i+5}$, where $\alpha_{i+2}\alpha_{i+5}\neq 0$,

Situation 5. $u=x_{i+1}$, $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$, where $\alpha_{i-1}\alpha_{i+2}\neq 0$,

Situation 6. $u=x_{i+2}$, $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$, where $\alpha_{i-1}\alpha_{i+2}\neq 0$,

Situation 7. $u=x_{i+3}$, $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$, where $\alpha_{i-1}\alpha_{i+2}\neq 0$,

Situation 8. $u=x_{i+4}$, $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$, where $\alpha_{i-1}\alpha_{i+2}\neq 0$,

The discussion are as in all the other cases. The rest of the cases, i.e. when we have indexes i and $j, j+1, j+2, j+3$, where $i<j$ we have totally analogous situation as in situation 4, which is true for every $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}+\alpha_{j+3}x_{j+3}$, $j>i$

Sub case 12. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}+\alpha_{j+3}x_{j+3}$, $i>1$ where $\alpha_j\alpha_{j+3}\neq 0$.

This case is possible because the element v is not a coordinate of none of the elements of the 2-subspace M , but it is still element of the vector space X . Same case can be considered also for a vector v, in the form $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}+\alpha_{j+3}x_{j+3}+...+\alpha_{j+\kappa}x_{j+\kappa}$, for any k which is bigger than 3, but here $\alpha_j\alpha_{j+k}\neq 0$. There is no essential difference from this aspect.

In all this situations, for the vector v, we have $\alpha_{i-1}\alpha_{i+2}\neq 0$, $\alpha_i\alpha_{i+3}\neq 0$, $\alpha_{i+1}\alpha_{i+4}\neq 0$, $\alpha_{i+2}\alpha_{i+5}\neq 0$ and $\alpha_{i-1}\alpha_{i+2}\neq 0$. With this it is generally obtained that this vector in neither of this situations can not belong. i.e. to be a coordinate of any 2-vector from M .

Situation 1. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$, where $\alpha_{i-1}\alpha_{i+2}\neq 0$.

The vector $u = ax_i + bx_{i+1}$,belongs to the 2-subspace $S = \left\{ A(x_i, x_{i+1}) / A = \begin{bmatrix} \alpha_{i-1} & 0 \\ 0 & \alpha_i \end{bmatrix} \right\}$

Since v doesn't belong in M , this situation is like the sub case 2 of case 2.

Situation 2. $u = \alpha_i x_i + \alpha_{i+1} x_{i+1}$, $v = \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2} + \alpha_{i+3} x_{i+3}$, where $\alpha_i \alpha_{i+3} \neq 0$.

The vector $u = ax_i + bx_{i+1}$, belongs to the 2-subspace $S = \left\{ A(x_i, x_{i+1}) / A = \begin{bmatrix} \alpha_i & 0 \\ 0 & \alpha_{i+1} \end{bmatrix} \right\}$. Since v doesn't belong in M , this situation is like the sub case 2 of case 2.

Situation 3. $u = \alpha_i x_i + \alpha_{i+1} x_{i+1}$, $v = \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2} + \alpha_{i+3} x_{i+3} + \alpha_{i+4} x_{i+4}$,where $\alpha_{i+1} \alpha_{i+4} \neq 0$

The vector $u = ax_i + bx_{i+1}$, belongs to the 2-subspace $S = \left\{ A(x_i, x_{i+1}) / A = \begin{bmatrix} \alpha_i & 0 \\ 0 & \alpha_{i+1} \end{bmatrix} \right\}$. Since v doesn't belong in M , this situation is like the sub case 2 of case 2.

Situation 4. $u = \alpha_i x_i + \alpha_{i+1} x_{i+1}$, $v = \alpha_{i+2} x_{i+2} + \alpha_{i+3} x_{i+3} + \alpha_{i+4} x_{i+4} + \alpha_{i+5} x_{i+5}$, where $\alpha_{i+2} \alpha_{i+5} \neq 0$.

Same as in situation 3.

Situation 5. $u = \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$, $v = \alpha_{i-1} x_{i-1} + \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$, where $\alpha_{i-1} \alpha_{i+2} \neq 0$.

Same as in situation 3. Here, for the vector $u = \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$ belongs in the 2-subspace $S = \left\{ A(x_{i+1}, x_{i+2}) / A = \begin{bmatrix} \alpha_{i+1} & 0 \\ 0 & \alpha_{i+2} \end{bmatrix} \right\}$.

Situation 6. $u = \alpha_{i+2} x_{i+2} + \alpha_{i+3} x_{i+3}$, $v = \alpha_{i-1} x_{i-1} + \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$, where $\alpha_{i-1} \alpha_{i+2} \neq 0$.

Same as in situation 3. Here, for the vector $u = \alpha_{i+2} x_{i+2} + \alpha_{i+3} x_{i+3}$ belongs in the 2-subspace $S = \left\{ A(x_{i+2}, x_{i+3}) / A = \begin{bmatrix} \alpha_{i+2} & 0 \\ 0 & \alpha_{i+3} \end{bmatrix} \right\}$.

Situation 7. $u = \alpha_{i+3} x_{i+3} + \alpha_{i+4} x_{i+4}$, $v = \alpha_{i-1} x_{i-1} + \alpha_i x_i + \alpha_{i+1} x_{i+1} + \alpha_{i+2} x_{i+2}$, where $\alpha_{i-1} \alpha_{i+2} \neq 0$.

Same as in situation 3. Here, for the vector $u = \alpha_{i+3} x_{i+3} + \alpha_{i+4} x_{i+4}$ belongs in the 2-subspace $S = \left\{ A(x_{i+3}, x_{i+4}) / A = \begin{bmatrix} \alpha_{i+3} & 0 \\ 0 & \alpha_{i+4} \end{bmatrix} \right\}$.

Sub case 12'. $u = \alpha_1 x_1 + \alpha_2 x_2$, $v = \alpha_j x_j + \alpha_{j+1} x_{j+1} + \alpha_{j+2} x_{j+2} + \alpha_{j+3} x_{j+3}$, where $\alpha_j \alpha_{j+3} \neq 0$.

In all next situations of this sub case we have that the vector belongs in the 2-subspace generated from one element, i.e. from the element $u = \alpha_i x_i + \alpha_{i+1} x_{i+1}$. The vector $v = \alpha_j x_j + \alpha_{j+1} x_{j+1} + \alpha_{j+2} x_{j+2} + \alpha_{j+3} x_{j+3}$ is not a coordinate of none of the vectors in the 2-subspace M' . Here, the vectors x_1, v, x_2 are linearly independent. In the beginning of the branch they form a loop 2-subspace. The sequence of vectors $x_3, x_4, x_5, x_6, ...$ form a branch 2-subspace. According to this, for any of the next four possible situations, we have that the 2vector subspace has form $M' = S' \cup S''$,

where $S' = \bigcup_{w \in L(x_1, v, x_2)} L(w,u) \times L(w,u)$, a S'' e $S'' = \bigcup_{k=4}^{+\infty} \bigcup_{\alpha_{i-1}, \alpha_{i+1}} L^2(\alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1}, x_2)$.

Situation 1. $u = \alpha_1 x_1 + \alpha_2 x_2$, $v = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4$, where $\beta_1 \beta_4 \neq 0$

Situation 2. $u=\alpha_1x_1+\alpha_2x_2$, $v=\beta_2x_2+\beta_3x_3+\beta_4x_4+\beta_5x_5$, where $\beta_2\beta_5\neq 0$

Situation 3. $u=\alpha_1x_1+\alpha_2x_2$, $v=\beta_3x_3+\beta_4x_4+\beta_5x_5+\beta_6x_6$, where $\beta_3\beta_6\neq 0$

Situation 4. $u=\alpha_1x_1+\alpha_2x_2$, $v=\beta_4x_4+\beta_5x_5+\beta_6x_6+\beta_7x_7$, where $\beta_4\beta_7\neq 0$

In all situations from situation 2 to situation 4 we have exactly the same position: one vector that belongs to the starting branch 2-subspace. The second vector is not a coordinate of none of the 2-vectors that belong in the starting 2-subspace. Everywhere we have a situation in which the sub case from case 2 is repeating.

Sub case 13. $u=\alpha_ix_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}+\alpha_{i+3}x_{i+3}+...+\alpha_{i+k}x_{i+k}$ and $v=\alpha_jx_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}+\alpha_{j+3}x_{j+3}+...+\alpha_{j+s}x_{j+s}$, where $k,s\geq 3$. Here i and j can be any positive integers.

In this situation, nether the vector u nor the vector v are not coordinates of a 2-vector from M, so, according to this, this sub case is the same as the case 1.

Sub case 14. $u=\alpha_1x_1+\alpha_2x_2$, $v=\beta_jx_j+\beta_{j+1}x_{j+1}+\beta_{j+2}x_{j+2}$, where $\alpha_i\alpha_{i+1}\neq 0$ and $\beta_j\beta_{j+2}\neq 0$

This sub case is absolutely possible, and u and v are vectors which are coordinates of some 2-vectors from the 2-subspace M. Now we have

Situation 1. $v=\beta_1x_1+\beta_2x_2+\beta_3x_3$, $\beta_1\beta_2\neq 0$

Certainly, we have here elements which are parts from the 2-subspace M. One such element is certainly the element (x_2,v). Now, because $(x_2,u),(u,v)$ are also from M', we get that the kernel 2-subspace $L^2(u,v,x_2)$ is a 2-subspace from M'. Now, additionally we have that the vector u is a loop for x_1,x_2,v which is a loop 2-subspace, and will be denoted with S. The vectors $x_3,x_4,....$ form a one-sided branch 2-subspace which will be denoted with S'. So,

$$M'=M\cup L^2(x_1,u,v)\cup S\cup S'.$$

Situation 2. $v=\beta_2x_2+\beta_3x_3+\beta_4x_4$, $\beta_2\beta_4\neq 0$.

It is clear that the vectors x_1,x_2,v form a loop 2-subspace around the loop vector u, which will be denoted with S. On the other hand, the 2-vectors $(u,x_2),(x_2,x_3),(x_3,v),(v,u)$ which belong in M', form a cyclic 2-subspace which will be denoted with S'. The vectors $x_5,x_6,....,x_n,...$ also form one-sided branch 2-subspace which will be denoted with S''. So, we have that

$$M'=M\cup S\cup S'\cup S''$$

Situation 3. $v=\beta_3x_3+\beta_4x_4+\beta_5x_5$, $\beta_3\beta_5\neq 0$,

In this situation we have totally analogous case to the previous situation, but here the cyclic 2-subspace now is generated by five vectors i.e. $(u,x_2),(x_2,x_3),(x_3,x_4),(x_4,v),(v,u)$. The rest of the elements from the structure are completely the same as in the previous situation. So,

$$M'=M\cup S\cup S'\cup S''$$

Sub case 14' $u=\alpha_ix_i+\alpha_{i+1}x_{i+1}$, $v=\beta_jx_j+\beta_{j+1}x_{j+1}+\beta_{j+2}x_{j+2}$, where $\alpha_i\alpha_{i+1}\neq 0$ and $\alpha_j\alpha_{j+2}\neq 0$, and $i>1$.

This case is absolutely possible, where u and v are vectors which are coordinates in some 2-vectors from the 2-subspace M.

Situation 1. $u=\alpha_ix_i+\alpha_{i+1}x_{i+1}$, $v=\beta_{i-1}x_{i-1}+\beta_ix_i+\beta_{i+1}x_{i+1}$

From the definition of u and v is clear that the 2-vectors $(x_i,u),(u,v),(v,x_i)$ form a kernel 2-subspace from M', which will be denoted with S. The vectors $x_1,x_2,...,x_{i-2}$ (if there's any, i.e. if $i>3$) form a finite branch 2-subspace which will be denoted with S'. The vector x_{i+1} will be considered for a kernel 2-subspace which in a loop connects any two vectors from the kernel of the extension, together with the vector x_{i+2}. This 2-subspace from M' will be denoted with S''. At the end, the vectors $x_{i+3},x_{i+4},...$ form one sided branch 2-subspace which will be denoted with S'''.

So, this extension is $M'=M\cup S'\cup S''\cup S'''\cup S$.

Situation 2. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\beta_i x_i+\beta_{i+1}x_{i+1}+\beta_{i+2}x_{i+2}$.

This situation is completely analogous to the previous situation 1.

Situation 3. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\beta_{i+1}x_{i+1}+\beta_{i+2}x_{i+2}+\beta_{i+3}x_{i+3}$.

This situation is analogous to the situation 2 from the previous sub case. The discussion is the same.

Situation 4. $u=\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $v=\beta_{i+2}x_{i+2}+\beta_{i+3}x_{i+3}+\beta_{i+4}x_{i+4}$.

This situation is analogous to the situation 3 from the previous sub case. The discussion is the same.

Sub case 15. $u=x_i$, $i>1$, $v=\alpha_j x_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}$, where $\alpha_j\alpha_{j+2}\neq 0$

This case is absolutely possible, and here u and v are vectors which are coordinates of some 2-vectors from the 2-subspace M.

Situation 1. $v=\alpha_{i-1}x_{i-1}+\alpha_i x_i+\alpha_{i+1}x_{i+1}$, $\alpha_{i-1}\alpha_{i+1}\neq 0$,

In this situation is enough to suppose that $u=x_2$. The rest of the situations are considered as an addition to this situation. If $u=x_2$, then from $v=\alpha_1x_1+\alpha_2x_2+\alpha_3x_3$, we get that essentially we do not have an extension of M, because $(u,v)=(x_2,\alpha_1x_1+\alpha_2x_2+\alpha_3x_3)\in M$. So, in this situation

$M'=M$

In the situation when $i>2$, the vector x_{i-1} is a loop element of the 2-subspace in which are included as 2-vectors $(x_{i-2},x_{i-1}),(v,x_{i-1}),(x_{i+1},x_{i-1})$. That 2-subspace will be denoted with S. Additionally one more 2-subspace can appear, and that is a finite branch 2-subspace, which will be generated from the vectors $x_1,x_2,....,x_{i-2}$, and will be denoted with S'. So, now we have that

$M'=M\cup S\cup S'$.

Situation 2. $v=\alpha_i x_i+\alpha_{i+1}x_{i+1}+\alpha_{i+2}x_{i+2}$, $\alpha_i\alpha_{i+2}\neq 0$.

In this situation we will make additional assumptions that $u=x_2$. In this situation we have that $v=\alpha_2x_2+\alpha_3x_3+\alpha_4x_4$, where $(x_3,v),(x_3,x_2)$ are vectors from M, and (v,u) is a 2-vector which we add. So, we have a real extension, where those three vectors form a kernel 2-subspace from X^2, denoted with S. Additionally, the vector u is a loop for the 2-subspace from M', in which the 2-vector (x_1,x_2) is included also, and is denoted with S'. In this 2-subspace there will be another loop element, and that is the vector x_3, which will be denoted with S'', and in it belong any two vectors from the kernel 2-subspace, together with the vector $(x_1,x_2)=(x_1,u)$. So, in this case we have that

$M'=M\cup S\cup S'\cup S''$.

We have a completely analogous situation when the vector $u = x_2$ will be any vector $u = x_i, i > 2$, and here additionally we will have one more branch 2-subspace S'''. In this case we would have that

$M' = M \cup S \cup S' \cup S'' \cup S'''$.

Situation 3. $v = \alpha_{i+1}x_{i+1} + \alpha_{i+2}x_{i+2} + \alpha_{i+3}x_{i+3}$, $\alpha_{i+1}\alpha_{i+3} \neq 0$.

In this situation we well make the same additional assumptions, for which $u = x_2$, and the case $u = x_i$ will be considered additionally. In this situation we have that the vector $u = x_2$ is a loop 2-vector for the 2-vectors $(u, x_1), (u, x_3), (u, v)$ and the 2-subspace that they generate will be denoted with S. The vectors $(u, x_3), (u, x_4), (x_4, v), (v, u)$ are four vectors which form a cyclic 2-subspace, denoted with S'. Additionally, the 2-vectors $x_5, x_6, ..., x_n, ...$ form one sided branch 2-subspace, denoted with S''. Finally, this extension of M will be

$M' = M \cup S \cup S' \cup S''$.

If we have extension with a vector $u = x_i$, $i > 2$ then additionally we will have one more 2-subspace which is a finite branch generated with the vectors $x_1, x_2, ..., x_{i-1}$ and which will be denoted with S'''. In that case we will have an extension in the following form

$M' = M \cup S \cup S' \cup S'' \cup S'''$.

Situation 4. $v = \alpha_{i+2}x_{i+2} + \alpha_{i+3}x_{i+3} + \alpha_{i+4}x_{i+4}$, $\alpha_{i+2}\alpha_{i+4} \neq 0$.

In this situation we have totally analogous picture as in the previous example, except that in this case we will have that the number of generator elements of the cyclic 2-subspace is for one 2-vector bigger than before.

Sub case 15'. $u = x_1$ $v = \alpha_j x_j + \alpha_{j+1}x_{j+1} + \alpha_{j+2}x_{j+2}$, where $\alpha_j \alpha_{j+2} \neq 0$

This case is absolutely possible, and here u and v are vectors which are coordinates of some 2-vectors from the 2-subspace M. Here, we should especially see the case when $u = x_1$, and $j \geq 1$ is any positive integer. The previous cases are completely analogous as in the example of two-sided branch.

Situation 1. $v = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, $\alpha_1 \alpha_3 \neq 0$.

In this situation we have that the 2-vectors $(u, x_2), (x_2, v), (v, u)$ which belong in M form also a kernel 2-subspace S in M'. According to this, in this situation we have a kernel 2-subspace, which is connected to a loop 2-subspace S' and at the end of the last kernel 2-subspace is attached a one sided 2-subspace. So,

$M' = M \cup S \cup S'$.

Situation 2. $v = \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$, $\alpha_2 \alpha_4 \neq 0$

In this situation we have one cyclic 2-subspace determined with the 2-vectors

$(u, x_2), (x_2, x_3), (x_3, v), (v, u)$,

which will be denoted as S. On the other hand the 2-vectors $(x_4, x_3), (x_4, x_5), (x_4, v)$ form a loop 2-subspace which will be denoted as S'. The rest of the vectors $x_5, x_6, x_7, x_8, ..., x_n, ...$, are certainly a part of M. According to this,

$M' = M \cup S \cup S'$

Situation 3. $v = \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5$, $\alpha_3 \alpha_5 \neq 0$.

In this situation we have only a movement for one element in comparison with the previous case. Here, the generator elements of the cyclic 2-subspace are the elements $(u,x_2),(x_2,x_3),(x_3,x_4),(x_4,v),(v,u)$, and this cyclic 2-subspase we will denote as S. The loop 2-subspace we will denote with S', and it is generated with the vectors $(x_5,x_4),(x_5,x_6),(x_5,v)$. The rest of the vectors $x_6,x_7,x_8,...,x_n,...$ form a one sided branch 2-subspace. So,

$M'=M\cup S\cup S'$.

Situation 4. $v=\alpha_4x_4+\alpha_5x_5+\alpha_6x_6$, $\alpha_4\alpha_6\neq 0$.

This case is the same as the previous case, just moved for one element to the right.

Sub case 16. $v=\alpha_jx_j+\alpha_{j+1}x_{j+1}+\alpha_{j+2}x_{j+2}$, $u=\beta_ix_i+\beta_{i+1}x_{i+1}+\beta_{i+2}x_{i+2}$, $i<j$, where $\alpha_j\alpha_{j+2}\neq 0$ and $\beta_i\beta_{i+2}\neq 0$.

It is clear that for any i and j we have that the smallest variant for u and v is that they have the following form $u=\alpha_jx_j+\alpha_{j+2}x_{j+2}$ and $v=\beta_ix_i+\beta_{i+2}x_{i+2}$.

Situation 1. $u=\beta_1x_1+\beta_2x_2+\beta_3x_3$, $v=\alpha_1x_1+\alpha_2x_2+\alpha_3x_3$.

The 2-vectors $(x_2,u),(x_2,v),(u,v)$ belong in the 2-subspace M', so, according to this also the kernel 2-subspace S generated by them is a subset of M'. Additionally the vectors $x_4,x_5,....$ determine a one sided branch 2-subspace, which is a subspace both from M' and from M . It will be denoted with S'. According to this,

$M'=M\cup S\cup S'$

Situation 2. $u=\beta_1x_1+\beta_2x_2+\beta_3x_3$, $v=\alpha_2x_2+\alpha_3x_3+\alpha_4x_4$.

The 2-vectors $(u,x_2),(x_2,x_3),(x_3,u),(u,v)$ are four vectors which form a cyclic 2-subspace from X^2, and because they belong in M', they are a 2-subspace from M', too, which we will denote as S. If we have in mind now also the 2-subspace M , then we get that

$M'=M\cup S$.

Situation 3. $u=\beta_1x_1+\beta_2x_2+\beta_3x_3$, $v=\alpha_3x_3+\alpha_4x_4+\alpha_5x_5$.

We will consider the 2-vectors $(u,x_2),(x_2,x_3),(x_3,x_4),(x_4,v),(v,u)$ which for itself form a cyclic 2-subspace, with five generator elements, which we will denote as S. With necessary calculations made, we get that the 2-subspace M' is in fact determined with

$M'=M\cup S$.

Situation 4. $u=\beta_1x_1+\beta_2x_2+\beta_3x_3$, $v=\alpha_4x_4+\alpha_5x_5+\alpha_6x_6$

In this situation we have a 2-subspace which is equal to the 2-subspace determined in the previous situation 3.

Completely analogous are considered all 2-subspaces which are determined with such 2-vectors. The most that can happen is to have at the beginning a finite branch 2-subspace generated by $x_1,x_2,...,x_{i-1}$.

Sub case 17. $u=\alpha_ix_i+\alpha_{i+1}x_{i+1}$, $v=\beta_jx_j+\beta_{j+1}x_{j+1}$, $i\leq j$, $\alpha_i\alpha_{i+1}\neq 0$, ${}_j\alpha_{j+1}\neq 0$.

Situation 1. $u=\alpha_1x_1+\alpha_2x_2$, $v=\beta_1x_1+\beta_2x_2$

In this situation we have a 2-vector which belong to the 2-subspace M', and that is the 2-vector (u,v), for which all is already defined, like the value $\Lambda(u,v)$. This is practically the same as the sub case 2 of case 3. In other words, there is no extension of M .

Situation 2. $u=\alpha_1x_1+\alpha_2x_2$, $v=\beta_2x_2+\beta_3x_3$,

In this situation, the two vectors belong in the 2-subspace M , but here we have an extension of M . Regarding the extension, we have a completely same situation as in the sub case 7. So, the vectors x_1,x_2,x_3 essentially form a kernel 2-subspace, and to it a one sided branch 2-subspace is attached. In this situation we would have that $M'=M\cup L(x_1,x_2,x_3)\times L(x_1,x_2,x_3)$.

Situation 3. $u=\alpha_1x_1+\alpha_2x_2$, $v=\beta_3x_3+\beta_4x_4$

In this we have that the four 2-vectors $(u,x_2),(x_2,x_3),(x_3,v),(v,u)$ form cyclic 2-subspace, denoted as S . Now, the extension of M would be

$$M'=M\cup\bigcup_{w\in L(v,x_1,x_2)}L(w,u)\times L(w,u)\cup\bigcup_{z\in L(u,x_3,x_4)}L(z,v)\times L(z,v)\cup S$$

Situation 4. $u=\alpha_1x_1+\alpha_2x_2$, $v=\beta_jx_j+\beta_{j+1}x_{j+1}$, $j>3$.

This situation is the same as the previous situation, but here the number of the 2-vectors that form the cyclic 2-subspace is bigger. Here, those 2-vectors that form the cyclic 2-subspace are $(u,x_2),(x_2,x_3),(x_3,x_4),....,(x_j,v)(v,u)$ and as before, it will be denoted with S . In this situation we would have an extension in the form

$$M'=M\cup\bigcup_{w\in L(v,x_1,x_2)}L(w,u)\times L(w,u)\cup\bigcup_{z\in L(u,x_j,x_{j+1})}L(z,v)\times L(z,v)\cup S$$

Situation 5. $u=\alpha_ix_i+\alpha_{i+1}x_{i+1}$, $v=\beta_ix_i+\beta_{i+1}x_{i+1}$, $i>1$.

In this situation, we don't have extension of the 2-subspace M , because (v,u) is a vector from M , which belongs to the 2-subspace $S=\{(x,y)/A(x_i,x_{i+1}),A\in M_2(\Phi)\}\subseteq M$.

Situation 6. $u=\alpha_ix_i+\alpha_{i+1}x_{i+1}$, $v=\beta_{i+1}x_{i+1}+\beta_{i+2}x_{i+2}$, $i>1$.

In this situation we have a real extension of the 2-subspace M . Now, the kernel 2-subspace generated by the 2-vectors $(u,x_{i+1}),(x_{i+1},v),(v,u)$ in fact, same as in sub case 7 from this case, is extended to 2-subspace which is determined with $(x_i,x_{i+1}),(x_{i+1},x_{i+2}),(x_{i+2},x_i)$ and it will be denoted with S . So, we have

$$M'=M\cup\bigcup_{w\in L(v,x_i,x_{i+1})}L(w,u)\times L(w,u)\cup\bigcup_{z\in L(u,x_{i+1},x_{i+2})}L(z,v)\times L(z,v)\cup S$$

Situation 7. $u=\alpha_ix_i+\alpha_{i+1}x_{i+1}$, $v=\beta_{i+2}x_{i+2}+\beta_{i+3}x_{i+3}$,, $i>1$.

In this situation we have an extension of the 2-subspace M . Now, the 2-vectors $(u,x_{i+1}),(x_{i+1},x_{i+2}),(x_{i+2},v),(v,u)$ form a cyclic 2-subspace. On the other hand, the vectors $x_{i+3},x_{i+4},...,x_n,...$ form a one-sided branch 2-subspace which will be denoted with S, and the vectors u and v are loop centres of the 2-subspaces $\bigcup_{w\in L(v,x_i,x_{i+1})}L(w,u)\times L(w,u)$ and $\bigcup_{z\in L(u,x_{i+2},x_{i+3})}L(z,v)\times L(z,v)$. If $i>2$, then we have also a starting 2-subspace, which is a branch 2-subspace and it will be denoted with S' . So,

$$M'=M\cup\bigcup_{w\in L(v,x_i,x_{i+1})}L(w,u)\times L(w,u)\cup\bigcup_{z\in L(u,x_{i+2},x_{i+3})}L(z,v)\times L(z,v)\cup S\cup S'$$

Situation 8.// $u=\alpha_ix_i+\alpha_{i+1}x_{i+1}$, $v=\beta_jx_j+\beta_{j+1}x_{j+1}$,, $i>1$, $j\geq i+3$.

This situation is completely analogous of the previous situation, and here only is enlarged the cyclic 2-subspace, where its generators are the 2-vectors determined with $(u,x_{i+1}),(x_{i+1},x_{i+2}),(x_{i+2},x_{i+3}),....(x_j,v),(v,u)$. Now, we have

$$M' = M \cup \bigcup_{w\in L(v,x_i,x_{i+1})} L(w,u)\times L(w,u) \cup \bigcup_{z\in L(u,x_j,x_{j+1})} L(z,v)\times L(z,v) \cup S \cup S'.$$

3. EXTENSION OF A 2-SKEW-SYMMETRIC LINEAR FORM

Theorem. *Let* $\Lambda : M \to \mathbb{R}$ *be a* 2*-skew-symmetric form such that* $\Lambda(x,y) \le p(x,y)$ *for every* $(x,y)\in M$, $p : X^2 \to \mathbb{R}$ *be a* 2*-semi norm and* M *is a branch* 2*-subspace of the* 2*-space* X^2. *Let* M' *be an extension of* M *as in sub case 1 of case 2. Then there exists a* 2*-skew-symmetric linear form* $\Lambda' : M' \to \mathbb{R}$ *such that*

$$\Lambda'/M = \Lambda$$

$$-p(-x,y) \le \Lambda(x,y) \le p(x,y)\,. \qquad (*)$$

Proof. We will consider the two cases separately.

Situation 1. Let $u = x_i$, where $i > 1$. In this situation we have a complete analogy with the one in the case of two sided branch 2-subspace. That is why it is enough to take it from there. [9]

Situation 2. Let $u = x_1$. The choice of two 2-vectors from the 2-subspace of u can be done in a way that it is done in the 2-subspaces generated in the paper for two sided branch 2-subspace. In other words, we can choose 2-vectors in the form $(u,\alpha_1 x_1 + \alpha_2 x_2)$ and $(u,\alpha_1' x_1 + \alpha_2' x_2)$. Further on, the proof is the same as the proofs in the theorems of Hahn-Banach in the part for a two sided branch 2-subspace. [9]

The rest of the cases are the same with the corresponding ones described in the paper for two sided branch 2-subspace. [9]

CONFLICT OF INTEREST

No conflict of interest was declared from the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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EXTENSION OF FINITE BRANCH 2-SUBSPACE AND SOME EXTENSIONS OF HAHN - BANACH TYPE FOR SKEW-SYMMETRIC 2-LINEAR FUNCTIONALS DEFINED ON IT

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Abstract. In this paper 2-subspaces from 2-space X^2, which are from finite branch 2-subspace type, generated with four 2-vectors, will be taken in consideration. Then all its possible extensions adding one element (u,v) and their complete description will be considered. Also, all extensions of 2-skew-symmetric linear form defined on 2-subspace M' of Hahn-Banach type will be considered, in the cases when one vector belongs in 2-vector from M, and the other does not belong in any 2-vector from M, as well as cases when the two coordinates of (u,v) do not belong in M.

1. INTRODUCTION

Extensions of mappings is something that is often looked at in various mathematical disciplines. One classical example of extension of a given mapping is of course the Hanh-Banach theorem for linear functionals. One version of it comprises the contents of the following theorem.

Theorem 1. *Let M be a vector subspace of the vector space X. The functional $p: X \to \mathbb{R}$ satisfies the conditions*

(a) $p(x+y) \le p(x) + p(y)$

(b) $p(tx) = tp(x)$,

for every $x, y \in X$ *and* $t \ge 0$.

The functional $f: M \to R$ *is linear and* $f(x) \le p(x)$. *There exists a linear functional* $\Lambda: X \to \mathbb{R}$ *such that* $\Lambda / M = f$ *and* $-p(-x) \le \Lambda(x) \le p(x)$.

Of course, it is worth mentioning here both the definitions for 2-norm and especially for 2 semi norm, which we will use many times further.

Definition 1. Let X be a vector space over the field Φ. The mapping $\|\bullet,\bullet\|: X^2 \to \mathbb{R}_{\ge 0}$ for which the following conditions are fulfilled

(i) $\|x,y\| = 0$ if and only if $\{x,y\}$ is a linear dependent set

(ii) $\|x,y\| = \|y,x\|$ for arbitrary $x, y \in X$

(iii) $\|\alpha x, y\| = |\alpha| \cdot \|x,y\|$ for arbitrary $\alpha \in \Phi$ and for arbitrary $x, y \in X$

(iv) $\|x+x', y\| \le \|x,y\| + \|x',y\|$, for arbitrary $x, y \in X$,

we call **2-norm**, and $(X^2, \|\bullet,\bullet\|)$ we call **2-normed space**.

Definition 2. Let X is a vector space over the field Φ. The mapping $p: X^2 \to \mathbb{R}_{\ge 0}$ for which the following conditions are fulfilled

(i) $p(x,y) \ge 0$ if and only if $\{x,y\}$ is a linear dependent set

(ii) $p(x,y) = p(y,x)$ for arbitrary $x, y \in X$

(iii) $p(\alpha x, y) = |\alpha| \cdot p(x,y)$ for arbitrary $\alpha \in \Phi$ and for arbitrary $x, y \in X$

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(iv) $p(x+x',y)\le p(x,y)+p(x',y)$, for arbitrary $x,y\in X$,
we call **2-semi norm**, and (X^2,p) we call **2-semi normed space**.

It is worth to note here that for any 2-norm the following equation is fulfilled $\|x,y\|=\|x,y+\alpha x\|$, for arbitrary $x,y\in X$ and for arbitrary scalar $\alpha\in\Phi$.

Due to the definition of an n-norm and the definition of an n-semi norm it turned out that, on the set X^2, where X is a vector space over the field Φ (Φ is the field of real numbers or the field of complex numbers), it is convenient to consider additional operations, two of which are partial and one of which is a complete operation, with the aim of making the notation and considerations easier.

Definition 3. Let X be a vector space over the field Φ. The set X^2 together with the operations

$$(x,z)+(y,z)=(x+y,z)$$
$$(z,x)+(z,y)=(z,x+y)$$
$$A(x,y)=A(x,y)^T$$

where $x,y,z\in X$ and $A\in M_2(\Phi)$ is called a **2-vector space** or **2-space**.

Comment. The third operation in the previous definition is a complete operation, and on the right-hand side of the equality is a multiplication of a matrix with a vector.

Definition 4. Let X be a vector space over the field Φ. The functional $\Lambda: X^2\to\Phi$ for which the following conditions hold

(a) $\Lambda(x+y,z)=\Lambda(x,z)+\Lambda(y,z)$, for arbitrary $x,y,z\in X$

(b) $\Lambda(x,y)=-\Lambda(y,x)$, for arbitrary $x,y\in X$

(c) $\Lambda(\alpha x,y)=\alpha\Lambda(x,y)$, for arbitrary $x,y\in X$ and $\alpha\in\Phi$.

is called **2-skew-symmetric linear form**.

It is not hard to prove that the previous definition (Definition 4) is equivalent with the following definition.

Definition 5. Let X be a vector space over the field Φ. The functional $\Lambda: X^2\to\Phi$ for which the following conditions hold

(a) $\Lambda(x+y,z)=\Lambda(x,z)+\Lambda(y,z)$, for arbitrary $x,y,z\in X$

(b) $\Lambda(A(x,y))=(\det A)\Lambda(x,y)$, for arbitrary $x,y\in X$ and $A\in M_2(\Phi)$.

is called skew-symmetric 2-linear form or simply 2-linear functional.

Completely analogously to the definition of 2-linear functional, which is essentially a definition of a skew-symmetric 2-form, the definitions of 2-seminorm and 2-norm are changing and adapting.

Definition 2'. Let X be a vector space over the field Φ. The mapping $p: X^2\to\mathbb{R}$ for which the following conditions hold

(a) $p(x+y,z)\le p(x,z)+p(y,z)$, for every $x,y,z\in X$

(b) $p(A(x,y))=|\det A|\,p(x,y)$, for every $x,y\in X$ and $A\in M_2(\Phi)$.

is called a **2-semi norm** and (X^2,p) is called a **2-semi normed space**.

Definition 6. The mapping $\|\cdot\|: X^n\to\mathbb{R}$, $n\ge 2$ for which it is fulfilled that:

(a) $\|x_1,x_2\|=0$ if and only if x_1,x_2 are linear dependant vectors;

(b) $\| A(x_1,x_2)\| = |\det A| \| x_1,x_2 \|$, for all $x_1,x_2 \in X$ and for all $A \in M_2(\Phi)$;

(c) $\| x_1+x_2,x_3 \| \leq \| x_1,x_3 \| + \| x_2,x_3 \|$, for all $x_1,x_2,x_3 \in X$,

we call **2-norm of the vector space** X, and the ordered pair $(X,\|\cdot,\cdot\|)$ we call **2-normed space**.

In this section a special type of subsets from X^2 will be considered separately.

Definition 7. The subset S, $S \subseteq X^2$ which is closed with respect to the operations of the 2-space X^2 is called **2-subspace** of X^2.

Of course in these considerations the following two theorems are important.

Theorem 2. *The intersection of an arbitrary family of 2-subspaces of the 2-vector space X^2 is a 2-subspace.*

According to the last theorem, each subset $A \subseteq X^2$ determines a 2-subspace S_A, the smallest 2-subspace of the 2-vector space X^2 which contains the set A. We will call the 2-subspace S_A the 2-subspace **generated by the set** A, and the set A - **the generating set**.

In this matter we will consider a special type of generating sets, i.e. a generating set of the form $M \cup \{(u,v)\}$, where M is a special type of a 2-subspace, and $(u,v) \in X^2$ is arbitrarily given where $\{u,v\}$ is a linearly independent set.

The basic question which we will consider here is whether it is possible to extend a 2-skew-symmetric linear form defined on some types, i.e. classes 2-subspaces to a bigger subspace, in the sense of extension of linear functionals, i.e. of the type of Hanh-Banach.

The main aim if all such considerations is whether we can prove the following theorem or some of its variants.

Theorem 3. *Let S be a 2-subspace of the 2-space X^2, $\Lambda : S \to \mathbb{R}$ be 2-skew-symmetric linear form, and $p : X^2 \to \mathbb{R}$ be a mapping for which*

(a) $p(x+y,z) \leq p(x,z)+p(y,z)$, *for all* $x,y,z \in X$

(b) $p(tx,y) = tp(x,y)$, *for all* $x,y \in X$ *and* $t>0$.

There exists 2-skew-symmetric linear form $\Lambda' : X^2 \to \mathbb{R}$, such that $\Lambda' / S = \Lambda$.

Each 2-semi norm satisfies the conditions a) and b) from the previous theorem.

Furthermore, in many parts we may come across a special kind of subset of X^2 .One type of them is given in the following definition.

Definition 8. The subset T, $T \subseteq X^2$ is called n**-invariant** if $AT \subseteq T$ for every $A \in M_2(\Phi)$, $\det A = 1$.

The general structure of 2-subspaces is, of course, not simple. The simplest forms of 2-subspaces are the kernel subspaces, loop subspaces, branch subspaces and cyclic subspaces. Those are discussed and described in [6].

Solving the problem presented in the last theorem is of course not simple. An affirmation of that is of course the complex structure of the 2-subspaces of the 2-space X^2. Due to this, we will discuss partial cases of this problem.

In this matter we will look at extension of 2-skew-symmetric form defined on a branch-2-subspace and extension of a 2-skew-symmetric form defined on a finite 2-subspace.

From here on, we will assume that the subset $\{x_1, x_2, ..., x_n, ...\}$ is a linearly independent subset of the vector space X, not excluding the case when it is finite.

Definition 9. Let X be a vector space over the field Φ. The 2-subspace S generated by the subset $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),....,(x_{n-1},x_n)\}$, where $\{x_1,x_2,...,x_n\}$ is linearly independent set, is called **finite branch 2-subspace.**

These 2-subspaces are also called finite branch, i.e. finite branch 2-subspace. In other papers one-sided and two-sided branch 2-subspaces, which are sets that are 2-subspaces generated with set in the form $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),....,(x_{n-1},x_n)\}$, will be also considered.

A detailed description of branch 2-subspaces is given in [7]. That is the content of the theorem that follows.

Theorem 4. *If M is a branch 2 subspace generated by the set $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),....,(x_{n-1},x_n)\}$ where $\{x_1,x_2,...,x_n\}$ is a linearly independent set, then* $M = \bigcup_{i=2}^{n-1} \bigcup_{a_{i-1},a_{i+1}\in\Phi} L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i)$.

In the following part we will consider extension of a finite branch 2-subspace M with the addition of one element (u,v) as well as extension of a 2-skew-symmetric form $\Lambda : M \to \mathbb{R}$, that is a 2-skew-symmetric form on $\Lambda' : M' \to \mathbb{R}$, where $M' = \langle M \cup \{(u,v)\}\rangle$.

The leading result in the description of the special 2-subspaces such as cyclic branch 2-subspaces, kernel 2-subspaces and loop 2-subspaces is the following lemma:

Lemma. *The subspace generated by the elements $(x_{i-1},x_i),(x_i,x_{i+1}),(x_{i+1},x_{i+2})$, where $\{x_{i-1},x_i,x_{i+1},x_{i+2}\}$ is a linearly independent set, is*

$L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \times L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \cup L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i)$

The idea for such lemma is because here it seems as if we have put together two branches, i.e.

$$L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \times L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \quad (1)$$

and $$L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i). \quad (2)$$

Here, as its 2-subspace appears a set determined with

$M = \{(A(x_i, x_{i+1})^T / A \in M_2(\Phi)\}$.

Addition of elements from (1) and (2) certainly is possible, but the result is always an element which can be considered that belongs in one of these 2-subspaces, i.e. either in (1) or in (2).

In this part, according to the considerations and operations that are subject of this analysis, it is completely clear that the following comment should be taken into consideration and be respected.

Comment. If in some part a scalar should come in front of an ordered pair, then it can be done and that scalar can be multiplied in some other pat in the ordered pair. Such pairs of elements belong in the same class of equivalence and in the same 2- subspace of the 2-vector space.

2. EXTENSION OF A FINITE BRANCH 2-SUBSPACE

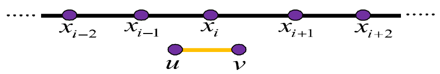
Let Λ be a 2-skew-symmetric linear form defined on a branch 2-subspace M which is generated by the elements of the set $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),....,(x_{m-1},x_m),....\}$, where $\{x_1,x_2,...,x_n,...\}$ is a linearly independent set. Let $(u,v)\in X^2$ be such that $\{u,v\}$ is a linearly independent set. We denote the 2-subspace of X^2 generated by $M\cup\{(u,v)\}$ with M'. Several cases are possible.

Case 1. $u,v\notin L(x_1,x_2,...,x_n)$, where $L(x_1,x_2,...,x_n)$ is the subspace of X generated by $\{x_1,x_2,...,x_n\}$.

The 2-subspace generated by $\{(u,v)\}$ is $L(u,v)\times L(u,v)$. Let us notice that $L(u,v)\cap L(x_1,x_2,...,x_n)=\{0\}(\subseteq\Delta_2)$. Accordingly,

$$M'=M\cup L(u,v)\times L(u,v),$$

where M is determined in theorem 3.



Case 2. Let $u\in L(x_1,x_2,...,x_n)$ and $v\not\in L(x_1,x_2,...,x_n)$.

In this case we will consider several sub cases.

Sub case 1. $u=x_i$ for some $1\le i\le n$.

Situation 1. $u=x_1$

Situation 2. $u=x_n$

Situation 3. $u=x_i$, каде $2\le i\le n-1$

The first and the second sub case are identical and because of that we will consider only the first one.

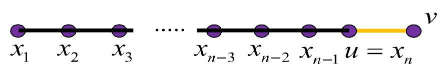
Situation 1. In this situation we have that $u=x_1$ is an element at the end of the sequence of vectors, so we will have that

$$v,x_1,x_2,....,x_n=u$$

is a sequence of $n+1$ vectors which as in the case when it is a beginning element of n vectors, form a branch 2- subspace which is a finite branch. Now, the extension of the beginning 2-subspace is given with

$$M'=L(u,v)\times L(u,v)\cup\left(\bigcup_{i=2}^{n-1}\bigcup_{a_{i-1},a_{i+1}\in\Phi}L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\times L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\right)=$$
$$=L(u,v)\times L(u,v)\cup M$$

Situation 3. $u=x_i$, where $2\le i\le n-1$



In this sub case the set $\{(x_{i-1},x_i),(x_i,x_{i+1}),(x_i,v)\}=\{(x_{i-1},u),(u,x_{i+1}),(u,v)\}$ generates a 2-subspace which is a loop subspace and its form is

$$L=\bigcup_{w\in L(x_{i-1},v,x_{i+1})}L(u,w)\times L(u,w).$$

Simultaneously the sets $P' = \{(x_1,x_2),(x_2,x_3),....,(x_{i-2},x_{i-1})\}$ and $P'' = \{(x_{i+1},x_{i+2}),(x_{i+2},x_{i+3}),...,(x_{n-1},x_n)\}$ generate 2-subspaces $S_{P'}$ and $S_{P''}$ respectively, which are finite branch 2-subspaces. At the same time, they, as well as L, are 2-subspaces from the required extension M'. The forms of $S_{P'}$ and $S_{P''}$ are

$$S_{P'} = \bigcup_{k=2}^{i-1} \bigcup_{a_{k-1},a_{k+1}\in\Phi} L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k) \times L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k)$$

$$S_{P''} = \bigcup_{k=i+1}^{n-1} \bigcup_{a_{k-1},a_{k+1}\in\Phi} L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k) \times L(a_{k-1}x_{k-1} + a_{k+1}x_{k+1}, x_k)$$

In order for us to see the form of M' it is enough to consider several types of addition of elements of $L, S_{P'}$ and $S_{P''}$. It is enough to consider the cases:

1° $(m,n)\in L$, $\quad (x,y)\in L((x_{i-2},x_{i-1}),(x_{i-1},x_i))$

2° $(m,n)\in L$, $\quad (x,y)\in L((x_{i-3},x_{i-2}),(x_{i-2},x_{i-1}))$

3° $(m,n)\in L$, $\quad (x,y)\in L((x_i,x_{i+1}),(x_{i+1},x_{i+2}))$

4° $(m,n)\in L$, $\quad (x,y)\in L((x_{i+1},x_{i+2}),(x_{i+2},x_{i+3}))$.

In case 1° we have

$$(m,n) = (b_1(\alpha_1 x_{i-1} + \alpha_2 v + \alpha_3 x_{i+1}) + b_2 x_i, b_3(\alpha_1 x_{i-1} + \alpha_2 v + \alpha_3 x_{i+1}) + b_4 x_i)$$

$$(x,y) = (a_1(\alpha x_{i-2} + \beta x_i) + a_2 x_{i-1}, a_3(\alpha x_{i-2} + \beta x_i) + a_4 x_{i-1}).$$

The sum of two such elements is possible in 2 cases:

a) $\alpha_2 = \alpha_3 = \alpha = 0$, $b_1\alpha_1 = a_2 = s$, $a_1\beta = b_2 = t$

b) $\alpha_2 = \alpha_3 = \alpha = 0$, $b_3\alpha_1 = a_4 = s$, $a_3\beta = b_4 = t$

In case a) the elements get the form

$$(b_1\alpha_1 x_{i-1} + b_2 x_i, b_3\alpha_1 x_{i-1} + b_4 x_i) = (sx_{i-1} + tx_i, b_3\alpha_1 x_{i-1} + b_4 x_i)$$

$$(a_1\beta x_i + a_2 x_{i-1}, a_3\beta x_i + a_4 x_{i-1}) = (sx_{i-1} + tx_i, a_3\beta x_i + a_4 x_{i-1}),$$

and their sum is $(sx_{i-1} + tx_i, (a_3\beta + b_4)x_i + (a_4 + b_3\alpha_1)x_{i-1}) \in L((x_{i-1},x_i)) \subset L$

We similarly get for case b)

In case 2° we have

$$(x,y) = (a_1(\alpha x_{i-3} + \beta x_{i-1}) + a_2 x_{i-2}, a_3(\alpha x_{i-3} + \beta x_{i-1}) + a_4 x_{i-2})$$

$$(m,n) = (b_1(\alpha_1 x_{i-1} + \alpha_2 v + \alpha_3 x_{i+1}) + b_2 x_i, b_3(\alpha_1 x_{i-1} + \alpha_2 v + \alpha_3 x_{i+1}) + b_4 x_i)$$

The sum of two such elements is possible in 2 cases:

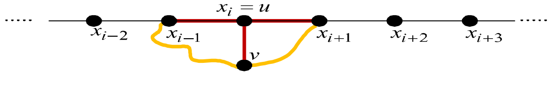
c) $\alpha_2 = \alpha_3 = \alpha = 0$, $a_2 = b_2 = 0$, $a_1\beta = b_1\alpha_1 = s$

d) $\alpha_2 = \alpha_3 = \alpha = 0$, $a_4 = b_4 = 0$, $a_3\beta = b_3\alpha_1 = s$

In case c) the elements get the form

$$(sx_{i-1}, a_3\beta x_{i-1} + a_4 x_{i-2})$$

$$(sx_{i-1}, b_3\alpha_1 x_{i-1} + b_4 x_i)$$



and their sum is

$$(sx_{i-1}, (a_3\beta + b_3\alpha_1)x_{i-1} + a_4 x_{i-2} + b_4 x_i) \in L((x_{i-2},x_{i-1}),(x_{i-1},x_i)) \subset M$$

We similarly get for case d)

According to that, in this sub case the extension is

$$M' = M \cup \bigcup_{w\in L(x_{i-1},x_i,x_{i+1})} L(x_i, w) \times L(x_i, w).$$

Sub case 2. $u \in L(x_j, x_{j+1})$ for some $j \in \{1,2,...,n-1\}$, where $u \neq x_j, x_{j+1}$.

In other words, we have here a situation in which $u = \alpha_j x_j + \alpha_{j+1} x_{j+1}$, $\alpha_j \alpha_{j+1} \neq 0$. In this sub case we have $u = \mu x_j + \nu x_{j+1}$, where $\mu, \nu \neq 0$. The sets $\{v, u, x_j\}$ and $\{v, u, x_{j+1}\}$ are linearly independent sets. The sets $K' = \{(u,v),(u,x_j)\}$ and $K' = \{(u,v),(u,x_{j+1})\}$ generate 2 -subspaces $S_{K'}$ and $S_{K''}$ and their forms are

$$S_{K'} = \bigcup_{\alpha,\beta \in \Phi} L(\alpha v + \beta x_j, u) \times L(\alpha v + \beta x_j, u)$$

$$S_{K''} = \bigcup_{\alpha,\beta \in \Phi} L(\alpha v + \beta x_{j+1}, u) \times L(\alpha v + \beta x_j, u)$$

The general form of the elements of $S_{K'}$ is

$$(a_1(\alpha v + \beta x_j) + a_2 u, a_3(\alpha v + \beta x_j) + a_4 u)$$

and of the elements of $S_{K''}$ is

$$(b_1(\gamma v + \delta x_{j+1}) + b_2 u, b_3(\gamma v + \delta x_{j+1}) + b_4 u).$$

Addition of the latter two forms of elements is possible in the following 2 cases:

a) $\beta = \delta = 0$, $a_2 = b_2 = t$, $a_1 \alpha = b_1 \gamma = s$

b) $\beta = \delta = 0$, $a_2 = b_2 = t$, $a_3 \alpha = b_3 \gamma = s$.

In case a) the elements get the form

$$(sv + tu, a_3 \alpha v + a_4 u)$$

$$(sv + tu, b_3 \gamma v + b_4 u)$$

and their sum is

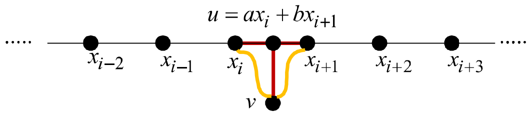
$$(sv + tu, (b_3 \gamma + a_3 \alpha) v + (a_4 + b_4) u) \in L((u,v)) \subset M'$$

The result in case b) is similar.

From the whole of the former discussion it is clear that

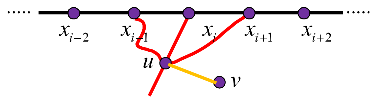
$M' = M \cup S_{K'} \cup S_{K''}$.

We consider the sub cases 3 and 4 similarly.



In this case are considered also the special cases, i.e. when $i = 1$ and when $i + 1 = n$.

Sub case 3. $u \in L(\alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1}, x_i)$, $v \notin L(x_1, x_2, ..., x_{n-1}, x_n)$.



In this situation we have a case when two elements belong in the 2-subspace generated by the elements $(u, x_i), (u, v)$. This 2-subspace can be treated as a branch 2-subspace generated by x_i and v, or we can treat it as a 2-subspace generated by the two dimensional space $L(x_i, v)$. According to this

$$M' = M \cup \bigcup_{w \in L(x_i, v)} L(w,u) \times L(w,u).$$

Here, we will consider also the special cases, i.e. when one of the scalars at u is zero, i.e. $\alpha_{i-1} = 0 \vee \alpha_{i+1} = 0$.

Position 1. $\alpha_{i-1} = 0$

In this situation we have that the vector u has the form $u = \alpha_i x_i + \alpha_{i+1} x_{i+1}$ where we have completely analogue situation as in the sub case 2 of this case, which is completely described.

Position 2. $\alpha_{i+1} = 0$

In this situation we have that the vector u has the form $u = \alpha_{i-1} x_{i-1} + \alpha_i x_i$ where we have completely analogue situation as in the sub case 2 of this case, which is completely described.

Sub case 4. $u \in L(x_i, ..., x_k)$, where $k > i+2$ and the coefficients in the representation before x_i and x_k are different from zero. In this situation we have that for example k can be at least $i+3$. In that situation u is not a coordinate vector of an 2-vector element which belongs in M. According to this, this is a case when for u we can consider that it is not in $L(x_1, x_2, ..., x_n)$. The extension here is the same as in the case 1.

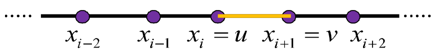
The case $u \notin L(x_1, x_2, ..., x_n)$ and $v \in L(x_1, x_2, ..., x_n)$ is completely analogously considered.

Case 3. Let $u, v \in L(x_1, x_2, ..., x_n)$.

We will consider several possibilities, i.e. sub cases.

Sub case 1. $u = x_i$, $v = x_{i+1}$, including also $i = 1$ or $i = n-1$

In this sub case $L(u, v) = L(x_j, x_{j+1})$, therefore we don't have a true extension of M. In other words $M = M'$, i.e. we do not have an extension.



Sub case 2. $u = x_i$, $v = x_{i+2}$, including also $i = 1$ and $i = n-2$.

In this sub case, the pairs (x_i, x_{i+1}), (x_{i+1}, x_{i+2}) and (x_i, x_{i+2}) are included in the generating of M' so, accordingly, they define a kernel subspace S which is of the form $L(x_i, x_{i+1}, x_{i+2}) \times L(x_i, x_{i+1}, x_{i+2})$. Now, the subspace M' is generated by one kernel subspace S, and two branch 2-subspaces, one generated by $(x_1, x_2), (x_2, x_3), ..., (x_{i-2}, x_{i-1}), (x_{i-1}, x_i)$ and the other by $(x_{i+2}, x_{i+3}), (x_{i+3}, x_{i+4}), ..., (x_m, x_{m+1}), (x_{m+1}, x_{m+2}), ...$

The form of S is

$$S = L(x_i, x_{i+1}, x_{i+2}) \times L(x_i, x_{i+1}, x_{i+2}).$$

The form of the 2-subspace S' is

$$S' = \bigcup_{k=2}^{i-1} \bigcup_{a_{k-1}, a_{k+1} \in \Phi} L(a_{k-1} x_{k-1} + a_{k+1} x_{k+1}, x_k) \times L(a_{k-1} x_{k-1} + a_{k+1} x_{k+1}, x_k)$$

The form of the 2-subspace S'' is

$$S'' = \bigcup_{k=i+3}^{n} \bigcup_{a_{k-1}, a_{k+1} \in \Phi} L(a_{k-1} x_{k-1} + a_{k+1} x_{k+1}, x_k) \times L(a_{k-1} x_{k-1} + a_{k+1} x_{k+1}, x_k)$$

Let us notice that the addition of elements of S or S' or S'' is again an element of S or S' or S'', respectively. Addition of elements of S' and S'', one from S' and the other from S'', is not possible.

We will determine when addition of elements of S and S' is possible and what is the result of that addition. Every element of S is of the form

$$(a_1x_i + b_1x_{i+1} + c_1x_{i+2}, a_2x_i + b_2x_{i+1} + c_2x_{i+2})$$

and the elements from S' for which addition is possible are of the form

$$(d_1(\alpha x_{i-2} + \beta x_i) + e_1x_{i-1}, d_2(\alpha x_{i-2} + \beta x_i) + e_2x_{i-1}).$$

Addition in this case is possible in the following two cases:

a) $b_1 = c_1 = 0$, $\alpha = 0$, $d_1\beta = a_1 = s$

b) $b_2 = c_2 = 0$, $\alpha = 0$, $d_2\beta = a_2$.

It is enough to consider the case a). Then the elements obtain the form

$$(sx_i, a_2x_i + b_2x_{i+1} + c_2x_{i+2}),\ (sx_i, d_2\beta x_i + e_2x_{i-1})$$

and their sum is

$$(sx_i, (a_2 + d_2\beta)x_i + b_2x_{i+1} + c_2x_{i+2} + e_2x_{i-1}).$$

Therefore, the sum of these elements is an element from the 2-subspace T defined by

$$T = \bigcup_{u \in L(x_{i-1}, x_{i+1}, x_{i+2})} L(x_1, u) \times L(x_1, u).$$

Now it is enough to determine the sum of the elements from the 2-subspace T with the elements of the 2-subspace generated by the elements of the set $\{(x_{i-3}, x_{i-2}), (x_{i-2}, x_{i-1})\}$. The former are of the form

$$A(x_i, \alpha_1x_{i-1} + \alpha_2x_i + \alpha_3x_{i+1} + \alpha_4x_{i+2}) = \\ = (b_1x_i + b_2(\alpha_1x_{i-1} + \alpha_2x_i + \alpha_3x_{i+1} + \alpha_4x_{i+2}), b_3x_i + b_4(\alpha_1x_{i-1} + \alpha_2x_i + \alpha_3x_{i+1} + \alpha_4x_{i+2})) \quad (*)$$

The subspace generated by the set $\{(x_{i-3}, x_{i-2}), (x_{i-2}, x_{i-1})\}$ is

$$\bigcup_{\alpha, \beta \in \Phi} L(\alpha x_{i-3} + \beta x_{i-1}, x_{i-2}) \times L(\alpha x_{i-3} + \beta x_{i-1}, x_{i-2}),$$

and its elements are of the form

$$(a_1(\alpha x_{i-3} + \beta x_{i-1}) + a_2x_{i-2}, a_3(\alpha x_{i-3} + \beta x_{i-1}) + a_4x_{i-2}). \quad (**)$$

Elements of the form (*) and (**) is feasible in two cases:

c) $b_1 = 0$, $\alpha_2 = \alpha_3 = \alpha_4 = 0$, $\alpha = 0$, $a_2 = 0$, $b_2\alpha_1 = a_1\beta = s$

d) $b_3 = 0$, $\alpha_2 = \alpha_3 = \alpha_4 = 0$, $\alpha = 0$, $a_4 = 0$, $b_4\alpha_1 = a_3\beta = s$.

In the case c) we have

$$(sx_i, b_3x_i + b_4\alpha_1x_{i-1})$$

$$(sx_i, a_3\beta x_{i-1} + a_4x_{i-2})$$

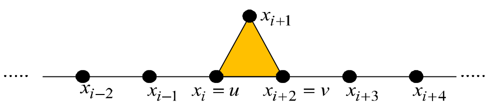
and their sum is

$$(sx_i, b_3x_i + (b_4\alpha_1 + a_3\beta)x_{i-1} + a_4x_{i-2}) \in L(\gamma x_{i-2} + \delta x_i, x_{i-1}) \times L(\gamma x_{i-2} + \delta x_i, x_{i-1}).$$

The case d) is considered analogously.

Accordingly, in this case M' is the 2-subspace

$$M' = M \cup (L(x_i, x_{i+1}, x_{i+2}))^2 \cup \bigcup_{u \in L(x_{i-1}, x_{i+1}, x_{i+2})} L(u, x_i) \times L(u, x_i) \cup \bigcup_{v \in L(x_{i+3}, x_{i+1}, x_i)} L(v, x_{i+2}) \times L(v, x_{i+2}).$$



In the beginning as well in the end situation we would have a loop 2-subspace, at which in the first situation, i.e. $i=1$ is followed by a finite branch 2-subspace, and in the other situation, i.e. $i=n-2$ when a finite branch 2-subspace ends with a loop 2-subspace generated by x_{n-2}, x_{n-1}, x_n.

In other words we would have

a) $M' = M \cup L^2(x_1,x_2,x_3) \cup \bigcup_{w\in L(x_4,x_2,x_1)} L(w,x_3)\times L(w,x_3)$

b) $M' = M \cup L^2(x_{n-2},x_{n-1},x_n) \cup \bigcup_{w\in L(x_{n-3},x_{n-1},x_n)} L(w,x_{n-2})\times L(w,x_{n-2})$

Sub case 3. $u=x_i$, $v=x_j$, $1<i<j<n$, $j>i+2$.

In this sub case the ordered pairs $(x_i,x_{i+1}),(x_{i+1},x_{i+2}),....,(x_{j-1},x_j),(x_j,x_i)$ form a cyclic subspace S. Now, the extension is generated by one cyclic subspace S, and two finite branch 2-subspaces, one S' generated by $(x_1,x_2),(x_2,x_3),....,(x_{i-2},x_{i-1}),(x_{i-1},x_i)$ and the other S'' generated by $(x_j,x_{j+1}),(x_{j+1},x_{j+2}),....,(x_{n-1},x_n)$.

The form of S is.

$$S=\bigcup_{i=1}^{n}\left[L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\times L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\right].$$

The form of S' is

$$S'=\bigcup_{k=2}^{i-1}\ \bigcup_{a_{k-1},a_{k+1}\in\Phi} L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)\times L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)$$

The form of S'' is

$$S''=\bigcup_{k=j}^{\infty}\ \bigcup_{a_{k-1},a_{k+1}\in\Phi} L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)\times L(a_{k-1}x_{k-1}+a_{k+1}x_{k+1},x_k)$$

Addition of elements of S' and S'', i.e. one element from S' and the other form S'' is not possible.

We will consider the remaining possibilities for addition of elements of S,S' and S''. Let us notice that the sets $K'=\{(x_{i-1},x_i),(x_i,x_{i+1}),(x_i,x_j)\}$ and $K''=\{(x_i,x_j),(x_{j-1},x_j),(x_j,x_{j+1})\}$ are generators of the 2-subspaces S_K and $S_{K'}$ which are subspaces of M'. At the same time they are loop 2-subspaces generated by three elements. We have:

$$S_{K'}=\bigcup_{u\in L(x_i,x_{j-1},x_{j+1})} L(u,x_j)\times L(u,x_j) \quad\text{and}\quad S_{K''}=\bigcup_{v\in L(x_{i-1},x_{i+1},x_j)} L(v,x_i)\times L(v,x_i)\ .$$

First we will determine when addition is possible between elements from $S_{K'}$ and $S_{K''}$ and what will the result from the addition be. The elements from $S_{K'}$ are of the form

$$(a_1(\alpha_1x_i+\alpha_2x_{j-1}+\alpha_3x_{j+1})+b_1x_j,a_2(\alpha_1x_i+\alpha_2x_{j-1}+\alpha_3x_{j+1})+b_2x_j)$$

and the elements of $S_{K''}$ are of the form

$$(c_1(\beta_1x_{i-1}+\beta_2x_{i+1}+\beta_3x_j)+d_1x_i,c_2(\beta_1x_{i-1}+\beta_2x_{i+1}+\beta_3x_j)+d_2x_i).$$

It is clear that addition is possible in two cases:

a) $\alpha_2=\alpha_3=\beta_1=\beta_2=0$, $a_1\alpha_1=d_1=s$, $c_1\beta_3=b_1=t$

b) $\alpha_2=\alpha_3=\beta_1=\beta_2=0$, $a_2\alpha_1=d_2=s$, $c_2\beta_3=b_2=t$.

In case a) we have the sum

$(sx_i + tx_j, (a_2\alpha_1 + d_2)x_i + (c_2\beta_3 + b_2)x_j) \in L((x_i, x_j))$

In case b) we have the sum

$((a_1\alpha_1 + d_1)x_i + (c_1\beta_3 + b_1)x_j, sx_i + tx_j) \in L((x_i, x_j))$

Therefore in each case the sum is an element from the 2 -subspace $L((x_i, x_j))$

We will determine the sums in the remaining possibilities for addition in M'. We have the following possibilities:

1° $(x, y) \in S_{K''}$ and $(m, n) \in L((x_{i-3}, x_{i-2}), (x_{i-2}, x_{i-1}))$

2° $(x, y) \in S_{K''}$ and $(m, n) \in L((x_{i+1}, x_{i+2}), (x_{i+2}, x_{i+3}))$

3° $(x, y) \in S_{K'}$ and $(m, n) \in L((x_{j-3}, x_{j-2}), (x_{j-2}, x_{j-1}))$

4° $(x, y) \in S_{K'}$ and $(m, n) \in L((x_{j+3}, x_{j+2}), (x_{j+2}, x_{j+1}))$

In 1° the elements from $S_{K''}$ are of the form

$(c_1(\beta_1 x_{i-1} + \beta_2 x_{i+1} + \beta_3 x_j) + d_1 x_i, c_2(\beta_1 x_{i-1} + \beta_2 x_{i+1} + \beta_3 x_j) + d_2 x_i$

and $(m, n) = (a_1(\alpha x_{i-3} + \beta x_{i-1}) + b_1 x_{i-2}, a_2(\alpha x_{i-3} + \beta x_{i-1}) + b_2 x_{i-2}))$.

Therefore, addition is possible in the following two cases:

c) $\beta_2 = \beta_3 = 0$, $\alpha = 0$, $b_1 = 0$, $d_1 = 0$, $c_1\beta_1 = a_1\beta = t$

d) $\beta_2 = \beta_3 = 0$, $\alpha = 0$, $b_1 = 0$, $d_1 = 0$, $c_2\beta_1 = a_2\beta = t$

In the case c) we get

$(tx_{i-1}, c_2\beta_1 x_{i-1} + d_2 x_i)$

$(tx_{i-1}, a_2\beta x_{i-1} + b_2 x_{i-2})$

and for the sum we get

$(tx_{i-1}, (c_2\beta_1 + a_2\beta)x_{i-1} + d_2 x_i + b_2 x_{i-2}) \in L((x_{i-2}, x_{i-1}), (x_{i-1}, x_i))$

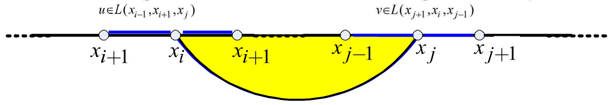
The case d) can be analogously considered.

Similar results are obtained in 2°, 3° and 4° with the results of the additions being elements of the 2 -subspaces $L((x_i, x_{i+1}), (x_{i+1}, x_{i+2}))$, $L((x_{j-2}, x_{j-1}), (x_{j-1}, x_j))$ and $L((x_{j+2}, x_{j+1}), (x_{j+1}, x_j))$ respectively, and also being elements of M.

The remaining cases for addition, when it is possible, are addition of elements M and they again belong to M.

Finally, we can conclude that in this sub case:

$$M' = M \cup \bigcup_{u \in L(x_{i-1}, x_{i+1}, x_j)} L(u, x_i) \times L(u, x_i) \cup \bigcup_{v \in L(x_{j+1}, x_i, x_{j-1})} L(v, x_j) \times L(v, x_j).$$

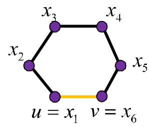


In case when we have exactly four or five generator elements (see drawing), we get that the new 2-subspace will have the form given in the next two positions.

Position 1. If we have 5 elements which generate the starting 2-subspace M then we have the following two situations: $u = x_2, x_3, x_4.x_5 = v$ generate cyclic 2-subspace, and $u = x_2$ is a center of a loop 2-subspace, or $u = x_1, x_2, x_3.x_4 = v$ form a cyclic 2-subspace, and $v = x_4$ is a centre of a loop 2-subspace. According to this, we will have one cyclic 2-subspace S and one loop 2-subspace S'. So, we have that

$M' = S \cup S'$.

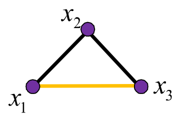
Position 2. In case when we have four generator elements, the case is not like the previous position at all, because then we have that $u=x_1,x_2,x_3.x_4=v$. According to that, it is made a cyclic 2-subspace generated by the same vectors that are mentioned here.



Sub case 4. $u=x_1$, $v=x_n$

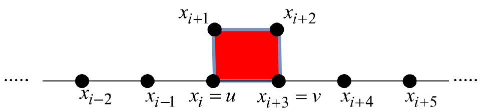
In this situation we get that the ends of the finite branch, i.e. x_1 and x_n now are u and v accordingly. So, the sequence of 2-vectors $(x_1=u,x_2),(x_2,x_3)...,(x_{n-1},x_n)(v=x_n,u)$ generate classical cyclic 2-subspace (see drawing, it is given for case $n=6$).

Here it is important to note that characteristic case is the one when the number of generator elements of the starting space is 2, i.e. for the set of linearly independent vectors $\{x_1,x_2,x_3\}$, with the 2-vectors (x_1,x_2) and (x_2,x_3) are chosen vectors $u=x_1$, $v=x_3$. In that case, the 2-subspace which will be generated is a kernel 2-subspace and that is the only such case (see drawing).



The cases $n=4,5,6,7,...$, i.e. for all $n\geq 4$ are the same, i.e. they are as illustrated on the drawing.

Sub case 5. $u=x_i$, $v=x_j$ where $j=i+3<n$ and $1<i$



In this situation, we have that the 2-vectors $(u=x_i,x_{i+1}),(x_{i+1},x_{i+2}),(x_{i+2},v=x_{i+3}),(v,u=x_i)$ form a four with which is determined one cyclic 2-subspace. But now, the vectors $(v,x_{i+4}),(v,x_{i+2}),(v,u)$ from one side and the three 2-vectors $(u,x_{i-1}),(u,x_{i+1}),(u,v)$, both for itself are loop 2-subspaces, i.e. two loop 2-subspaces. Also we have two branch 2-subspaces, which are finite branch 2-subspaces.

Sub case 6. $u=x_1$, $v=x_{n-1}$, where $n\geq 4$.

In this case we have forming of one cyclic 2-subspace and one loop 2-subspace. This can be seen from the drawing. Here, the cyclic 2-subspace in the case $n>4$ is fully cyclic 2-subspace as it is shown on the drawing.

In context of the previous discussion, we will have here:

Situation 1. $n=4$

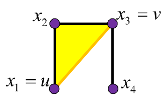
First, the kernel 2-subspace has the form $M=L(x_1,x_2,x_3)\times L(x_1,x_2,x_3)$ whilst the loop 2-subspace has the form $S=\bigcup_{w\in L(x_1,x_2,x_4)} L(w,v)\times L(w,v)$ and $M'=M\cup S$.

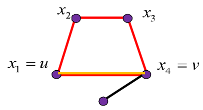
Situation 2. $n\geq 5$

We should note that here it is enough to consider only the case $n=5$. All other cases are completely analogous. Now,

$$M=\bigcup_{\substack{i=1,i+2>4\\ i+2\equiv 1\,(\mathrm{mod}\,4)\\ i+2\equiv 2\,(\mathrm{mod}\,4)}}^{4}\ \bigcup_{\alpha_i,\alpha_{i+2}\in\Phi} L(\alpha_i x_i+\alpha_{i+2}x_{i+2},x_{i+1})\times L(\alpha_i x_i+\alpha_{i+2}x_{i+2},x_{i+1})\ ,$$

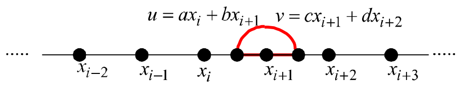
while $S=\bigcup_{w\in L(x_1,x_4,x_3)} L(w,v)\times L(w,v)$. It is clear that $M'=M\cup S$.





Sub case 7. $u=ax_i+bx_{i+1}$, $v=cx_{i+1}+dx_{i+2}$ where $ab\neq 0$ and $cd\neq 0$

In this situation, we have that the three 2-vectors $(u,x_{i+1}),(x_{i+1},v),(v,u)$ which are determined, determine three vectors v,u,x_{i+1} which are linearly independent. They determine a kernel 2-subspace which is fully contained in the extended subspace M'.



Procedure 1. The vectors are (u,v) and (u,x_{i+1}). Now, we have that

$$\begin{bmatrix}1 & 0\\ 0 & \frac{1}{d}\end{bmatrix}\left((u,v)+\begin{bmatrix}1 & 0\\ 0 & -c\end{bmatrix}(u,x_{i+1})\right)=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{d}\end{bmatrix}\big((u,cx_{i+1}+dx_{i+2})+(u,-cx_{i+1})\big)=$$

$$=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{d}\end{bmatrix}(u,dx_{i+2})=(u,x_{i+2})$$

So, we have that the 2-vector $(u,x_{i+2})\in M'$.

Procedure 2. The vectors $(x_{i+2},u),(x_{i+2},x_{i+1})$ are 2-vectors which are from M'. But, now

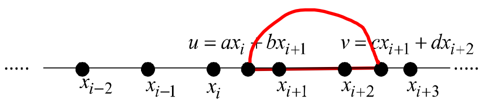
$$\begin{bmatrix}1 & 0\\ 0 & \frac{1}{a}\end{bmatrix}\left((x_{i+2},u)+\begin{bmatrix}1 & 0\\ 0 & -b\end{bmatrix}(x_{i+2},x_{i+1})\right)=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{a}\end{bmatrix}\big((x_{i+2},ax_i+bx_{i+1})+(x_{i+2},-bx_{i+1})\big)=$$

$$=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{a}\end{bmatrix}(x_{i+2},ax_i)=(x_{i+2},x_i)$$

So, we finally get the 2-vectors $(x_i,x_{i+1}),(x_{i+1},x_{i+2}),(x_{i+2},x_i)$ which are part of the 2-subspace M'. According to this, this 2-subspace is consisted from two 2-subspaces which are finite branch 2-subspaces, one of them is kernel 2-subspace and eventually two loop subspaces. This happens when $i>1$ and $i+2<n$.

In the cases when $i=1$ or $i+2=n$, then we have one starting/ending kernel 2-subspace, one branch less and one loop less.

Sub case 8. $u=ax_i+bx_{i+1}$, $v=cx_{i+2}+dx_{i+3}$, where $ab\neq 0$ and $cd\neq 0$



Here we have a clear picture that the vectors $u = ax_i + bx_{i+1}$ and $v = cx_{i+2} + dx_{i+3}$ are vectors that belong in the current 2-subspace M. But, here in fact is formed one cyclic 2-subspace which is generated by the elements $(u, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, v), (v, u)$, which will be denoted with S. So, here we have additional elements which should be taken in consideration. Certainly, the most interesting part in this context is whether the vectors $u = ax_i + bx_{i+1}$ and $v = cx_{i+2} + dx_{i+3}$ are loops, i.e. if they are loops of two loop 2-subspaces. One of them is generated by $(x_{i+2}, v), (v, x_{i+3}), (v, u)$, and the other one is generated by $(x_i, u), (u, x_{i+1}), (u, v)$.

The answer is yes, they are loop 2-subspaces which are generated with loop elements as said, i.e. by $u = ax_i + bx_{i+1}$ and $v = cx_{i+2} + dx_{i+3}$

According to this,

$$M' = M \cup S \cup \bigcup_{w \in L(x_i, x_{i+1}, v)} L(w,u) \times L(w,u) \cup \bigcup_{z \in L(x_{i+2}, x_{i+3}, u)} L(z,v) \times L(z,v)$$

Comment. Similarly as in the part at cyclic 2-subspace generated by elements in this form, if we choose two scalars $\delta, \gamma \in \Phi$, then for them we get that the 2-vector

$$\begin{bmatrix} \delta & 0 \\ 0 & \gamma \end{bmatrix} (u, v) = (\delta u, \gamma v) \in M'.$$

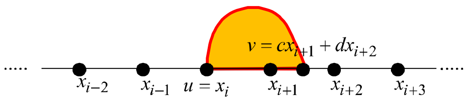
From the other side we have that

$$\begin{bmatrix} \delta & 0 \\ 0 & 1 \end{bmatrix} (u, x_i) = (\delta u, x_i) \in M \subseteq M', \qquad \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} (x_{i+2}, v) = (x_{i+2}, \gamma v) \in M \subseteq M'$$

According to this, for the set of four 2-vectors $(\delta u, \gamma v)$, $(\delta u, x_i)$, (x_{i+1}, x_{i+2}), $(x_{i+2}, \gamma v)$ one 2-subspace is generated which is cyclic 2-subspace. That is fulfilled for any $\delta, \gamma \in \Phi$. But, even if they form cyclic 2-subspace, they are a part of already formed cyclic 2-subspace and it doesn't make sense to consider it separately.

Sub case 9. $u = x_i$, $v = ax_{i+1} + bx_{i+2}$, $ab \neq 0$

In this case we have that the three vectors $u = x_i$, x_{i+1}, $v = cx_{i+1} + dx_{i+2}$ are linearly



independent. According to this, the triple of vectors

$$(u = x_i,\ x_{i+1}),\ (x_{i+1}, v = cx_{i+1} + dx_{i+2}), (u = x_i, v = cx_{i+1} + dx_{i+2}),$$

because of the fact that they are all in the newly generated 2-subspace, they form just for themselves a kernel 2-subspace, as from the 2-space X^2, as from the new subspace. Now, the vector x_i which is linearly independent with the vectors x_{i-1}, x_{i+1} and $v = cx_{i+1} + dx_{i+2}$ is a loop center of a loop 2-subspace, which at the same time is also a subspace from X^2 as well as from the new 2-subspace. The problem what happens with $v = cx_{i+1} + dx_{i+2}$ remains, and will it be a loop from the new 2-subspace, which at the same time is a 2-subspace from X^2 as from the new 2-subspace.

The question is whether this kernel 2-subspace can be extended to the kernel 2-subspace in the form $S = L(x_i, x_{i+1}, x_{i+2}) \times L(x_i, x_{i+1}, x_{i+2})$. It is clear that

$(x_i, \alpha x_{i+1} + \beta x_{i+2})$ and (x_i, x_{i+1}) are two 2-vectors from the given subspace. Now, it is clear that $(x_i, \alpha x_{i+1} + \beta x_{i+2} - \alpha x_{i+1}) = (x_i, \beta x_{i+2}) \in M$, where from we get that

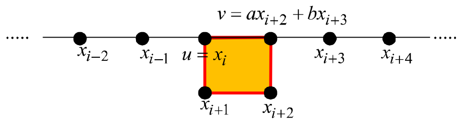
$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\beta} \end{bmatrix} (x_i, \beta x_{i+2}) = (x_i, x_{i+2}) \in M'.$$

According to this, $(x_i, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, x_i)$ are three 2-vectors which are in M', so, the kernel 2-subspace generated from them is consisted in it.

Now, this case overlaps with the sub case 2.

But here, there are many ambiguities regarding what can x_i be? If for example $x_i = x_1$, Tthen kernel 2-subspace make the vectors $(x_1, x_2), (x_2, x_3), (x_3, x_1)$ There is only one loop, and that is the vector x_3, and in the continuation there is one branch, and that are the 2-vectors $(x_3, x_4), ..., (x_{n-1}, x_n)$. Also if $x_{i+2} = x_n$, then kernel 2-subspace make the vectors $(x_n, x_{n-1}), (x_{n-1}, x_{n-2}), (x_{n-2}, x_n)$, a finite branch 2-subspace make the vectors $(x_1, x_2), (x_2, x_3), ..., (x_{n-3}, x_{n-2})$ and we have one loop 2-vector, and that is the vector x_{n-2}. Certainly, the previous two comments apply when n is greater or equal than 4, with absent of the finite branch when $n = 4$.

Sub case 10. $u = x_i$, $v = ax_{i+2} + bx_{i+3}$



In this case we have four vectors, x_i, x_{i+1}, x_{i+2}, v which according to the conditions are linearly independent. But, they form four 2-vectors, and they are

$(x_i = u, x_{i+1}), (x_{i+1}, x_{i+2}), (x_{i+2}, v = ax_{i+2} + bx_{i+3}), (v = ax_{i+2} + bx_{i+3}, u)$.

These four 2-vectors form a cyclic 2-subspace when $i = 1$ as well as when $i + 3 = n$. The vector $u = x_i$ is a loop centre when $i > 1$, and the vector $v = ax_{i+2} + bx_{i+3}$ is always a loop centre.

The vectors $x_1, x_2, ..., x_{i-1}$ form a finite branch 2-subspace when $i > 2$.

Sub case 11. $u = ax_i + bx_{i+1}$, $v \in L(ax_{i-1} + bx_{i+1}, x_i)$ **.**

In this sub case are covered all situations when $i = 1$ and when $i + 1 = n$. In this sub case we have that the 2-vectors $(u, x_i), (u, v)$ and (v, x_i) form a kernel 2-subspace. The vector u is at least a loop of a 2-subspace, and so is the vector v.

In this context should be considered all extensions which are considered in the paper [8] for one-sided branch, and that are the sub cases from 11 to 17 from that paper, including all sub cases which have ' sign (ex. 11').

3. EXTENSION OF A 2-SKEW-SYMMETRIC LINEAR FORM

Theorem. *Let* $\Lambda : M \to \mathbb{R}$ *be a* 2 *-skew-symmetric form such that* $\Lambda(x, y) \leq p(x, y)$ *for every* $(x, y) \in M$, $p : X^2 \to \mathbb{R}$ *be a* 2 *-semi norm and* M *is a branch* 2 *-subspace of* *the 2 -space* X^2. *Let* M' *be an extension of* M *as in sub case 3 of case 2. Then there exists a 2 -skew-symmetric linear form* $\Lambda': M' \to \mathbb{R}$ *such that* $\Lambda'/M = \Lambda$

$$-p(-x,y) \le \Lambda(x,y) \le p(x,y) . \qquad (*)$$

Proof. It is enough to consider the case when $\alpha_{i-1} \neq 0$ and $\alpha_{i+1} \neq 0$. According to this, we can consider that the vector u to have the form $u = b_{11}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{12}x_i$, which is first coordinate or second one from the 2-vector $(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i)$. We choose two such vectors, i.e.

$$(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i)$$
$$(u, b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i) .$$

According to the conditions of the theorem, we have that

$$\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i) + \Lambda(u, b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i) =$$
$$= \Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i + b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i) \le$$
$$= p(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i - v + b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i + v) \le$$
$$= p(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i - v) + p(b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i + v)$$

So, we get that

$$\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1})b_{22}x_i) - p(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i - v) \le$$
$$\le p(b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i + v) - \Lambda(u, b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i) .$$

Now, we get that

$$\sup_{\alpha_{i-1}, \alpha_{i+1}} [\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i) - p(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i - v)] = d \le$$
$$\le p(b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i + v) - \Lambda(u, b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i).$$

From the last equality it is obvious that

$$\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i) - p(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i - v) \le d \le$$
$$\le p(b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i + v) - \Lambda(u, b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i).$$

i.e. the following inequalities are fulfilled

$$\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i) - d \le p(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i - v) \quad (1)$$
$$\Lambda(u, b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i) + d \le p(b_{21}(\alpha'_{i-1}x_{i-1} + \alpha'_{i-1}x_{i+1}) + b_{22}x_i + v). \quad (2)$$

Now, we will determine the 2-linear skew-symmetric form $\Lambda': M' \to \mathbb{R}$ with

$$\Lambda'[A(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i-1}x_{i+1}) + b_{22}x_i + \gamma v)] = (\det A)[\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i-1}x_{i+1}) + b_{22}x_i) + \gamma d]$$
$$\Lambda'(x,y) = \Lambda(x,y), \ (x,y) \in M$$

So, $\Lambda'/M = \Lambda$.

From the other side, if in (1) instead α_{i-1} and α_{i+1} we choose $\frac{\alpha_{i-1}}{t}$ and $\frac{\alpha_{i+1}}{t}$, $t > 0$ and if we use the properties of Λ and p accordingly, we get that

$$\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i) - td \le p(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1}) + b_{22}x_i - tv) \ (3)$$

Completely analogous, if in (2) instead α'_{i-1} and α'_{i+1} we choose $\frac{\alpha'_{i-1}}{t}$ and $\frac{\alpha'_{i+1}}{t}$, $t > 0$ accordingly, and if we use the properties of Λ and p, again, we get that

$$\Lambda(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i-1}x_{i+1}) + b_{22}x_i) + td \le p(b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i-1}x_{i+1}) + b_{22}x_i + tv). \quad (4)$$

Now, from (3) and (4) we see that

$$\Lambda'(u, b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i-1}x_{i+1}) + b_{22}x_i + \gamma v) \leq p(b_{21}(\alpha_{i-1}x_{i-1} + \alpha_{i-1}x_{i+1}) + b_{22}x_i + \gamma v)$$

where from it is clear that in general case $\Lambda' \leq p$ on M'. In other words, the inequality (*) is fulfilled.

CONFLICT OF INTEREST

No conflict of interest was declared from the authors.

AUTHOR'S CONTRIBUTIONS

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EXTENSION OF A CYCLIC 2-SUBSPACE GENERATED BY FOUR 2-VECTORS AND SOME EXTENSIONS OF HAHN-BANACH TYPE FOR SKEW-SYMMETRIC 2-LINEAR FUNCTIONALS DEFINED ON IT

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Abstract. In this paper 2-subspaces from 2-space X^2, which are from the type - cyclic 2-subspace generated by four 2-vectors, will be taken in consideration. Then all its possible extensions adding one element (u,v) and their complete description will be considered. Also, all extensions of 2-skew-symmetric linear form defined on 2-subspace M' Hahn-Banach type will be considered, in the cases when one vector belongs in 2-vector from M, and the other does not belong in any 2-vector, as well as cases when the two coordinates of (u,v) do not belong in M.

1. INTRODUCTION

Extensions of mappings is something that is often looked at in various mathematical disciplines. One classical example of extension of a given mapping is of course the Hanh-Banach theorem for linear functionals. One version of it comprises the contents of the following theorem.

Theorem 1. *Let M be a vector subspace of the vector space X. The functional $p: X \to \mathbb{R}$ satisfies the conditions*

a) $p(x+y) \le p(x) + p(y)$

b) $p(tx) = tp(x)$,

for every $x, y \in X$ and $t \ge 0$.

The functional $f: M \to R$ is linear and $f(x) \le p(x)$. There exists a linear functional $\Lambda: X \to \mathbb{R}$ such that $\Lambda / M = f$ and $-p(-x) \le \Lambda(x) \le p(x)$.

Due to the definition of an n-norm and the definition of an n-semi norm it turned out that, on the set X^2, where X is a vector space over the field Φ (Φ is the field of real numbers or the field of complex numbers), it is convenient to consider additional operations, two of which are partial and one of which is a complete operation, with the aim of making the notation and considerations easier.

Of course, it is worth mentioning here both the definitions, for 2-norm, and especially for 2 semi-norm, which we will use many times further.

Definition 1. Let X be a vector space over the field Φ. The mapping $\|\bullet,\bullet\|: X^2 \to \mathbb{R}_{\ge 0}$ for which the following conditions are fulfilled

(i) $\|x,y\| = 0$ if and only if $\{x,y\}$ is a linear dependent set

(ii) $\|x,y\| = \|y,x\|$ for arbitrary $x, y \in X$

(iii) $\|\alpha x, y\| = |\alpha| \cdot \|x,y\|$ for arbitrary $\alpha \in \Phi$ and for arbitrary $x, y \in X$

(iv) $\|x+x', y\| \le \|x,y\| + \|x',y\|$, for arbitrary $x, y \in X$,

we call **2-norm**, and $(X^2, \|\bullet,\bullet\|)$ we call **2-normed space**.

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Definition 2. Let X is a vector space over the field Φ. The mapping $p: X^2 \to \mathbb{R}_{\geq 0}$ for which the following conditions are fulfilled

(i) $p(x,y) \geq 0$ if and only if $\{x,y\}$ is a linear dependent set

(ii) $p(x,y) = p(y,x)$ for arbitrary $x,y \in X$

(iii) $p(\alpha x, y) = |\alpha| \cdot p(x,y)$ for arbitrary $\alpha \in \Phi$ and for arbitrary $x,y \in X$

(iv) $p(x+x',y) \leq p(x,y) + p(x',y)$, for arbitrary $x,y \in X$,

we call **2-semi norm**, and (X^2, p) we call **2-semi normed space**.

It is worth to note here that for any 2-norm the following equation is fulfilled $\| x,y \| = \| x, y+\alpha x \|$, for arbitrary $x,y \in X$ and for arbitrary scalar $\alpha \in \Phi$.

With further consideration and development of the previous equation, the following definitions for 2-semi-normed space, for 2-norm and for 2-semi-norm were directly imposed.

Definition 3. Let X be a vector space over the field Φ. The mapping $p: X^2 \to \mathbb{R}$ for which the following conditions hold

(a) $p(x+y,z) \leq p(x,z) + p(y,z)$, for every $x,y,z \in X$

(b) $p(A(x,y)) = |\det A| \, p(x,y)$, for every $x,y \in X$ and $A \in M_2(\Phi)$.

is called a **2-semi norm** and (X^2, p) is called a **2-semi normed space**.

Definition 4. The mapping $\| \cdot \| : X^n \to \mathbb{R}$, $n \geq 2$ for which it is fulfilled that:

(a) $\| x_1, x_2 \| = 0$ if and only if x_1, x_2 are linear dependant vectors;

(b) $\| A(x_1,x_2) \| = |\det A| \, \| x_1, x_2 \|$, for all $x_1, x_2 \in X$ and for all $A \in M_2(\Phi)$;

(c) $\| x_1 + x_2, x_3 \| \leq \| x_1, x_3 \| + \| x_2, x_3 \|$, for all $x_1, x_2, x_3 \in X$,

we call **2-norm** of the vector space X, and the ordered pair $(X, \|\cdot,\cdot\|)$ we call **2-normed space**.

Of course this in itself has led to consideration of definitions for addition operations and 2-vector multiplication operations with scalar, which are the basic operations in X^2. In other words, we will consider, i.e. under consideration is the following definition.

Definition 5. Let X be a vector space over the field Φ. The set X^2 together with the operations

$$(x,z) + (y,z) = (x+y,z)$$

$$(z,x) + (z,y) = (z,x+y)$$

$$A(x,y) = A(x,y)^T$$

where $x,y,z \in X$ and $A \in M_2(\Phi)$ is called a **2-vector space** or **2-space**.

Comment. The third operation in the previous definition is a complete operation, and on the right-hand side of the equality is a multiplication of a matrix with a vector.

Definition 6. Let X be a vector space over the field Φ. The functional $\Lambda: X^2 \to \Phi$ for which the following conditions hold

(a) $\Lambda(x+y,z) = \Lambda(x,z) + \Lambda(y,z)$, for arbitrary $x,y,z \in X$

(b) $\Lambda(x,y) = -\Lambda(y,x)$ for arbitrary $x,y \in X$

(c) $\Lambda(\alpha x, y) = \alpha \Lambda(x,y)$, for arbitrary $x,y \in X$ and $\alpha \in \Phi$.

is called **skew-symmetric 2-linear form**.

It is not hard to prove that the previous definition (Definition 6) is equivalent with the following definition.

Definition 7. Let X be a vector space over the field Φ. The functional $\Lambda : X^2 \to \Phi$ for which the following conditions hold

(a) $\Lambda(x+y,z) = \Lambda(x,z) + \Lambda(y,z)$, for arbitrary $x,y,z \in X$

(b) $\Lambda(A(x,y)) = (\det A)\Lambda(x,y)$, for arbitrary $x,y \in X$ and $A \in M_2(\Phi)$

is called **skew-symmetric 2-linear form** or simply **2-linear functional**.

This makes levelling to all previous definitions for such considerations.

The very definition of a 2-semi-norm and the numerous examples that were constructed after that led to a situation, analogous as in vector space, to consider different subsets. Among them are of course the sets that are closed in relation to the addition and multiplication operations with matrix as basic operations in X^2.

Definition 8. The subset S, $S \subseteq X^2$ which is closed with respect to the operations of the 2-space X^2 is called **2-subspace** of X^2.

Comment. Of course, one of the most bitter problems associated with the operations on X^2 and the subsets of X^2 at the given moment is the complete description of the structure of the 2-subspaces of the 2-space X^2. Due to this, we will focus our attention on only one special type of 2-subspaces of the 2-vector space X^2.

In the following considerations the following theorem about subspaces is useful.

Theorem 2. *The intersection of an arbitrary family of 2-subspaces of the 2-vector space X^2 is a 2-subspace.*

According to the last theorem, each subset $A \subseteq X^2$ determines a 2-subspace S_A, the smallest 2-subspace of the 2-vector space X^2 which contains the set A. We will call the 2-subspace S_A the 2-subspace **generated by the set** A, and the set A **- the generating set**.

In this matter we will consider a special type of generating sets, i.e. a generating set of the form $M \cup \{(u,v)\}$, where M is a special type of a 2-subspace, and $(u,v) \in X^2$ is arbitrarily given where $\{u,v\}$ is a linearly independent set.

The basic question which we will consider here is whether it is possible to extend a 2-skew-symmetric linear form defined on some types, i.e. classes 2-subspaces to a bigger subspace, in the sense of extension of linear functionals, i.e. of the type of Hanh-Banach.

The main aim if all such considerations is whether we can prove the following theorem or some of its variants.

Theorem 3. *Let S be a 2-subspace of the 2-space X^2, $\Lambda : S \to \mathbb{R}$ be 2-skew-symmetric linear form, and $p : X^2 \to \mathbb{R}$ be a mapping for which*

(a) $p(x+y,z) \le p(x,z) + p(y,z)$, *for all* $x,y,z \in X$

(b) $p(tx,y) = tp(x,y)$, *for all* $x,y \in X$ *and* $t > 0$.

There exists 2-skew-symmetric linear form $\Lambda' : X^2 \to \mathbb{R}$ *, such that* $\Lambda'/S = \Lambda$.

Each 2-seminorm satisfies the conditions a) and b) from the previous theorem.

Furthermore, in many parts we may come across a special kind of subset of X^2. One type of them is given in the following definition.

Definition 9. The subset $T, T \subseteq X^2$ is called n**-invariant** if $AT \subseteq T$ for every $A \in M_2(\Phi)$, $\det A = 1$.

Solving the problem presented in the last theorem is of course not simple. An affirmation of that is of course the complex structure of the 2-subspaces of the 2-space X^2. Due to this, we will discuss partial cases of this problem.

In this matter we will look at extension of a 2-skew-symmetric form defined on a cyclic 2-subspace.

From here on, we will assume that the subset $\{x_1, x_2, ..., x_n\}$ is a linearly independent subset of the vector space X.

Definition 10. Let X be a vector space over the field Φ. The 2-subspace S generated by the set $\{(x_1,x_2),(x_2,x_3),...,(x_n,x_1)\}, n \geq 3$, is called a **cyclic 2-subspace.**

In the next part we will deal with 2-subspaces that are generated with 4 elements. They are quite characteristic and in many ways significantly different from other cyclic 2-subspaces that are generated with 5 or more elements. Therefore, definition 10 will get the following form

Definition 10'. Let X be a vector space over the field Φ. The 2-subspace S generated by the set $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),(x_4,x_1)\}$, is called a **cyclic 2-subspace generated with four elements.**

A detailed description of this kind of a 2-subspace is given in [7]. That is the content of the theorem that follows.

Theorem 4. *The cyclic 2-subspace generated by the elements of the set* $\{(x_1,x_2),(x_2,x_3),...,(x_n,x_1)\}, n \geq 5$ *is*

$$M = \bigcup_{i=1}^{n} [L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i)],$$

where $x_{n+1} \equiv x_1$ *and* $x_0 \equiv x_n$.

In the case $n = 4$ this theorem has the following form

Theorem 4'. *The cyclic 2-subspace generated by the elements of the set* $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),(x_4,x_1)\}$ *is*

$$M = \bigcup_{\substack{i=1 \\ x_0 = x_4, x_5 = x_1}}^{4} [L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i)].$$

The leading result in the description of the special 2-subspaces such as cyclic, branch 2-subspaces, kernel 2-subspaces and loop 2-subspaces is the following lemma:

Lemma. *The subspace generated by the elements* $(x_{i-1},x_i),(x_i,x_{i+1}),(x_{i+1},x_{i+2})$, *where* $\{x_{i-1},x_i,x_{i+1},x_{i+2}\}$ *is a linearly independent set is*

$$L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \times L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \cup L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i)$$

The idea for such lemma is because here it seems as if we have put together two branches, i.e.

$$L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \times L(b_{i+2}x_{i+2} + b_i x_i, x_{i+1}) \qquad (1)$$

and $L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\times L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)$. (2)

Here, as its 2-subspace appears a set determined with

$M=\{(A(x_i,x_{i+1})^T \,/\, A\in M_2(\Phi)\}$.

Addition of elements from (1) and (2) certainly is possible, but the result is always an element which can be considered that belongs in one of these 2-subspaces, i.e. either in (1) or in (2), except maybe in some special cases when it belongs in the 2-subspace $M=\{(A(x_i,x_{i+1})^T \,/\, A\in M_2(\Phi)\}$.

2. EXTENSION OF A CYCLIC 2-SUBSPACE

Let Λ be a skew-symmetric linear form defined on a branch 2-subspace M, generated by the elements of the set $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),....,(x_{n-1},x_n),(x_n,x_1)\}$ where $\{x_1,x_2,...,x_n\}$ is a linearly independent set. Let $(u,v)\in X^2$ be such that $\{u,v\}$ is a linearly independent set. We denote the 2-subspace of X^2 generated by $M\cup\{(u,v)\}$ by M'. Several cases are possible.

Case 1. $u,v\notin L(x_1,x_2,x_3,x_4)$, where $L(x_1,x_2,x_3,x_4)$ is the subspace of X generated by $\{x_1,x_2,x_3,x_4\}$.

Case 2. $u\in L(x_1,x_2,x_3,x_4)$ and $v\notin L(x_1,x_2,x_3,x_4)$. The case $u\notin L(x_1,x_2,x_3,x_4)$ and $v\in L(x_1,x_2,x_3,x_4)$ is completely analogous due to symmetry.

Case 3. $u,v\in L(x_1,x_2,x_3,x_4)$.

In cases 2 and 3 there are several sub cases:

Sub cases of case 1

The 2-subspace that is determined here is

$M'=M\cup\left\{A(u,v)^T \,/\, A\in M_2(\Phi)\right\}$

Indeed, from the fact that $u,v\notin L(x_1,x_2,x_3,x_4)$, there exist vectors y,z which are not equal to zero vector and for which

$u=\alpha_1x_1+\alpha_2x_2+\alpha_3x_3+\alpha_4x_4+\gamma y=x+\gamma y$

$v=\beta_1x_1+\beta_2x_2+\beta_3x_3+\beta_4x_4+\delta z=w+\delta z$,

where $\gamma\delta\neq 0$. But then for any matrix $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ which is nonsingular, we have the following possibilities

1° $a_{11}\neq 0$, $a_{12}=0$

2° $a_{11}=0$, $a_{12}\neq 0$

3° $a_{11}\neq 0$, $a_{12}\neq 0$

For 1° we have the following three additional possibilities:

a) $a_{21}\neq 0$, $a_{22}=0$, which is not possible because in this situation we would have that $\det A=0$, which is not possible.

b) $a_{21}=0$, $a_{22}\neq 0$, which is possible. In this situation $\det A=a_{11}a_{22}\neq 0$. Here

$A(u,v)^T=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}=(a_{11}u,a_{22}v)=(a_{11}(x+\gamma y),a_{22}(w+\delta z))=(a_{11}x+a_{11}\gamma y,a_{22}w+a_{22}\delta z)$, where from because of the condition $a_{11}\gamma a_{22}\delta \neq 0$, we get that $A(u,v)^T \notin M$. This element will belong in the new set which is added.

c) $a_{21} \neq 0$, $a_{22} \neq 0$, which is possible. In this situation $\det A = a_{11}a_{22} \neq 0$. Here, same as the previous case

$$A(u,v)^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = (a_{11}u, a_{21}u + a_{22}v) = (a_{11}(x+\gamma y), a_{21}(x+\gamma y) + a_{22}(w+\delta z)) =$$

$$= (a_{11}x + a_{11}\gamma y, a_{21}x + a_{22}w + a_{21}\gamma y + a_{22}\delta z)$$

where from because of the condition $a_{11}\gamma \neq 0$ we have that $A(u,v)^T \notin M$. This element belongs in the new set $\{A(u,v)^T \,/\, A \in M_2(\Phi)\}$.

For 2° we have the following three additional possibilities:

a) $a_{21} \neq 0$, $a_{22} = 0$, which is possible, because in this case we have that

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = -a_{12}a_{21} \neq 0 .$$

But, in this case we have that

$$A(u,v)^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = (a_{12}v, a_{21}u) = (a_{12}(w+\delta z), a_{21}(x+\gamma y)) = (a_{12}w + a_{12}\delta z, a_{21}x + a_{21}\gamma y)$$

and now because of the condition $a_{12}\delta a_{21}\gamma \neq 0$, $A(u,v)^T \notin M$.

b) $a_{21} = 0$, $a_{22} \neq 0$, which is possible from technical view. But,

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = 0 \cdot a_{22} - 0 \cdot a_{21} = 0 ,$$

And by starting assumption we have that $\det A \neq 0$. Because of this contradiction, this case is not possible in this situation.

c) $a_{21} \neq 0$, $a_{22} \neq 0$. This case is possible from technical view. Indeed,

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = 0 \cdot a_{22} - a_{12} \cdot a_{21} = -a_{12} \cdot a_{21} \neq 0 .$$

In this case we have that

$$A(u,v)^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = (a_{12}v, a_{21}u + a_{22}v) = (a_{12}(w+\delta z), a_{21}(x+\gamma y) + a_{22}(w+\delta z)) =$$

$$= (a_{12}w + a_{12}\delta z, a_{21}x + a_{21}\gamma y + a_{22}w + a_{22}\delta z) =$$

$$= a_{12}(w+\delta z, a_{21}x + a_{21}\gamma y + a_{22}w + a_{22}\delta z) \notin M$$

because the first component $w + \delta z \in L(x_1, x_2, x_3, x_4)$.

For 3° we have the following three additional possibilities:

a) $a_{21} \neq 0$, $a_{22} = 0$, which is possible, because in this case we have that

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -a_{21}a_{12} \neq 0 ,$$

and the matrix is nonsingular. According to that

$$A(u,v)^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = (a_{12}v, a_{21}u) = (a_{12}(w+\delta z), a_{21}(x+\gamma y)) =$$

$$= a_{12}a_{21}(w + a_{12}\delta z, x+\gamma y) \notin M$$

b) $a_{21} = 0$, $a_{22} \neq 0$, which is also possible, where

$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} \neq 0$. Here,

$$A(u,v)^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = (a_{11}u, a_{22}v) = (a_{11}(x+\gamma y), a_{22}(w+\delta z)) =$$

$$= a_{11}a_{22}(x+\gamma y, w + a_{12}\delta z) \notin M$$

c) $a_{21} \neq 0$, $a_{22} \neq 0$. This case because of its nature is the most radical one. But, here, if we use the technique which we have in the 2-normed spaces, we have that

$$A(u,v)^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = (a_{11}u + a_{12}v, a_{21}u + a_{22}v) =$$

$$= (a_{11}(x+\gamma y) + a_{12}(w+\delta z), a_{21}(x+\gamma y) + a_{22}(w+\delta z)) =$$

$$= (a_{11}x + a_{11}\gamma y + a_{12}w + a_{12}\delta z, a_{21}x + a_{22}w + a_{21}\gamma y + a_{22}\delta z) \sim (=)$$

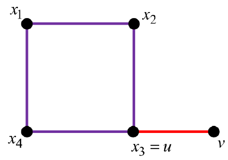
$$= (\det A)(x+\gamma y, w+\delta z) = ((\det A)x + (\det A)\gamma y, w+\delta z) \notin M$$

because $(\det A)\gamma \neq 0$, where from we get that the first element does not belong in any element of M and with that the whole element does not belong in M. Also, let us note that in the part where we have $\sim(=)$ it is a sign of equality. But, that is not a problem, because from that element till the ending element we constantly multiply with a matrix that has a determinant equal to 1, so according to this if one element does not belong in M, then also any element multiplied with a matrix that has a determinant equal to 1 does not belong in M. The last equality may not be used, because $\det A \neq 0$ and $\gamma \neq 0$ where from follows the proof.

Sub cases of case 2

Sub case 1. $u = x_i$, for any $i = 1,2,3,4$, which in the discussion is a fixed element.

In this case we have a situation as given in the scheme in case when we have 4 generator 2-vectors; the rest of the cases are equal to this one that is illustrated (see the drawing; in other words, the vector u can be any of the vectors x_1, x_2, x_3, x_4; here, only the case $u = x_3$ is considered, and all other cases are equivalent to that one).



In this situation, we have the existing cyclic 2-subspace, which has undistorted structure with the addition of one element and a structure of one pure loop 2-subspace S, generated by the elements $(u,v) = (x_3, v), (u, x_2) = (x_3, x_2)$ and $(u, x_4) = (x_3, x_4)$. Let us mention here that

$$S = \bigcup_{w \in L(v, x_2, x_4)'} L(w,u) \times L(w,u).$$

We are interested especially the cases when $w=\alpha v+\beta x_2+\gamma x_4$, where especially $\alpha\neq 0$. Since

$$M=\bigcup_{\substack{i=1\\ x_0=x_4,x_5=x_1}}^{4}\left[L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\times L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\right]$$

we can prove directly that

$$M'=M\cup S=\bigcup_{\substack{i=1\\ x_0=x_4,x_5=x_1}}^{4}\left[L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\times L(a_{i+1}x_{i+1}+a_{i-1}x_{i-1},x_i)\right]\cup\left[\bigcup_{w\in L(v,x_2,x_4)'}L(w,u)\times L(w,u)\right]$$

Indeed, if we have an element from the set S, it has the form $A(u,\alpha v+\beta x_2+\gamma x_4)$ where we consider that $\det A\neq 0$, and let us have an element from the cyclic 2-subspace. Then, that element has to have the form

a) $B(u,\alpha_2x_2+\alpha_4x_4)$, $B\in M_2(\Phi)$

b) $C(x_4,\alpha_3u+\alpha_1x_1)$, $C\in M_2(\Phi)$

c) $D(x_1,\alpha_2x_2+\alpha_4x_4)$, $D\in M_2(\Phi)$

d) $E(x_2,\alpha_1x_1+\alpha_4x_4)$, $E\in M_2(\Phi)$.

Everyone of this cases will be considered separately, where we will consider that $\det B,\det C,\det D,\det E\neq 0$

Case c) In this case every vector from $D(x_1,\alpha_2x_2+\alpha_4x_4)$ has the form

$$D(x_1,\alpha_2x_2+\alpha_4x_4)=\begin{bmatrix}d_{11}&d_{12}\\d_{21}&d_{22}\end{bmatrix}(x_1,\alpha_2x_2+\alpha_4x_4)=$$
$$=(d_{11}x_1+d_{12}(\alpha_2x_2+\alpha_4x_4),d_{21}x_1+d_{22}(\alpha_2x_2+\alpha_4x_4)).$$

Since $\det D\neq 0$, then either $d_{11}\neq 0$ or $d_{21}\neq 0$, so, according to this either in the first coordinate or in the second coordinate there will exist somewhere in the adding a vector x_1 and because of that cannot be equal with any element from S, which has the form

$$A(u,\alpha v+\beta x_2+\gamma x_4)=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}(u,\alpha v+\beta x_2+\gamma x_4)=$$
$$=(a_{11}x_3+a_{12}(\alpha v+\beta x_2+\gamma x_4),a_{21}x_3+a_{22}(\alpha v+\beta x_2+\gamma x_4))$$

In the 2-vector $D(x_1,\alpha_2x_2+\alpha_4x_4)$ in one of the coordinates always there will appear x_1. So, according to this, $S\cap M=\varnothing$ (or $S\cap M\subseteq\Delta_2$) in this situation.

Case b) In this case, we have that the elements from $C(x_4,\alpha_3u+\alpha_1x_1)$ has the form

$$C(x_4,\alpha_3u+\alpha_1x_1)=C(x_4,\alpha_3x_3+\alpha_1x_1)=\begin{bmatrix}c_{11}&c_{12}\\c_{13}&c_{14}\end{bmatrix}(x_4,\alpha_3x_3+\alpha_1x_1)=$$
$$=(c_{11}x_4+c_{12}(\alpha_3x_3+\alpha_1x_1),c_{21}x_4+c_{22}(\alpha_3x_3+\alpha_1x_1))=(\sim)$$
$$=(\det C)(x_4,\alpha_3x_3+\alpha_1x_1)=$$

But, the matrix C is nonsingular, so according to this either $c_{11}\neq 0$ or $c_{12}\neq 0$, so in order to have operation addition, it has to $c_{12}\neq 0$, and $c_{11}=0$. In this case we have that α_1 has to be equal to zero, $a_{12}=0$, where $c_{12}\alpha_3=a_{11}$. Then, we would have that

$$A(u,\alpha v+\beta x_2+\gamma x_4)=\begin{bmatrix}0 & a_{12}\\ a_{21} & a_{22}\end{bmatrix}(u,\alpha v+\beta x_2+\gamma x_4)=(a_{12}x_3,a_{21}x_3+a_{22}(\alpha v+\beta x_2+\gamma x_4))$$

$$C(x_4,\alpha_3 u+\alpha_1 x_1)=\begin{bmatrix}0 & c_{12}\\ c_{21} & c_{22}\end{bmatrix}(x_4,\alpha_3 x_3)=(c_{12}\alpha_3 x_3,c_{21}x_4+c_{22}(\alpha_3 x_3+\alpha_1 x_1))$$

and the addition can be done. The result is always a 2-vector is both in S and in M, as well.

Case d) This case due to the complete symmetry is fully analogous to the previous Case b) and there is no need to be considered.

Case a) It is clear that this case is absolutely possible. The general member from the subspace generated from the element $(u,\alpha_2 x_2+\alpha_4 x_4)=(x_3,\alpha_2 x_2+\alpha_4 x_4)$ is equal to

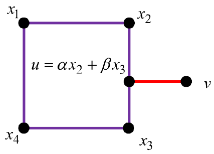
$$B(u,\alpha_2 x_2+\alpha_4 x_4)=(b_{11}x_3+b_{12}(\alpha_2 x_2+\alpha_4 x_4),b_{21}x_3+b_{22}(\alpha_2 x_2+\alpha_4 x_4)).$$

It is clear that he elements from the 2-subspace generated from $(u,\alpha v+\beta x_2+\gamma x_4)=(x_3,\alpha v+\beta x_2+\gamma x_4)$, have the following form

$$A(u,\alpha v+\beta x_2+\gamma x_4)=A(x_3,\alpha v+\beta x_2+\gamma x_4)=$$
$$=(a_{11}x_3+a_{12}(\alpha v+\beta x_2+\gamma x_4),a_{21}x_3+a_{22}(\alpha v+\beta x_2+\gamma x_4))$$

It is obvious that addition is possible in more than one variant and the result is always an element from the new 2-subspace described before.

Sub case 2. $u=\alpha x_i+\beta x_{i+1}$ where $\alpha,\beta\neq 0$



In this situation, the element u can belong in any 2-dimensional vector subspace $L(x_i,x_{i+1})$ for $u=1,2,3,4$ where $x_5\equiv x_1$. Because of the previous considerations we have that $u=\alpha x_2+\beta x_3,\alpha\beta\neq 0$.

Here, because of determination, we choose $u\in L(x_2,x_3)$, i.e. $u=\alpha x_2+\beta x_3,\alpha\beta\neq 0$ (because of the condition $\alpha\beta\neq 0$ we have that $\alpha\neq 0$ and $\beta\neq 0$; it means that $u\neq x_2$ and $u\neq x_3$; so, u is any element, which is fixed, from the 2-dimensional vector subspace from the vector space X except the vectors x_2 and x_3, but the vector u is a fixed vector. Completely analogous in total, are considered the following three cases:

a) $u\in L(x_1,x_2)$, i.e. $u=\alpha x_1+\beta x_2$, $\alpha\beta\neq 0$

b) $u\in L(x_3,x_4)$, i.e. $u=\alpha x_3+\beta x_4$, $\alpha\beta\neq 0$

c) $u\in L(x_4,x_1)$, i.e. $u=\alpha x_4+\beta x_1$, $\alpha\beta\neq 0$

In this situation which we will consider, we have a situation in which $(u,v),(u,x_2)$ forms one branch , from one side, and $(u,v),(u,x_3)$ another branch, from the other side, where $u=\alpha x_2+\beta x_3$, $\alpha\beta\neq 0$.

Can we consider, in this case, the vector $u=\alpha x_2+\beta x_3$ as a loop in one loop 2-subspace which is generated by the three 2-vectors $(u,v),(u,x_2),(u,x_3)$ where $u=\alpha x_2+\beta x_3,\alpha\beta\neq 0$? It is clear that u is generated from two vectors which build the starting cyclic 2-subspace, i.e.

$$(u,m)=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}(x_2,x_3)^T=(\alpha x_2+\beta x_3,\gamma x_2+\delta x_3).$$

Now, from formal point of view we have that:

- All elements from the 2-subspace generated by $(u,v),(u,x_2)$ are 2-vectors in the following form

$$A(\alpha_1 v+\alpha_2 x_2,u)^T=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}(\alpha_1 v+\alpha_2 x_2,\alpha x_2+\beta x_3)^T=$$

$$=(a_{11}(\alpha_1 v+\alpha_2 x_2)+a_{12}(\alpha x_2+\beta x_3),a_{21}(\alpha_1 v+\alpha_2 x_2)+a_{22}(\alpha x_2+\beta x_3))=$$

$$=(a_{11}\alpha_1 v+(a_{11}\alpha_2+a_{12}\alpha)x_2+a_{12}\beta x_3,a_{21}\alpha_1 v+(a_{21}\alpha_2+a_{22}\alpha)x_2+a_{22}\beta x_3)$$

All elements from the 2-subspace generated by $(u,v),(u,x_3)$ are 2-vectors in the following form

$$B(\beta_1 v+\beta_3 x_3,u)^T=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}(\beta_1 v+\beta_3 x_3,\alpha x_2+\beta x_3)^T=$$

$$=(b_{11}(\beta_1 v+\beta_3 x_3)+b_{12}(\alpha x_2+\beta x_3),b_{21}(\beta_1 v+\beta_3 x_3)+b_{22}(\alpha x_2+\beta x_3))=$$

$$=(b_{11}\beta_1 v+b_{12}\alpha x_2+(b_{11}\beta_3+b_{12}\beta)x_3,b_{21}\beta_1 v+b_{22}\alpha x_2+(b_{21}\beta_3+b_{22}\beta)x_3)$$

These are the two most general elements from the two subspaces and over them addition can be done only in different cases, which will be considered further in the text. Here in order addition to be possible, one of the coordinates has to be the same. We will choose only the first coordinate to be the same. The considerations for the second coordinate is the same as for the first one and in given moment we may include it too, parallel to the first one, if there is a need for it. Here we have the equation for the first two coordinates which has to be fulfilled:

$$a_{11}(\alpha_1 v+\alpha_2 x_2)+a_{12}(\alpha x_2+\beta x_3)=b_{11}(\beta_1 v+\beta_3 x_3)+b_{12}(\alpha x_2+\beta x_3) \qquad (*)$$

i.e. $a_{11}\alpha_1 v+(a_{11}\alpha_2+a_{12}\alpha)x_2+a_{12}\beta x_3=b_{11}\beta_1 v+b_{12}\alpha x_2+(b_{11}\beta_3+b_{12}\beta)x_3$, (**)

having in mind that $\alpha\beta\neq 0$, i.e. $\alpha,\beta\neq 0$.

We will consider all possible cases for equality of the two vectors by the first coordinate. Here we get the system of equations

$$\begin{cases} a_{11}\alpha_1=b_{11}\beta_1 \\ a_{11}\alpha_2+a_{12}\alpha=b_{12}\alpha \\ a_{12}\beta=b_{11}\beta_3+b_{12}\beta \end{cases}.$$

We will determine when this system has solutions.

Case 1. $a_{11}=0$.

Then the system gets the form

$$\begin{cases} b_{11}\beta_1=0 \\ a_{12}\alpha=b_{12}\alpha \\ a_{12}\beta=b_{11}\beta_3+b_{12}\beta \end{cases}$$ and because $\alpha,\beta\neq 0$ we get the system $$\begin{cases} b_{11}\beta_1=0 \\ a_{12}=b_{12} \\ b_{11}\beta_3=0 \end{cases}$$

Then (*) gets the form

$$a_{12}(\alpha x_2+\beta x_3)=b_{11}(\beta_1 v+\beta_3 x_3)+b_{12}(\alpha x_2+\beta x_3)$$

(*),

i.e. $b_{11}(\beta_1 v+\beta_3 x_3)=0$.

Sub case 1. Now, if $b_{11}=0$, then for the first coordinate we have that $a_{12}(\alpha x_2+\beta x_3)=b_{12}(\alpha x_2+\beta x_3)$ and addition is possible and the sum is clear.

Sub case 2. If $b_{11} \neq 0$, then the two scalars β_1 and β_3 are zero. But then practically we get an element from the set Δ_2, i.e. $B(\beta_1 v + \beta_3 x_3, u)^T = B(o,u)^T \in \Delta_2$. Now, because $a_{12} = b_{12}$ addition is possible and the sum is clear.

Case 2. $a_{11} \neq 0$.

Then, the system gets the form

$$\begin{cases} a_{11}\alpha_1 = b_{11}\beta_1 \\ a_{11}\alpha_2 = (b_{12} - a_{12})\alpha \\ b_{11}\beta_3 = (a_{12} - b_{12})\beta \end{cases}$$

Sub case 1. In this situation it is possible $b_{11} = 0$, but then because of the condition $\beta \neq 0$, we would have that $a_{12} = b_{12}$. Then, for the first coordinate of the 2-vectors we would have that the equality $a_{12}(\alpha x_2 + \beta x_3) = b_{12}(\alpha x_2 + \beta x_3)$ holds.

So, addition is possible and here we would have a 2-vector which belongs same as when the vector u is a loop element.

Sub case 2. Also, in this situation it is possible that $b_{11} \neq 0$, too. But, then we would have a system which will come down to the following

$$\begin{cases} \alpha_1 = \dfrac{b_{11}\beta_1}{a_{11}} \\ \dfrac{a_{11}\alpha_2}{\alpha} = b_{12} - a_{12} \\ \dfrac{b_{11}\beta_3}{\beta} = a_{12} - b_{12} \end{cases}$$

In such a situation we would have that $\alpha_1 = \dfrac{b_{11}\beta_1}{a_{11}}$, $a_{12} = \dfrac{b_{11}\beta_3}{\beta} + b_{12}$ and because of the equality $\dfrac{a_{11}\alpha_2}{\alpha} = -\dfrac{b_{11}\beta_3}{\beta}$, we get that $\alpha_2 = -\dfrac{b_{11}}{a_{11}}\dfrac{\alpha}{\beta}\beta_3$. Now, if we make a substitution in (*) or in (**) we would get that

$$\begin{aligned} a_{11}(\alpha_1 v + \alpha_2 x_2) + a_{12}(\alpha x_2 + \beta x_3) &= a_{11}\left(\frac{b_{11}}{a_{11}}\beta_1 v - \frac{b_{11}}{a_{11}}\frac{\alpha}{\beta}\beta_3 x_2\right) + a_{12}(\alpha x_2 + \beta x_3) = \\ &= b_{11}\beta_1 v - b_{11}\frac{\alpha}{\beta}\beta_3 x_2 + \left(b_{12} + \frac{b_{11}\alpha_3}{\beta}\right)(\alpha x_2 + \beta x_3) \\ &= b_{11}\beta_1 v - b_{11}\frac{\alpha}{\beta}\beta_3 x_2 + b_{12}(\alpha x_2 + \beta x_3) + b_{11}\frac{\alpha}{\beta}\beta_3 x_2 + b_{11}\alpha_3 x_3 = \\ &= b_{11}\beta_1 v + b_{11}\alpha_3 x_3 + b_{12}(\alpha x_2 + \beta x_3) = \\ &= b_{11}(\beta_1 v + \alpha_3 x_3) + b_{12}(\alpha x_2 + \beta x_3) \end{aligned}$$

According to that, the 2-vectors are additive. Their sum is not hard to calculate.

From these reasons, from formal point of view, regarding the addition of the elements from X^2, the element u will be called as a **loop/sub loop element**.

So, finally we have that

$M' = M \cup S$,

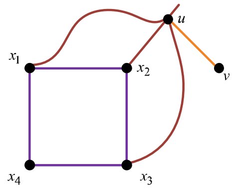
where S is that loop-sub loop 2-subspace with loop u.

Sub case 3. The vector u is a coordinate of some vector $(u,w)\in L(\alpha_1x_1+\alpha_3x_3,x_2)\times L(\alpha_1x_1+\alpha_3x_3,x_2)$ and $v\notin L(x_1,x_2,x_3,x_4)$, additional restrictions for u.

In this situation we have only one case which can be seen in the drawing. In this situation the vector u can be shown in the following form

$$(u,w)=A(\alpha_1x_1+\alpha_3x_3,x_2)^T=(\underbrace{a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2}_{u},\underbrace{a_{21}(\alpha_1x_1+\alpha_3x_3)+a_{22}x_2}_{w}),$$

i.e. $u=a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2$, where $a_{11},\alpha_1,\alpha_3,a_{12}$ are arbitrary scalars which at this moment are fixed. All elements which would be considered are from this type. Now, the 2-vector (u,v) has the following form $(a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2,v)$ and the elements from this 2-subspace which is generated from this 2-vector have the following form $B(a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2,v)^T$. So, their general form would be



$$B(a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2,v)^T=(b_{11}((\alpha_1x_1+\alpha_3x_3)+a_{12}x_2)+b_{12}v,b_{21}((\alpha_1x_1+\alpha_3x_3)+a_{12}x_2)+b_{22}v)$$

On the other hand, the vector v does not belong in $L(x_1,x_2,x_3,x_4)$, so according to that $v\neq\beta_1x_1+\beta_2x_2+\beta_3x_3+\beta_4x_4$, for any $\beta_1,\beta_2,\beta_3,\beta_4$ from the field Φ. According to this, for the vector v we have a form $v=x+\alpha y$, where $x=\beta_1x_1+\beta_2x_2+\beta_3x_3+\beta_4x_4$ for fixed scalars $\beta_1,\beta_2,\beta_3,\beta_4$, and y is a vector which is not from $L(x_1,x_2,x_3,x_4)$.

Now, it is unclear if the vector u is a loop element.

From the definition of this 2-subspace is clear that (u,v) is its element. But, also an element is the 2-vector (u,x_2), too, which can be presented in the form

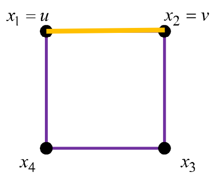
$$(u,x_2)=A(\alpha_1x_1+\alpha_3x_3,x_2)^T=\begin{bmatrix}a_{11}&a_{12}\\0&1\end{bmatrix}(\alpha_1x_1+\alpha_3x_3,x_2)^T=(\underbrace{a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2}_{u},x_2).$$

This 2-vector together with the 2-vector (x_2,x_1) form a finite branch 2-subspace which we had at the beginning of the discussion. Here finally we have that the vector u is a loop of two elements from M', or it can be considered as a finite branch 2-subspace.

Comment. Sub case 3 is of course much more complicated than sub case 2.

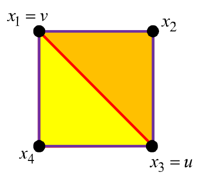
Sub cases of case 3

Sub case 1. $u=x_i$, $v=x_{i+1}$ for some $i\in\{1,2,..n\}$.



In this case we do not have extension. According to the definition of M and the definition of the 2-subspace generated by the element $(u,v)\equiv(x_i,x_{i+1})$, i.e. $M=M'$

Sub case 2. $u=x_i$, $v=x_{i+2}$ where without loss of generality we can assume that $i+2\leq n$.



Especially, only in this case, with addition of one element we get two 2-subspaces which are kernel 2-subspaces and which touch each other. In this case, as shown in the drawing, only one element is enough (in this case it is (x_2, x_4) so that we get a complete kernel 2-subspace). Now it is important to determine the general type of elements from this new 2-subspace.

We will consider now the two kernel 2-subspaces, generated by $(u,x_2),(x_2,v),(v,u)$ and $(u,x_4),(x_4,v),(v,u)$ separately, as well as the loop 2-subspaces which are 2-subspaces from the new 2-subspace M', generated by $(x_3,x_2),(x_3,x_4),(x_3,x_1)$ and $(x_1,x_2),(x_1,x_4),(x_1,x_3)$.

Situation a)

A kernel 2-subspace generated by $(u,x_2),(x_2,v),(v,u)$, i.e. by $(x_3,x_2),(x_2,x_1),(x_1,x_3)$.

The elements from the kernel 2-subspace generated by $(u,x_2),(x_2,v),(v,u)$, because of the way of denotation and formation have the form $(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3)$.

Situation b)

A kernel 2-subspace generated by $(u,x_4),(x_4,v),(v,u)$, i.e. by $(x_3,x_4),(x_4,x_1),(x_1,x_3)$

Again because of the way of denotation and formation of the elements of this 2-subspace, we have that they have the following form $(\gamma_1 x_1 + \gamma_3 x_3 + \gamma_4 x_4, \delta_1 x_1 + \delta_3 x_3 + \delta_4 x_4)$.

Between this two elements we can determine addition only in the case when

i) $\alpha_2 = 0$ and $\gamma_4 = 0$.

ii) $\beta_2 = 0$ and $\delta_4 = 0$.

Due to symmetry it is enough to consider only one of this cases.

Sub case *i*)

Let $\alpha_2 = 0$ and $\alpha_4 = 0$. Then, we have that the elements get the form

$(\alpha_1 x_1 + \alpha_3 x_3, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3)$

$(\alpha_1 x_1 + \alpha_3 x_3, \beta_1 x_1 + \beta_3 x_3 + \beta_4 x_4)$

If we apply addition between them, we get that the sum is equal to

$$(\alpha_1 x_1 + \alpha_3 x_3, \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) + (\alpha_1 x_1 + \alpha_3 x_3, \delta_1 x_1 + \delta_3 x_3 + \delta_4 x_4) =$$
$$= (\alpha_1 x_1 + \alpha_3 x_3, (\beta_1 + \delta_1) x_1 + \beta_2 x_2 + (\beta_3 + \delta_3) x_3 + \delta_4 x_4)$$

or has a form

$$A(\alpha_1 x_1 + \alpha_3 x_3, (\beta_1 + \delta_1) x_1 + \beta_2 x_2 + (\beta_3 + \delta_3) x_3 + \delta_4 x_4),$$

for arbitrary matrix $A \in M_2(\Phi)$.

Sub case *ii)*

It is completely analogous to the sub case *i*).

Now, we will consider the loop 2-subspaces, in which as loops appear the elements x_1 and x_3. Certainly, between these two 2-subspaces it can happen that addition may be done, because both of them as its own 2-subspace have the 2-subspace $M = \{(A(x_1, x_3)^T / A \in M_2(\Phi)\}$. But now, in this situation we have the following possibilities:

- if we have two elements and both of them belong in the loop with a loop centre x_1, where they can be added, then, their sum is again an element of that loop. And the sum is either in the first kernel 2-subspace, or in the other kernel 2-subspace.

- If one of them belongs in the loop in x_1 and the other one in the loop of x_1 then the sum belongs in the 2-subspace generated from (x_1, x_3), i.e. in $\{A(x_1, x_3) / A \in M_2(\Phi)\}$

In some way, it seems as if we have a basis in the 2-dimensional subspace (like a line in a bundle of planes) and on it, for every point on it as if four dimensional subspaces are set. From every one of them, only two points are taken and are built four dimensional subspaces. All together make this 2-subspace.

For example, the element (x_2, x_4) does not belong in this new 2-subspace M'. But then, in this 2-dimensional subspace does not belong not anyone of the elements in the following form

$$A(x_2, x_4) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} (x_2, x_4) = (a_{11}x_2 + a_{12}x_4, a_{21}x_2 + a_{22}x_4),$$

for any $A \in M_2(\Phi)$.

Sub case 3. $u = x_i$, $v = x_{i+k}$ for some $k > 2$, where without loss of generality we can assume that $i + k \leq n$. If $i + k > n$, then we consider it that $i + k = p$, $p < i - 1$ where $i + k \equiv p \ (\text{mod } n)$.

This case for $n = 4$ is not possible at all, because for $k = 3$ and $i = 1$ we get the vectors x_1 and x_4 and it comes to the case 3.1. (sub case 3.1.) . All other similar situations are considered analogously.

Sub case 4. $u \in L(\alpha_1 x_1 + \alpha_3 x_3, x_2) \times L(\alpha_1 x_1 + \alpha_3 x_3, x_2)$, $v = x_4$.

Analogously to this case are considered also the following cases:

a) $u \in L(\alpha_2 x_2 + \alpha_4 x_4, x_3) \times L(\alpha_2 x_2 + \alpha_4 x_4, x_3)$, $v = x_1$

b) $u \in L(\alpha_1 x_1 + \alpha_3 x_3, x_4) \times L(\alpha_1 x_1 + \alpha_3 x_3, x_4)$, $v = x_2$

c) $u \in L(\alpha_2 x_2 + \alpha_4 x_4, x_1) \times L(\alpha_2 x_2 + \alpha_4 x_4, x_1)$, $v = x_3$

In this situation, it is clear to consider the case that is given initially. Here we should note that the vector u has the form of a real coordinate from the 2-vector

$$A(\alpha_1 x_1 + \alpha_3 x_3, x_2)^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} (\alpha_1 x_1 + \alpha_3 x_3, x_2) = (\underbrace{a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12}x_2}_{u}, \underbrace{a_{21}(\alpha_1 x_1 + \alpha_3 x_3) + a_{22}x_2}_{w}).$$

Now, it is obvious that the vector $u = a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12}x_2$ and the vector $v = x_4$ are linearly independent. So, the pair (u, v) until now did not a part of M .

It is worth mentioning that the 2-vector (u, x_2) is a 2-vector which belongs in the 2-subspace M. Indeed, from the assumptions that $\begin{cases} (x_1, x_2) \\ (x_3, x_2) \end{cases} \in M$, we get that $\begin{cases} \alpha_1(x_1, x_2) = (\alpha_1 x_1, x_2) \\ \alpha_3(x_3, x_2) = (\alpha_3 x_3, x_2) \end{cases} \in M$, and because M is a 2-subspace, we get that $(u, x_2) = (\alpha_1 x_1 + \alpha_3 x_3, x_2) \in M$.

Now, it is clear that $(u, x_2), (x_2, x_3), (x_3, x_4 = v) \in M$, so, adding of (u, v) we get a cycle from four 2-vectors. They form for themselves a cycle 2-subspace. Parallel to this cycle 2-subspace we have one more cycle 2-subspace, generated by the 2-

vectors $(u,x_2),(x_2,x_1),(x_1,v=x_4),(v=x_4,u)$. If the new cycle 2-subspaces are denoted with S and S', we will have that

$M'=M\cup S\cup S'$

Sub case 5. $u\in L(\alpha_1x_1+\alpha_3x_3,x_2)\times L(\alpha_1x_1+\alpha_3x_3,x_2)$, $v=\alpha_3x_3+\alpha_4x_4$.

All other analogous cases, which can be gotten with cyclic shift of the indexes are considered analogously to this case. Those cases are:

a) $u\in L(\alpha_2x_2+\alpha_4x_4,x_3)\times L(\alpha_2x_2+\alpha_4x_4,x_3)$, $v=\beta_1x_1+\beta_4x_4$

b) $u\in L(\alpha_3x_3+\alpha_1x_1,x_4)\times L(\alpha_3x_3+\alpha_1x_1,x_4)$, $v=\beta_1x_1+\beta_2x_2$

c) $u\in L(\alpha_2x_2+\alpha_4x_4,x_1)\times L(\alpha_2x_2+\alpha_4x_4,x_1)$, $v=\beta_2x_2+\beta_3x_3$

In this situation it is important to consider the initial case, where we will get the vector u from the equality

$$A(\alpha_1x_1+\alpha_3x_3,x_2)^T=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}(\alpha_1x_1+\alpha_3x_3,x_2)=(\underbrace{a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2}_{u},\underbrace{a_{21}(\alpha_1x_1+\alpha_3x_3)+a_{22}x_2}_{w}),$$

where from we see that $u=a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2$ is a coordinate of a 2-vector from M'. Additionally, $(\underbrace{a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2}_{u},x_2)$ we have it in M. It is clear that this vector without adding of $(u,v=x_3)$, as we said, we have it in M, because

$$A(\alpha_1x_1+\alpha_3x_3,x_2)^T=\begin{bmatrix}a_{11}&a_{12}\\0&1\end{bmatrix}(\alpha_1x_1+\alpha_3x_3,x_2)=(\underbrace{a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2}_{u},x_2).$$

Now $v=\alpha_3x_3+\alpha_4x_4$ so, according to this the pair (u,v) is consisted of linearly independent elements. It remains to see what kind of structure will build all this vectors.

But, now, it is important to see what will happen with the vectors $u=a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2,x_2,x_3,v=\alpha_3x_3+\alpha_4x_4$. This four vectors are linearly independent. Indeed they form a cyclic 2-subspace.

So, according to that, the 2-vectors $(u,x_2),(x_2,x_3),(x_3,v),(v,u)$ are vectors which all of them belong in the new 2-subspace M', so, they for themselves form a cyclic 2-subspace built with four elements, and we will denote it with S.

Similarly, it is worth considering also the vectors $u=a_{11}(\alpha_1x_1+\alpha_3x_3)+a_{12}x_2,x_2,x_1,x_4,v=\alpha_3x_3+\alpha_4x_4$ (which are not linearly independent), and of course the 2-vectors $(u,x_2),(x_2,x_1),(x_1,x_4),(x_4,v),(v,u)$, which are five and form cyclic 2-subspace M' with five elements which we will denote with S'. Now it is clear that

$M'=M\cup S\cup S'$. .

Sub case 6. $u\in L(\alpha_1x_1+\alpha_3x_3,x_2)\times L(\alpha_1x_1+\alpha_3x_3,x_2)$, $v=x_3$.

Analogously to this case, is considered also the case when the vector $u\in L(\alpha_1x_1+\alpha_3x_3,x_2)\times L(\alpha_1x_1+\alpha_3x_3,x_2)$ remains the same and $v=x_1$. Analogously to this case are considered also the following cases:

a) $u\in L(\alpha_2x_2+\alpha_4x_4,x_3)\times L(\alpha_2x_2+\alpha_4x_4,x_3)$, $v=x_2$, i.e.

$u\in L(\alpha_2x_2+\alpha_4x_4,x_3)\times L(\alpha_2x_2+\alpha_4x_4,x_3)$, $v=x_4$

b) $u \in L(\alpha_1 x_1 + \alpha_3 x_3, x_4) \times L(\alpha_1 x_1 + \alpha_3 x_3, x_4)$, $v = x_1$, i.e.
$u \in L(\alpha_1 x_1 + \alpha_3 x_3, x_4) \times L(\alpha_1 x_1 + \alpha_3 x_3, x_4)$, $v = x_3$
c) $u \in L(\alpha_2 x_2 + \alpha_4 x_4, x_1) \times L(\alpha_2 x_2 + \alpha_4 x_4, x_1)$, $v = x_2$,
$u \in L(\alpha_2 x_2 + \alpha_4 x_4, x_1) \times L(\alpha_2 x_2 + \alpha_4 x_4, x_1)$, $v = x_4$.

Now, it is clear that it is enough to consider only the initial case from this vectors, mentioned in the beginning. Certainly, in this case most valuable is to consider the case when $\alpha_1 \neq 0, \alpha_3 \neq 0$. We can get the vector u in this case as a member of the 2-vector from the equality

$$A(\alpha_1 x_1 + \alpha_3 x_3, x_2)^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} (\alpha_1 x_1 + \alpha_3 x_3, x_2) = (\underbrace{a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12} x_2}_{u}, \underbrace{a_{21}(\alpha_1 x_1 + \alpha_3 x_3) + a_{22} x_2}_{w}),$$

i.e. $u = a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12} x_2$, where $a_{11}, \alpha_1, \alpha_3, a_{12}$ are arbitrary, but fixed scalars from Φ.

Now, u, x_2, x_3 are three vectors which already belong in M, as coordinates of 2-vectors. But, we have a situation here when $(u, v = x_3)$ as 2-vector, we insert the 2-vector (x_3, x_2) which already exists in M, and it is a question whether the 2-vector $(\underbrace{a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12} x_2}_{u}, x_2)$ we have in M. It is clear that also that vector without adding of $(u, v = x_3)$, we already have it in M, because

$$A(\alpha_1 x_1 + \alpha_3 x_3, x_2)^T = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 1 \end{bmatrix} (\alpha_1 x_1 + \alpha_3 x_3, x_2) = (\underbrace{a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12} x_2}_{u}, x_2).$$

According to this, the 2-vectors $(u, v = x_3)$, (x_3, x_2) and $(\underbrace{a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12} x_2}_{u}, x_2)$ build a kernel 2-subspace in M', which we will denote with S.

According to this, now
$M' = M \cup L(x_2, x_3, u) \times L(x_2, x_3, u)$

Now, it is worth to consider also the special cases of this case, i.e. "playing" with α_1 and α_3 in already made signature, in order to get more precise picture in this case.

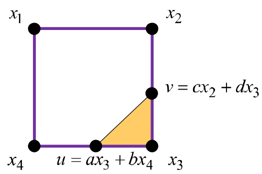
Situation 1. $\alpha_1 = 0, \alpha_3 \neq 0$.

In this situation we have that the vector $u = a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12} x_2$ gets the form $u = a_{11}\alpha_3 x_3 + a_{12} x_2$, and now in fact there is no extension of M, and the vectors $v = x_3, u = a_{11}\alpha_3 x_3 + a_{12} x_2, x_2$ are three linearly independent vectors.

Situation 2. $\alpha_1 =\neq 0, \alpha_3 = 0$.

In this situation, we have a vector which has the form $u = a_{11}\alpha_1 x_1 + a_{12} x_2$. Now, we have the same situation as in the sub case 9. So, we have connected two kernel 2-subspaces from already existing elements from M, i.e.
$M' = L(x_1, x_3, x_2) \times L(x_1, x_3, x_2) \cup L(x_3, x_4, x_1) \times L(x_3, x_4, x_1)$

Sub case 7. $u = \alpha_3 x_3 + \alpha_4 x_4$, $v = \alpha_2 x_2 + \alpha_3 x_3$, In this case it is clear that the vectors $u=\alpha_3x_3+\alpha_4x_4$, x_3 and $v=\alpha_2x_2+\alpha_3x_3$ are linearly independent vectors, and the pairs $(u,x_3),(x_3,v),(v,u)$ belong in the new 2-subspace. But, these 2-vectors for themselves form a kernel 2-subspace. The form of this kernel 2-subspace is $\{(\gamma_1u+\delta_1v+\lambda_1x_3,\gamma_2u+\delta_2v+\lambda_2x_3)/\gamma_i,\delta_i,\lambda_i\in\Phi\}$.



But, let us note here that (u,v) and (u,x_3) are two 2-vectors that belong in M', so, according to that

$$\begin{bmatrix}1&0\\0&\frac{1}{\alpha_2}\end{bmatrix}\left((u,v)+\begin{bmatrix}1&0\\0&-\alpha_3\end{bmatrix}(u,x_3)\right)=\begin{bmatrix}1&0\\0&\frac{1}{\alpha_2}\end{bmatrix}\left((u,\alpha_2x_2+\alpha_3x_3)+(u,-\alpha_3x_3)\right)=$$

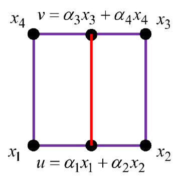
$$=\begin{bmatrix}1&0\\0&\frac{1}{\alpha_2}\end{bmatrix}(u,\alpha_2x_2)=(u,x_2),$$

because (v,u) and (x_3,u) belong in M' that also the 2-vector (x_2,u) belongs in M' Now, according the sub case 3.9 we get that also the 2-vector (x_2,x_4) also belongs in M'. Now, it is clear that this sub case comes to the sub case 3.2, i.e. in this situation the 2-subspaces which are kernel 2-subspaces and one of them is generated by $(x_1,x_2),(x_2,x_4),(x_4,x_1)$, and the other is generated by $(x_2,x_3),(x_3,x_4),(x_4,x_2)$ are glued one to another. Their common 2-subspace is the 2-subspace generated with the 2-vector (x_2,x_4).

Sub case 8. $u=\alpha_1x_1+\alpha_2x_2$, $v=\beta_3x_3+\beta_4x_4$, where $\alpha_1\alpha_2\neq0$, $\beta_3\beta_4\neq0$

Completely analogous to this case is

a) $u=\alpha_2x_2+\alpha_3x_3$, $v=\beta_1x_1+\beta_4x_4$



Now, we have a situation in which the vectors x_1,u,v,x_4 are linearly independent. They, for themselves form a cyclic 2-subspace. Completely analogous, in the same time also the vectors u,x_2,x_3,v are linearly independent. So, according to that, they also for themselves form a cyclic 2-subspace. The basic question is what kind of relationship have these cyclic 2-subspaces. Certainly u and v are loops of the loop 2-subspaces. So,

$M'=M\cup S\cup S'\cup K\cup K'$,

where S,S' are loop/ sub loop 2-subspaces, and K,K' are the cyclic 2-subspaces.

From technical point of view, the 2-vectors $(v,x_4),(v,x_3),(v,u)$ are three 2-vectors from the 2-vector subspace M'. So, they build one loop 2-subspace with loop centre v (completely analogous is for the vector u). Only disputable is the fact that x_3,v,x_4 are linearly dependent vectors.

On the other hand, for any δ and γ we would have that the 2-vector

$$(\delta u,\gamma v)=\begin{bmatrix}\delta&0\\0&\gamma\end{bmatrix}(u,v)\in M'.$$

But, since $(u,x_2),(u,x_1),(v,x_3),(v,x_4)\in M$ and with that also in M', we have that

$$(\delta v,x_4)=\begin{bmatrix}\delta & 0\\ 0 & 1\end{bmatrix}(v,x_4),\ (\delta v,x_3)=\begin{bmatrix}\delta & 0\\ 0 & 1\end{bmatrix}(v,x_3),$$

$$(\gamma u,x_1)=\begin{bmatrix}\gamma & 0\\ 0 & 1\end{bmatrix}(u,x_1),\ (\gamma u,x_2)=\begin{bmatrix}\gamma & 0\\ 0 & 1\end{bmatrix}(u,x_2)$$

are all 2-vectors from M'. But, then we have that

$(\gamma u,x_2),(x_2,x_3),(x_3,\delta v),(\delta v,\gamma u)$ and $(\gamma u,x_1),(x_1,x_4),(x_4,\delta v),(\delta v,\gamma u)$,

are two groups both of four 2-vectors, and every group makes one cycle. Practically, with the vectors u and v, i.e. with their one dimensional subspaces generated in the corresponding subspaces from X in one dimension, are determined sequence of cyclic 2-subspaces which are glued one to another. In the place where they are glued, i.e. in their endings δu and γv are determined one loop 2-subspace each.

Sub case 9. $u=\alpha_1x_1+\alpha_2x_2$, $v=x_3$.

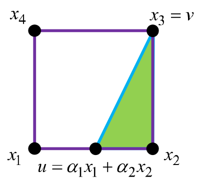
In this situation, we have three vectors $u=\alpha_1x_1+\alpha_2x_2,x_2,v=x_3$ which are linearly independent. Because the 2-vectors $(u,x_2),(x_2,v=x_3),(v,u)$ belong in the new 2-subspace, we get that the kernel 2-subspace generated from these three 2-vectors is a 2-subspace consisted in it.

But, let us note here that $(\alpha_1x_1+\alpha_2x_2,x_3),(x_2,x_3)\in M'$, and M' is a 2-subspace, we get that

$$\begin{bmatrix}\frac{1}{\alpha_1} & 0\\ 0 & 1\end{bmatrix}\left[(\alpha_1x_1+\alpha_2x_2,x_3)+\begin{bmatrix}-\alpha_2 & 0\\ 0 & 1\end{bmatrix}(x_2,x_3)\right]=\begin{bmatrix}\frac{1}{\alpha_1} & 0\\ 0 & 1\end{bmatrix}\left[(\alpha_1x_1+\alpha_2x_2,x_3)+(-\alpha_2x_2,x_3)\right]=$$

$$=\begin{bmatrix}\frac{1}{\alpha_1} & 0\\ 0 & 1\end{bmatrix}(\alpha_1x_1+\alpha_2x_2-\alpha_2x_2,x_3)=\begin{bmatrix}\frac{1}{\alpha_1} & 0\\ 0 & 1\end{bmatrix}(\alpha_1x_1,x_3)=(x_1,x_3)$$

i.e. $(x_1,x_3)\in M'$.



Now, we have a situation $(x_1,x_3),(x_3,x_2),(x_2,x_1)\in M'$, i.e. here a whole one kernel 2-subspace is consisted in M'. Similarly, $(x_1,x_3),(x_3,x_4),(x_4,x_1)\in M'$, so, according to this, we have a situation completely the same as in the sub case 3.2.

Sub case 10.

$u\in L(\alpha_1x_1+\alpha_3x_3,x_2)\times L(\alpha_1x_1+\alpha_3x_3,x_2)$, $v\in L(\alpha_2x_2+\alpha_4x_4,x_3)\times L(\alpha_2x_2+\alpha_4x_4,x_3)$.

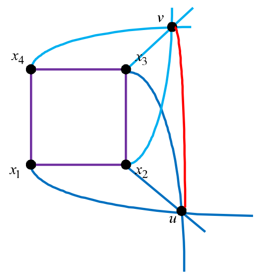
We have three more completely analogous 2-subspaces of the 2-subspace from this case, i.e.

a) $u\in L(\alpha_2x_2+\alpha_4x_4,x_3)\times L(\alpha_2x_2+\alpha_4x_4,x_3)$, $v\in L(\alpha_3x_3+\alpha_1x_1,x_4)\times L(\alpha_3x_3+\alpha_1x_1,x_4)$

b) $u\in L(\alpha_1x_1+\alpha_3x_3,x_4)\times L(\alpha_1x_1+\alpha_3x_3,x_4)$, $v\in L(\alpha_2x_2+\alpha_4x_4,x_1)\times L(\alpha_2x_2+\alpha_4x_4,x_1)$

c) $u\in L(\alpha_2x_2+\alpha_4x_4,x_1)\times L(\alpha_2x_2+\alpha_4x_4,x_1)$, $v\in L(\alpha_1x_1+\alpha_3x_3,x_2)\times L(\alpha_1x_1+\alpha_3x_3,x_2)$.

That is why, it is enough to consider only the first case.



It is clear that the 2-vectors (x_3, v) and (u, x_2) belong in the new 2-subspace M'. But then we have that (v, x_3), (x_2, x_3), (x_2, u) and (u, v) are four vectors from the new subspace, so, according to the definition of cyclic 2-subspace, they form such cyclic 2-subspace. But then we would have in fact two cyclic 2-subspaces, and one of them is the starting 2-subspace. Their common part is the 2-vector (x_2, x_3).

Practically, we have here two cyclic 2-subspaces, which have one common 2-subspace $\{A(x_2, x_3) / A \in M_2(\Phi)\}$. Let us denote this subspace with S. According to this, we have an extension from the type

$M' = M \cup S$.

The previous considerations are in case when $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \neq 0$.

Now, we will consider situations when some of this scalars may be zero. Then, we will have some more specific cases. In fact we will deal with scalars which are zero, and belong in one of these two subspaces. We will choose the first of them.

Situation 1. $\alpha_1 = 0$

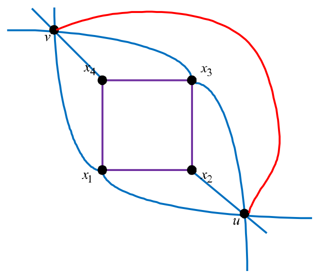
In general case $u = a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12} x_2$, but in this situation it is in the form $u = a_{11}\alpha_3 x_3 + a_{12} x_2$, i.e. without loss of generality we can consider that it is in the form $u = \alpha_2 x_2 + \alpha_3 x_3$. Now, it is clear that in this situation, we can get the 2-vector (v, x_2) as a 2-combination of the 2-vectors (v, x_3) and (v, u), in the following way:

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha_2} \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & -\alpha_3 \end{bmatrix}(v, x_3) + (v, \alpha_2 x_2 + \alpha_3 x_3)\right) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha_2} \end{bmatrix}\left((v, -\alpha_3 x_3) + (v, \alpha_2 x_2 + \alpha_3 x_3)\right) =$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha_2} \end{bmatrix}(v, \alpha_2 x_2 + \alpha_3 x_3 - \alpha_3 x_3) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha_2} \end{bmatrix}(v, \alpha_2 x_2) = (v, x_2)$$

So, the 2-vectors $(v, x_3), (x_3, x_2), (x_2, v)$ are three vectors which belong in M', so, according to that, the kernel 2-subspace generated by them also belongs in M'. In this situation, we have connected one cycle 2_subspace (which in fact is the starting one) and one kernel 2-subspace (which we described totally) i.e..

$M' = M \cup S$.



Situation 2. $\alpha_3 = 0$

In general case $u = a_{11}(\alpha_1 x_1 + \alpha_3 x_3) + a_{12} x_2$, but in this situation it is in the form $u = a_{11}\alpha_1 x_1 + a_{12} x_2$, i.e. without loss of generality, we can consider that it has the form $u = \alpha_2 x_2 + \alpha_1 x_1$. In this situation we have that the four vectors $(u, x_2), (x_2, x_3), (x_3, v), (v, u)$ form a cyclic 2-

subspace, which is a part from M'. We will denote it with S. So,

$M' = M \cup S$.

Sub case 11.

$u \in L(\alpha_1 x_1 + \alpha_3 x_3, x_2) \times L(\alpha_1 x_1 + \alpha_3 x_3, x_2)$,

$v \in L(\alpha_1 x_1 + \alpha_3 x_3, x_4) \times L(\alpha_1 x_1 + \alpha_3 x_3, x_4)$.

Completely analogous to this 2-subspace there is one more 2-subspace and it is determined in the following way:

$u \in L(\alpha_2 x_2 + \alpha_4 x_4, x_3) \times L(\alpha_2 x_2 + \alpha_4 x_4, x_3)$,

$v \in L(\alpha_2 x_2 + \alpha_4 x_4, x_1) \times L(\alpha_2 x_2 + \alpha_4 x_4, x_1)$

In this sub case we have two cyclic 2-subspaces which are generated each with five elements, i.e. those 2-subspaces are generated by

a) $(v, x_4), (x_4, x_1), (x_1, x_2), (x_2, u), (u, v)$, denoted as S

b) $(v, x_4), (x_4, x_3), (x_3, x_2), (x_2, u), (u, v)$, denoted as S'.

Now, it is clear that

$M' = M \cup S \cup S'$.

Sub case 12. In this sub case we have a situation when both vectors u and v belong in $L(x_1, x_2, x_3, x_4)$, but they are not a part from neither 2-subspace from the form

$L(\alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1}, x_i) \times L(\alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1}, x_i)$,

for $i = 1,2,3,4$. This situation even though is possible, technically is feasible, but still the result is as in the case 1.

Sub case 13. $u \in L(\alpha_1 x_1 + \alpha_3 x_3, x_2) \times L(\alpha_1 x_1 + \alpha_3 x_3, x_2)$, $v = \alpha_2 x_2 + \alpha_3 x_3$

In this sub case we have the 2-vectors $(u, x_2), (x_2, v), (v, u)$, which form for themselves a kernel 2-subspace from M', which will be denoted as S. So, according to this we have that

$M' = M \cup S$.

Now, we will consider this kernel 2-subspace. Let us note that $(u, x_2), (u, v) = (u, \alpha_2 x_2 + \alpha_3 x_3)$ are two vectors from M'. But, then we have that

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha_3} \end{bmatrix} \left((u, \alpha_2 x_2 + \alpha_3 x_3) + \begin{bmatrix} 1 & 0 \\ 0 & -\alpha_2 \end{bmatrix} (u, x_2) \right) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha_3} \end{bmatrix} \left((u, \alpha_2 x_2 + \alpha_3 x_3) + (u, -\alpha_2 x_2) \right) =$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha_3} \end{bmatrix} (u, \alpha_2 x_2 + \alpha_3 x_3 - \alpha_2 x_2) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha_3} \end{bmatrix} (u, \alpha_3 x_3) = (u, x_3) \in M'$$

According to this, the loop 2-subspace is generated from the 2-vectors $(u, x_2), (x_2, x_3), (x_3, u)$.

In the following part we will present one elaborated case of extension of 2-subspace and extension of a 2-skew-symmetric form defined on it.