18th Junior Balkan
Mathematical Olympiad
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MATHEMATICAL OLYMPIADS

Macedonian Mathematical Olympiads
Balkan Mathematical Olympiads

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Macedonian Mathematical Olympiad 2017

Balkan Mathematical Olympiad 2017

Mediterranean Mathematical Olympiad 2017

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Contests

21 – ва JMMO

21-th Junior Macedonian Mathematical Olympiad, JMMO 2017

Faculty for mechanical engineering-Skopje
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1. Let p is a prime number and let $3p+10$ is the sum of the squares of six consecutive positive integers. Prove that $36|p-7$.

Solution. From the conditions of the problem, we have that

$$3p+10=(n-2)^2+(n-1)^2+n^2+(n+1)^2+(n+2)^2+(n+3)^2=6n^2+6n+19,$$

so, we have that

$$3p=6n^2+6n+9,$$

and

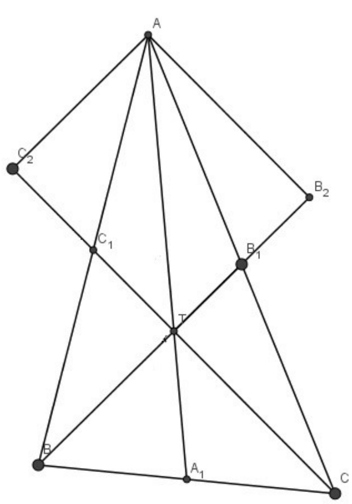
$$p=2n^2+2n+3=2n(n+1)+3.$$

If one of the numbers n or $n+1$ is divisible with 3, then we have a contradiction with the condition that p is a prime number. So, $n=3k+1$. Then,

$$p=2(3k+1)(3k+1+1)+3=2(3k+1)(3k+2)+3=2(9k^2+9k+2)+3=18k(k+1)+7.$$

Since $k(k+1)$ is an even number, we have that $36|p-7$.

2. Let be given ΔABC and let AA_1, BB_1 and CC_1 are the medians in the triangle which intersect in the point T and $\overline{BA_1}=\overline{A_1T}$. On the continuation of CC_1 we choose a point C_2 such that $\overline{C_1C_2}=\frac{\overline{CC_1}}{3}$, and on the continuation of BB_1 we choose a point B_2 such that $\overline{B_1B_2}=\frac{\overline{BB_1}}{3}$. Prove that the quadrilateral TB_2AC_2 is a rectangle.



Solution. Since AA_1 is a median in the ΔABC and $\overline{BA_1}=\overline{A_1T}$, we get that $\overline{A_1T}=\frac{\overline{BC}}{2}$ i.e. A_1 is the circumcenter of the circumcircle of ΔBCT. So according to the Thales theorem $\angle BTC=90^\circ$. We have $\angle B_2TC_2=90^\circ$ (as vertically opposite angles). Since T is the barycenter of ΔABC we have $\overline{C_1T}=\frac{\overline{CC_1}}{3}=\overline{C_1C_2}$. From $\overline{BC_1}=\overline{C_1A}$ we get that the quadrilateral $BTAC_2$ is a parallelogram. Then $BT\|AC_2$, so $\angle TC_2A=\angle CTB_2=180^\circ-\angle B_2TC_2=90^\circ$ (as angles on the transversal).

With analogy, we can prove that the quadrilateral TCB_2A is a parallelogram i.e. $\angle TB_2A = \angle C_2TB_2 = 90°$ (as angles on the transversal). So, we get that $\angle C_2AB_2 = 360° - 270° = 90°$ i.e. the quadrilateral is a rectangle.

Second proof for the statement that $\angle TB_2A = 90°$.

We look at the ΔBB_2C_1. Since C_1T is a median and an altitude in the triangle, we get that ΔBB_2C_1 is an isosceles triangle, so $\overline{BC_1} = \overline{C_1B_2}$ (it can be proven with the SAS sign: C_1T is a common side, $\angle BTC_1 = \angle B_2TC_1 = 90°$ and $\overline{BT} = \overline{TB_2}$). We get that the point C_1 is a circumcenter of the circumcircle of ΔBB_2A, so according to the Thales theorem $\angle BB_2A = 90°$. We have $\angle C_2AB_2 = 360° - 270° = 90°$ i.e. the quadrilateral is a rectangle.

3. Let x, y, z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^2+y^2+z}{x^2+2} + \frac{y^2+z^2+x}{y^2+2} + \frac{z^2+x^2+y}{z^2+2} \geq 3.$$

When does the equality holds?

Solution. From the AM-GM inequality and the condition of the problem, we have that

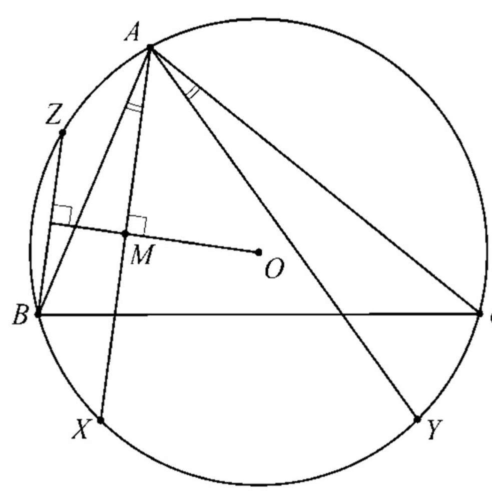
$$\frac{x^2+y^2+z}{x^2+2} + \frac{y^2+z^2+x}{y^2+2} + \frac{z^2+x^2+y}{z^2+2} \geq \frac{2xy+z}{x^2+2} + \frac{2yz+x}{y^2+2} + \frac{2zx+y}{z^2+2} =$$

$$= \frac{2xyz+z^2}{z(x^2+2)} + \frac{2xyz+x^2}{x(y^2+2)} + \frac{2xyz+y^2}{y(z^2+2)} = \frac{z^2+2}{z(x^2+2)} + \frac{x^2+2}{x(y^2+2)} + \frac{y^2+2}{y(z^2+2)} \geq$$

$$\geq 3\sqrt[3]{\frac{z^2+2}{z(x^2+2)} \cdot \frac{x^2+2}{x(y^2+2)} \cdot \frac{y^2+2}{y(z^2+2)}} = 3\sqrt[3]{\frac{1}{xyz}} = 3.$$

4. Let be given the ΔABC. On the arc $\overset{\frown}{BC}$ of the circumcircle of $\triangle ABC$, which does not contain the point A, points X and Y are chosen, such that $\measuredangle BAX = \measuredangle CAY$. Let M be the middle point of the chord AX. Prove that $\overline{BM} + \overline{CM} > \overline{AY}$.

Solution. Let O be the circumcenter of the circumcircle of ΔABC. Then $OM \perp AX$. We draw a normal line from the point B at OM and let it intersect the circumcircle in the point Z. Since $BZ \perp OM$ we have that OM is a line of symmetry of BZ. According to this, $\overline{MZ} = \overline{MB}$. Now, from the triangle inequality we have that



$$\overline{BM} + \overline{MC} = \overline{ZM} + \overline{MC} > \overline{CZ}.$$

But, $BZ \parallel AX$, so

$$\overset{\frown}{AZ} = \overset{\frown}{BX} = \overset{\frown}{CY}$$

where from we get

$$\overset{\frown}{ZAC} = \overset{\frown}{ZA} + \overset{\frown}{AC} = \overset{\frown}{YC} + \overset{\frown}{CA} = \overset{\frown}{YCA}$$

i.e. $\overline{CZ}=\overline{AY}$. That is why $\overline{BM}+\overline{CM}>\overline{AY}$.

5. Find all positive integers n such that n has number of ciphers which is the same as the number of its different prime divisors and the sum of the different prime divisors is equal to the sum of their powers.

Solution. Let $n=p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}$. From the condition of the problem

$$p_1+p_2+...+p_k=\alpha_1+\alpha_2+...+\alpha_k.$$

We discuss the number of ciphers of the number n. If n has 4 ciphers, then he has 4 different prime divisors. Then $n\geq 2^{14}\cdot 3\cdot 5\cdot 7>10^4$ which is not possible. If n has $k>4$ ciphers, then

$$n\geq 2^{2+3+5+7+.p_5+..+p_k-(k-1)}\cdot 3\cdot 5\cdot 7\cdot p_5\cdot ...\cdot p_k=2^{14}\cdot 3\cdot 5\cdot 7\cdot 2^{\cdot p_5+..+p_k-(k-4)}p_5\cdot ...\cdot p_k$$
$$>10^4\cdot 10^{k-4}=10^k$$

which again is not possible.

So, we get that n has at most three ciphers.

Let n have three ciphers. Then $n=p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$. If $5|n$, then $n\geq 2^8\cdot 3\cdot 5>10^3$.

We get that the prime divisors of the number n are ≤ 3. But, prime numbers ≤ 3 are 2 and 3, and in the factorization of the number n there are 3 prime numbers, which is a contradiction.

Let n has two ciphers. Then $n=p_1^{\alpha_1}p_2^{\alpha_2}$. If $5|n$, then $n\geq 2^6\cdot 5>10^2$. Remains $n=2^{\alpha_1}3^{\alpha_2}$ where $\alpha_1+\alpha_2=5$. With direct checking we get that $n=2^4\cdot 3=48, n=2^3\cdot 3^2=72$ are solutions of the problem.

Let n has one cipher. Then only $n=2^2$ fulfils the condition of the problem.

24_та ММО

24-th Macedonian mathematical olympiad
Faculty for electrical engineering and information technologies - Skopje
08.04.2017

1. Determine all functions $f:\mathbb{N}\to\mathbb{N}$ such that for every positive integer $n>1$ and every $x,y\in\mathbb{N}$

$$f(x+y)=f(x)+f(y)+\sum_{k=1}^{n-1}\binom{n}{k}x^{n-k}y^k.$$

2. Determine all positive integers n such that $(n^3+39n-2)n!+17\cdot 21^n+5$ is a full square.

3. Let x,y,z are positive real numbers such that $xyz=1$. Prove that

$$\left(x^4+\frac{z^2}{y^2}\right)\left(y^4+\frac{x^2}{z^2}\right)\left(z^4+\frac{y^2}{x^2}\right)\geq\left(\frac{x^2}{y}+1\right)\left(\frac{y^2}{z}+1\right)\left(\frac{z^2}{x}+1\right).$$

4. Let O is the circumcenter of the circumcircle of the acute triangle ABC $(\overline{AB}<\overline{AC})$. Let A_1 and P are the intersection points of the normal lines through the points A and O and the side BC, correspondingly. The lines BO and CO intersect with the line AA_1 in the points D and E, correspondingly. The circumcircles of the triangles ABD and ACE again intersects in the point F. Prove that the symmedian of the $\measuredangle FAP$ passes through the center of the incircle of the triangle ABC.

5. Let $n>1$ is a positive integer and $a_1,a_2,...,a_n$ is a sequence of n positive integers. Let

$$b_1=\left[\frac{a_2+...+a_n}{n-1}\right], b_i=\left[\frac{a_1+a_2+...+a_{i-1}+a_{i+1}+...+a_n}{n-1}\right], 1<i<n, b_n=\left[\frac{a_1+a_2+...+a_{n-1}}{n-1}\right].$$

Let f is a mapping such that $f(a_1,a_2,...,a_n)=(b_1,b_2,...,b_n)$.

a) Let the function $g:N\to N$ is defined such that $g(1)$ is the number of different elements in the sequence $f(a_1,a_2,...,a_n)$ and $g(m)$ is the number of different elements in the sequence $f^m(a_1,a_2,...,a_n)=f(f^{m-1}(a_1,a_2,...,a_n)),m>1$. Prove that there is a positive integer k_0 such that for $m\geq k_0$ the function $g(m)$ is periodical.

b) Prove that $\sum_{m=1}^{k}\frac{g(m)}{m(m+1)}<C$ for any positive integer k, where the constant C does not depend on k.

SOLUTIONS

1. Determine all functions $f:\mathbb{N}\to\mathbb{N}$ such that for every positive integer $n>1$ and every $x,y\in\mathbb{N}$

$$f(x+y)=f(x)+f(y)+\sum_{k=1}^{n-1}\binom{n}{k}x^{n-k}y^k .$$

Solution. From the condition of the problem we have that

$$f(x+y)-\sum_{k=1}^{n-1}\binom{n}{k}x^{n-k}y^k=f(x)+f(y)\Leftrightarrow$$

$$f(x+y)-\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k=f(x)+f(y)-x^n-y^n\Leftrightarrow$$

$$f(x+y)-(x+y)^n=f(x)-x^n+f(y)-y^n .$$

Let $f_1:\mathbb{N}\to\mathbb{N}_0$ is a mapping defined with $f_1(x)=f(x)-x^n$. Then

$$f_1(x+y)=f(x+y)-(x+y)^n=f(x)-x^n+f(y)-y^n=f_1(x)+f_2(x).$$

Using the induction method can be proven that $f_1(nx)=nf_1(x)$ for every $n\in\mathbb{N}$. For $x=1$ we have $f_1(n)=nf_1(1)=n\alpha$ for every $n\in\mathbb{N}$, where $\alpha=f_1(1)$. So, $f(x)=x^n+\alpha x$.

2. Determine all positive integers n such that $(n^3+39n-2)n!+17\cdot 21^n+5$ is a full square.

Solution. Lets denote $a_n=(n^3+39n-2)n!+17\cdot 21^n+5$.

If $n\geq 4$, then $8|n!$. Moreover,

$a_n\equiv 5^n+5\,(\text{mod}8)$.

If n is an even number, then $5^n\equiv 1(\text{mod}8)$, so $a_n\equiv 6(\text{mod}8)$. But, all full squares have remaining $0,1$ or 4 when divided with 8. So, if $n\geq 4$ and n is even, then a_n is not a full square.

Let $n\geq 7$. It is clear that $7|n!$. Then $a_n\equiv 5\,(\text{mod}7)$. On the other side, the remainings of the full squares when divided with 7 are $0,1,2$ or 4. So, a_n is not a full square for $n\geq 7$. Having in mind the previous discussion, it remains to check for $n=1$, $n=2$, $n=3$ and $n=5$.

If $n=5$, $a_5\equiv 2\cdot 1^5+5\equiv 2\,(\text{mod}5)$.

Since the remainings of a full square when divided with 5 are $0,1$ or 4, a_5 is not a full square.

For $n=3$, we have $a_3\equiv 3\,(\text{mod}7)$, so a_3 is not a full square.

For $n=2$, we have $a_2\equiv 1+5\equiv 2\,(\text{mod}4)$, so a_2 is not a full square.

For $n=1$, $a_1=(1+39-2)\cdot 1+17\cdot 21+5=400$.

This means that only for $n=1$, a_n is a full square.

3. Let x,y,z are positive real numbers such that $xyz=1$. Prove that

$$\left(x^4+\frac{z^2}{y^2}\right)\left(y^4+\frac{x^2}{z^2}\right)\left(z^4+\frac{y^2}{x^2}\right)\geq\left(\frac{x^2}{y}+1\right)\left(\frac{y^2}{z}+1\right)\left(\frac{z^2}{x}+1\right).$$

Solution. Using the Cauchy-Schwartz inequality, we get

$$\sqrt{(x^2)^2+\left(\frac{z}{y}\right)^2}\cdot\sqrt{(y^2)^2+\left(\frac{x}{z}\right)^2}\geq\left(x^2y^2+\frac{z}{y}.\frac{x}{z}\right)=x^2\left(y^2+\frac{1}{xy}\right)$$

$$\sqrt{(y^2)^2+\left(\frac{x}{z}\right)^2}\cdot\sqrt{(z^2)^2+\left(\frac{y}{x}\right)^2}\geq\left(y^2z^2+\frac{x}{z}.\frac{y}{x}\right)=y^2\left(z^2+\frac{1}{yz}\right)$$

$$\sqrt{(x^2)^2+\left(\frac{z}{y}\right)^2}\cdot\sqrt{(z^2)^2+\left(\frac{y}{x}\right)^2}\geq\left(x^2z^2+\frac{z}{y}.\frac{y}{x}\right)=z^2\left(x^2+\frac{1}{xz}\right).$$

If we multiply the last three inequalities, we get that,

$$\left(x^4+\frac{z^2}{y^2}\right)\left(y^4+\frac{x^2}{z^2}\right)\left(z^4+\frac{y^2}{x^2}\right)\geq x^2y^2z^2\left(y^2+\frac{1}{xy}\right)\left(z^2+\frac{1}{yz}\right)\left(x^2+\frac{1}{xz}\right)=$$

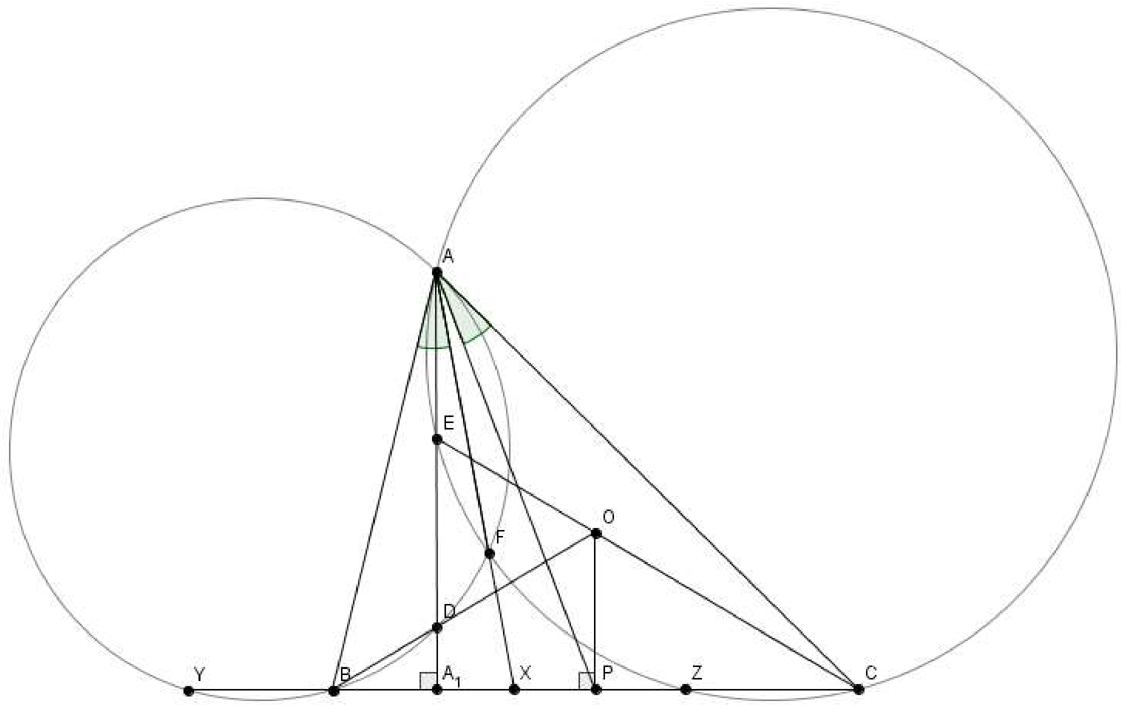
$$=(xyz)^3\left(\frac{y^2}{z}+1\right)\left(\frac{z^2}{x}+1\right)\left(\frac{x^2}{y}+1\right)=\left(\frac{x^2}{y}+1\right)\left(\frac{y^2}{z}+1\right)\left(\frac{z^2}{x}+1\right),$$

which we had to prove.

4. Let O is the circumcenter of the circumcircle of the acute triangle ABC $\left(\overline{AB}<\overline{AC}\right)$. Let A_1 and P are the intersection points of the normal lines through the points A and O and the side BC, correspondingly. The lines BO and CO intersect with the line AA_1 in the points D and E, correspondingly. The circumcircles of the triangles ABD and ACE again intersects in the point F. Prove that the symmedian of the $\measuredangle FAP$ passes through the center of the incircle of the triangle ABC.

Solution 1. We need to prove that $\measuredangle BAF = \measuredangle CAP$. Since OP is perpendicular to BC and O is the circumcenter, then P is the midpoint of BC. Since AP is the median from A, we need to prove that AF is the symmedian from A.

Let the line AF intersect the side BC at X and let the circumcircles of ABD and ACE meet the line BC again at Y and Z, respectively. Then, by the intersecting secant theorem, we have:



$$\overline{XB} \cdot \overline{XY} = \overline{XF} \cdot \overline{XA} = \overline{XZ} \cdot \overline{XC}$$

$$\frac{\overline{XB}}{\overline{XC}} = \frac{\overline{XZ}}{\overline{XY}} = \frac{\overline{XB} + \overline{XZ}}{\overline{XC} + \overline{XY}} = \frac{\overline{BZ}}{\overline{CY}} \qquad \dots \qquad (1)$$

$$\measuredangle ACE \equiv \measuredangle ACO = \frac{1}{2}(180^\circ - \measuredangle AOC) = \frac{1}{2}(180^\circ - 2\measuredangle ABC) = 90^\circ - \measuredangle ABC \equiv$$
$$\equiv 90^\circ - \measuredangle ABA_1 = \measuredangle BAA_1 \equiv \measuredangle BAE$$

so BA is tangent to the circumcircle of ACE.
Similarly, CA is tangent to the circumcircle of ABD. By the tangent-secant theorem, we have:

$$\overline{BA}^2 = \overline{BZ} \cdot \overline{BC}$$
$$\overline{CA}^2 = \overline{CB} \cdot \overline{CY}$$

By dividing these two equations and using (1), we get:

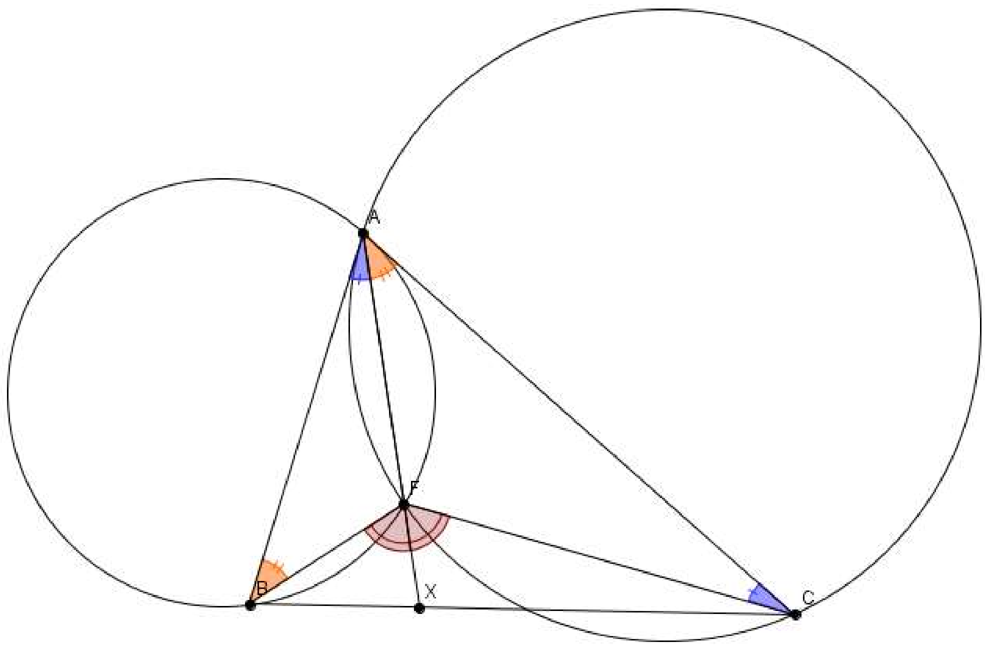
$$\frac{\overline{BA}^2}{\overline{CA}^2}=\frac{\overline{BZ}}{\overline{CY}}=\frac{\overline{XB}}{\overline{XC}}$$

We proved that AX divides the side BC in the ratio of the squares of the sides AB and AC, so by *Lemma 1* we get that $AF\equiv AX$ is the $A-$symmedian in the triangle ABC. •

Solution 2. We need to prove that $\measuredangle BAF=\measuredangle CAP$. Since OP is perpendicular to BC and O is the circumcenter, then P is the midpoint of BC. Since AP is the median from A, we need to prove that AF is the symmedian from A.

By some angle chasing:

$$\measuredangle ACE\equiv\measuredangle ACO=\frac{1}{2}(180^\circ-\measuredangle AOC)=\frac{1}{2}(180^\circ-2\measuredangle ABC)=90^\circ-\measuredangle ABC\equiv$$
$$\equiv 90^\circ-\measuredangle ABA_1=\measuredangle BAA_1\equiv\measuredangle BAE$$



we get that BA is tangent to the circumcircle of ACF.
Similarly, CA is tangent to the circumcircle of ABF.

Now, we use the fact that the angle between a tangent and a chord is equal to any inscribed angle over the same chord:

$$\measuredangle BAF=\measuredangle ACF$$
$$\measuredangle ABF=\measuredangle CAF$$

So, the triangles BAF and ACF are similar which gives:

$$\frac{\overline{BF}}{\overline{CF}}=\frac{\overline{BF}/\overline{AF}}{\overline{CF}/\overline{AF}}=\frac{\overline{BA}/\overline{AC}}{\overline{AC}/\overline{AB}}=\frac{\overline{AB}^2}{\overline{AC}^2}$$

Also, $\measuredangle BFX=180^\circ-\measuredangle BFA=180^\circ-\measuredangle AFC=\measuredangle CFX$, so FX is an angle bisector in BFC, so:

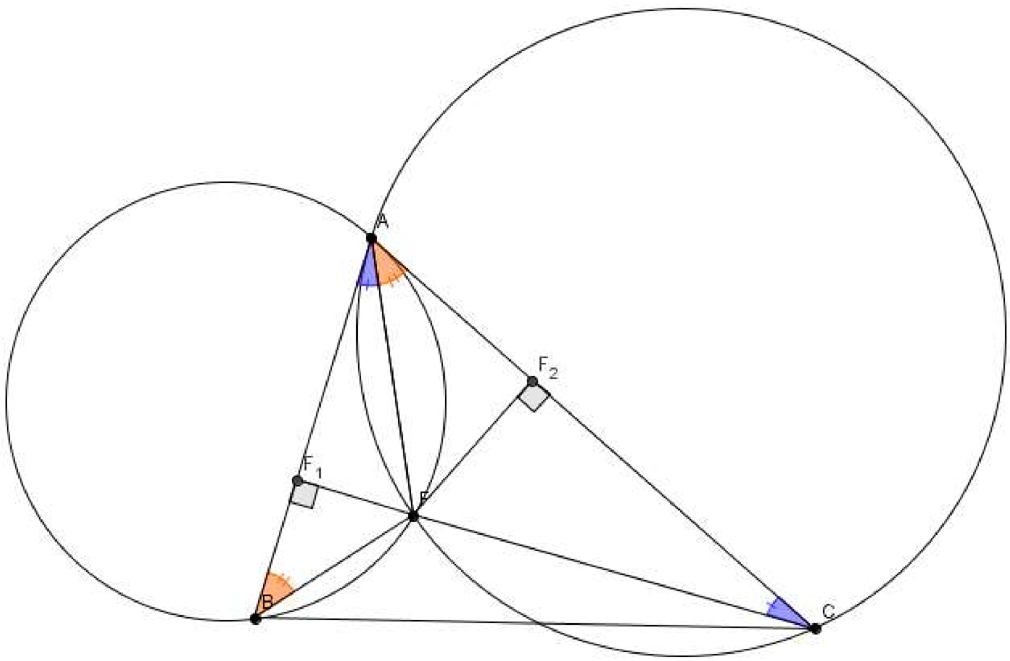
$$\frac{\overline{BF}}{\overline{CF}}=\frac{\overline{BX}}{\overline{CX}}$$

From these two equalities, we get that

$$\frac{\overline{BX}}{\overline{CX}}=\frac{\overline{AB}^2}{\overline{AC}^2}$$

So, the line AX divides the side BC in the ratio of the squares of the sides AB and AC, so by *Lemma 1* we get that $AF \equiv AX$ is the symmedian from the vertex A in the triangle ABC. ●

Solution 3. We need to prove that $\measuredangle BAF=\measuredangle CAP$. Since OP is perpendicular to BC and



O is the circumcenter, then P is the midpoint of BC. Since AP is the median from A, we need to prove that AF is the symmedian from A.

By some angle chasing:

$$\measuredangle ACE \equiv \measuredangle ACO=\frac{1}{2}(180^\circ-\measuredangle AOC)=\frac{1}{2}(180^\circ-2\measuredangle ABC)=90^\circ-\measuredangle ABC \equiv$$

$$\equiv 90^\circ-\measuredangle ABA_1=\measuredangle BAA_1 \equiv \measuredangle BAE$$

we get that BA is tangent to the circumcircle of ACF.

Similarly, CA is tangent to the circumcircle of ABF.

Now, we use the fact that the angle between a tangent and a chord is equal to any inscribed angle over the same chord:

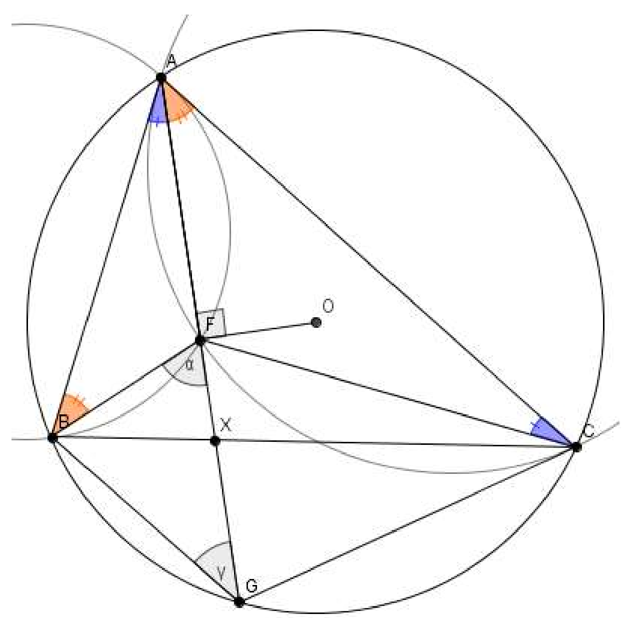
$$\measuredangle BAF=\measuredangle ACF$$

$$\measuredangle ABF=\measuredangle CAF$$

So, the triangles BAF and ACF are similar.

Let F_1 and F_2 be the feet of the perpendiculars from F to the sides AB and AC, respectively. Then, from the similarity we have:

$$\frac{\overline{FF_1}}{\overline{FF_2}}=\frac{\overline{AB}}{\overline{AC}}$$



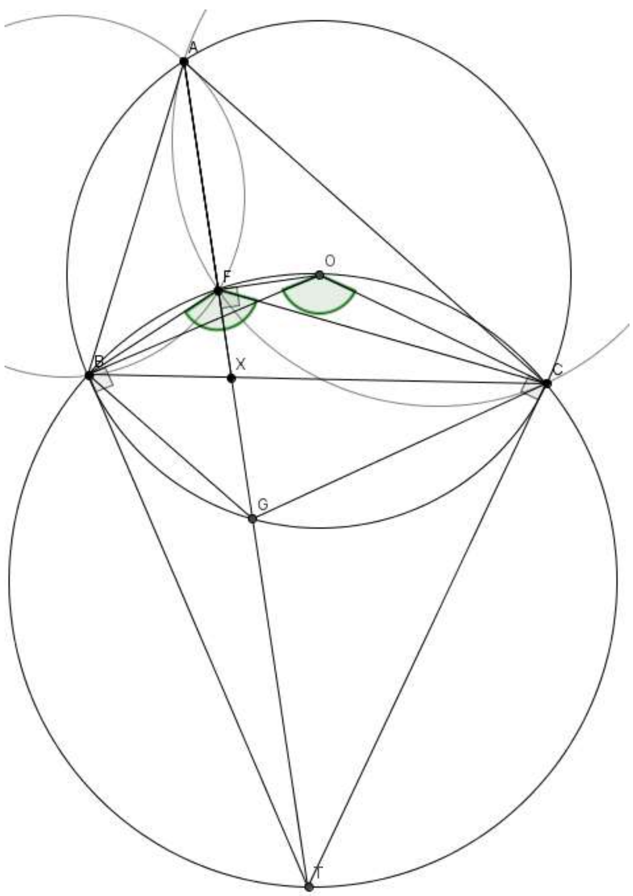
which means that the distances from F to the sides AB and AC are proportional to the lengths AB and AC, so by *Lemma 2b*, F lies on the symmedian from the vertex A in the triangle ABC. ●

Solution 4. We need to prove that $\measuredangle BAF = \measuredangle CAP$.

Since OP is perpendicular to BC and O is the circumcenter, then P is the midpoint of BC. Since AP is the median from A, we need to prove that AF is the symmedian from A.

By some angle chasing:

$$\measuredangle ACE \equiv \measuredangle ACO = \frac{1}{2}(180^\circ - \measuredangle AOC) = \frac{1}{2}(180^\circ - 2\measuredangle ABC) = 90^\circ - \measuredangle ABC \equiv$$
$$\equiv 90^\circ - \measuredangle ABA_1 = \measuredangle BAA_1 \equiv \measuredangle BAE$$



we get that BA is tangent to the circumcircle of ACF.

Similarly, CA is tangent to the circumcircle of ABF.

Now, we use the fact that the angle between a tangent and a chord is equal to any inscribed angle over the same chord:

$$\measuredangle BAF = \measuredangle ACF$$
$$\measuredangle ABF = \measuredangle CAF$$

So, the triangles BAF and ACF are similar and:

$$\frac{\overline{BA}}{\overline{BF}} = \frac{\overline{AC}}{\overline{AF}} \quad \dots \quad (1)$$

Let AX intersect the circumcircle of ABC again at G.

$$\measuredangle BFG = 180^\circ - \measuredangle BFA = \measuredangle FBA + \measuredangle FAB =$$

$$= \measuredangle FAC + \measuredangle FAB = \measuredangle BAC = \alpha$$

$$\measuredangle BGF = \measuredangle BGA = \measuredangle BCA = \gamma$$

So, the triangles ABC and FBG are also similar and:

$$\frac{\overline{AB}}{\overline{FB}} = \frac{\overline{AC}}{\overline{FG}} \quad \dots \quad (2)$$

From (1) and (2) we get that $\overline{AF} = \overline{FG}$ and because O is the circumcenter, we get that $\measuredangle OFG = 90^\circ$.

Now, let's draw the tangents at B and C to the circumcircle of ABC and let them intersect at T. The quadrilateral $OBTC$ is a cyclic quadrilateral with diameter OT.

Earlier in this solution, we proved that $\measuredangle BFG = \alpha$. Similarly, $\measuredangle CFG = \alpha$.

$\measuredangle BFC = \measuredangle BFG + \measuredangle CFG = \alpha + \alpha = 2\alpha = \measuredangle BOC$, so F lies on the circumcircle of BOC (with diameter OT). Because $\measuredangle OFG = 90^\circ$ and OT is the diameter of the circle, then T must lie on the line $FG \equiv AF$.

In conclusion, AF passes through the intersection of the tangents at B and C to the circumcircle of ABC, so by *Lemma 3b* we get that AF is the symmedian from the vertex A in the triangle ABC •

<u>Lemma 1:</u> The line AX divides the opposite side BC in the ratio of the squares of the sides AB and AC if and only if AX is a symmedian in the triangle ABC.

$$\frac{\overline{BX}}{\overline{CX}}=\frac{\overline{AB}^2}{\overline{AC}^2}$$

Proof: Let AM and AX, be the median and symmedian from the vertex A, respectively.

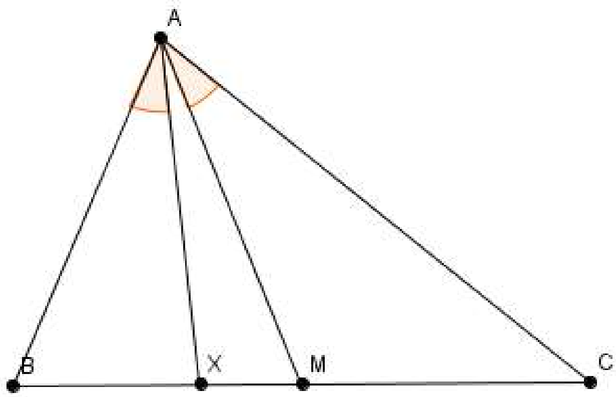
$$\frac{\overline{BX}}{\overline{MC}}=\frac{Area(BAX)}{Area(MAC)}=\frac{\overline{BA}\cdot\overline{AX}}{\overline{AM}\cdot\overline{AC}}$$

$$\frac{\overline{BM}}{\overline{XC}}=\frac{Area(BMA)}{Area(CXA)}=\frac{\overline{BA}\cdot\overline{AM}}{\overline{AX}\cdot\overline{AC}}$$

By multiplying these equalities we get:

$$\frac{\overline{BX}}{\overline{CX}}=\frac{\overline{AB}^2}{\overline{AC}^2}$$

Since there is only one point on the line segment BC that divides it in a given ratio, the "only if" part is also true •



<u>Lemma 2a:</u> The $A-$median is the locus of the points M in the interior of $\measuredangle BAC$ such that

$$\frac{d(M,AB)}{d(M,AC)}=\frac{\overline{AC}}{\overline{AB}}.$$

Proof:

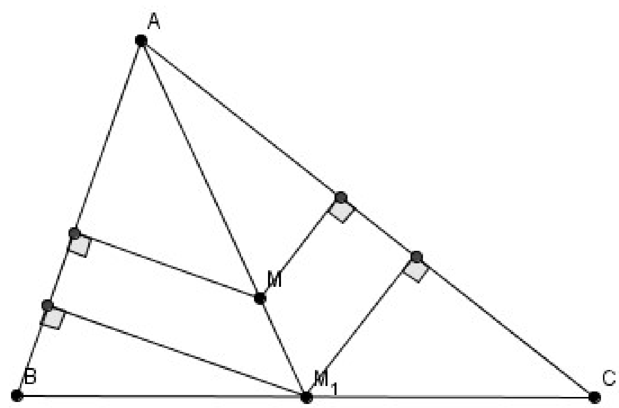
Let M be a point in the interior of $\measuredangle BAC$. Let AM meet BC at M_1. Then,

$$\frac{d(M_1,AB)}{d(M_1,AC)}=\frac{d(M,AB)}{d(M,AC)}=\frac{\overline{AC}}{\overline{AB}}$$

$$\Leftrightarrow d(M_1,AB)\cdot\overline{AB}=d(M_1,AC)\cdot\overline{AC}$$

$$\Leftrightarrow Area(ABM_1)=Area(ACM_1)$$

$$\Leftrightarrow \overline{BM_1}=\overline{M_1C}$$ •



<u>Lemma 2b:</u> The $A-$ is the locus of the points L in the interior of $\measuredangle BAC$ such that:

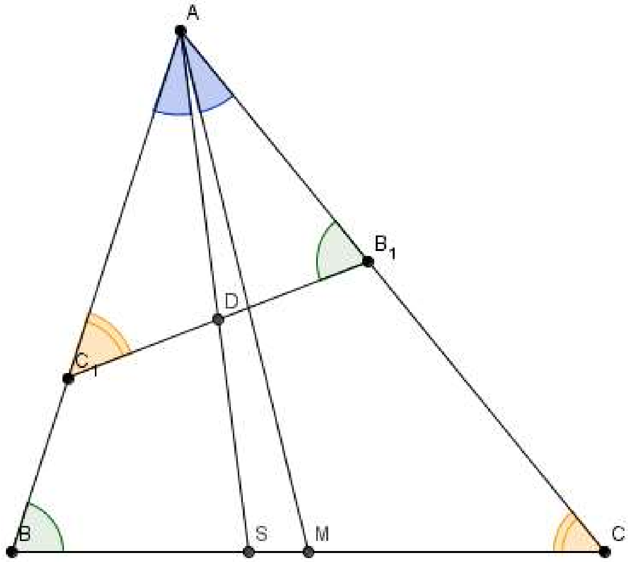
$$\frac{d(L,AB)}{d(L,AC)}=\frac{\overline{AB}}{\overline{AC}}.$$

Proof: The symmedian is the reflection of the median with respect to the angle bisector, so by symmetry:

$$\frac{d(L,AB)}{d(L,AC)}=\frac{d(M,AC)}{d(M,AB)}=\frac{\overline{AB}}{\overline{AC}}$$

which means that the $A-$symmedian is the locus of the points L in the interior of $\measuredangle BAC$ such that:

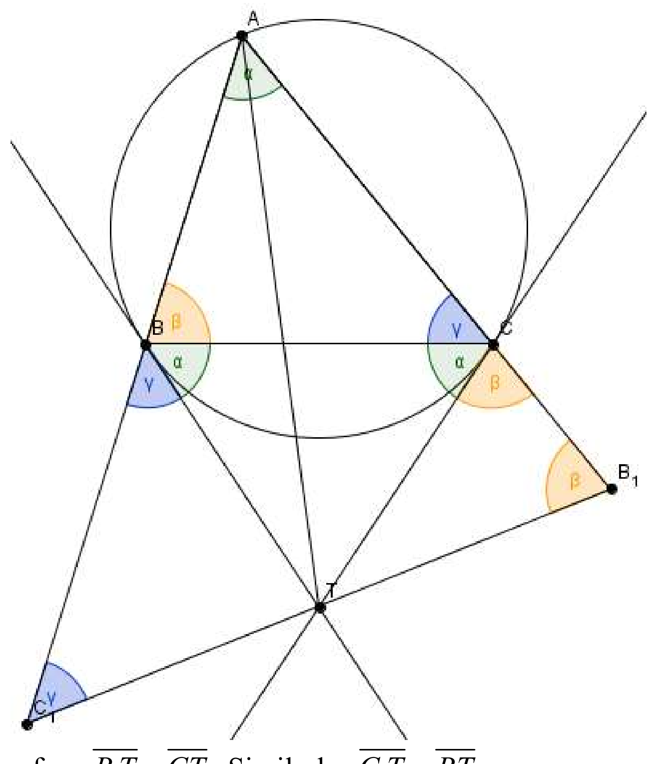
$$\frac{d(L,AB)}{d(L,AC)}=\frac{\overline{AB}}{\overline{AC}} \bullet$$



Lemma 3a: A symmedian drawn from a vertex of a triangle divides the antiparallels to the opposite side in half.

Proof: Let AS and AM be the symmedian and the median from the vertex A, respectively. Then, by the definition of symmedian, $\measuredangle BAS=\measuredangle CAM$ … (1)

Let D be the intersection of the lines AS and B_1C_1. By definition of antiparallel lines, the triangles ABC and $A_1B_1C_1$ are similar. Using (1) we get that the similarity maps AM to AD, so the symmedian AS passes through the midpoint of the side B_1C_1 which is antiparallel to BC (with respect to the lines AB and AC $\bullet$



Lemma 3b: A symmedian through one of the vertices of a triangle passes through the point of intersection of the tangents to the circumcircle at the other two vertices.

Proof: Let BT and CT be the tangents to the circumcircle of ABC at B and C. Then, because the angle between a tangent and a chord is equal to any inscribed angle over the same chord, $\measuredangle CBT=\measuredangle CAB=\alpha$ and $\measuredangle BCT=\measuredangle BAC=\alpha$, so the triangle BCT is isosceles and therefore $\overline{BT}=\overline{CT}$.

Let B_1C_1 be an antiparallel line to BC (with respect to the lines AB and AC) that passes through T. Then, $\measuredangle AB_1C_1=\measuredangle ABC=\beta$. Now,

$\measuredangle TCB_1=180^\circ-\measuredangle ACB-\measuredangle BCT=$

$=180^\circ-\gamma-\alpha=\beta=\measuredangle AB_1C_1\equiv\measuredangle CB_1T$

so the triangle TCB_1 is isosceles and therefore $\overline{B_1T}=\overline{CT}$. Similarly, $\overline{C_1T}=\overline{BT}$.

In conclusion, $\overline{C_1T}=\overline{BT}=\overline{CT}=\overline{B_1T}$, so T is the midpoint of B_1C_1. By *Lemma 3a*, AT is the symmedian from the vertex A $\bullet$

5. Let $n>1$ is a positive integer and $a_1,a_2,...,a_n$ is a sequence of n positive integers. Let

$$b_1=\left[\frac{a_2+...+a_n}{n-1}\right], b_i=\left[\frac{a_1+a_2+...+a_{i-1}+a_{i+1}+...+a_n}{n-1}\right], 1<i<n, b_n=\left[\frac{a_1+a_2+...+a_{n-1}}{n-1}\right].$$

Let f is a mapping such that $f(a_1,a_2,...,a_n)=(b_1,b_2,...,b_n)$.

a) Let the function $g:N\to N$ is defined such that $g(1)$ is the number of different elements in the sequence $f(a_1,a_2,...,a_n)$ and $g(m)$ is the number of different elements in the sequence $f^m(a_1,a_2,...,a_n)=f(f^{m-1}(a_1,a_2,...,a_n)),m>1$. Prove that there is a positive integer k_0 such that for $m\geq k_0$ the function $g(m)$ is periodical.

b) Prove that $\sum_{m=1}^{k}\frac{g(m)}{m(m+1)}<C$ for any positive integer k, where the constant C does not depend on k.

Solution. a) Let $n>2$. We will show that for m big enough, $g(m)=1$.

Let $a_1,a_2,...,a_n$ is a sequence of positive integers. Then

$$f(a_1,a_2,...,a_n)=\left(\left[\frac{a_2+a_3+...a_n}{n-1}\right],\left[\frac{a_1+a_3+...a_n}{n-1}\right],...,\left[\frac{a_1+a_2+...a_{n-1}}{n-1}\right]\right),$$

where some of the elements in the family can be equal, where from $g(1)\leq n$. By analogy we have that $g(m)\leq n$ for every positive integer m. Let S_r is a sum of the elements in the sequence $f^r(a_1,a_2,...,a_n)$. For the sum of the elements of the sequences $(b_1,...,b_n), f(b_1,...,b_n)$ from n elements, we have

$$S_{r+1}=\left[\frac{b_2+b_3+...b_n}{n-1}\right]+\left[\frac{b_1+b_3+...b_n}{n-1}\right]+...+\left[\frac{b_1+b_2+...b_{n-1}}{n-1}\right]$$
$$\leq\frac{b_2+b_3+...b_n}{n-1}+\frac{b_1+b_3+...b_n}{n-1}+...+\frac{b_1+b_2+...b_{n-1}}{n-1}=b_1+b_2+...+b_n=S_r \quad ....(*)$$

for every positive integer r. It is clear that $0\leq S_r\leq S$, for every positive integer r. We get that there is a positive integer k_0 such that for $m\geq k_0$, $\kappa=S_m=S_{m+1}=...$ is a number that is greater or equal to 0....(1). Equality in (*) holds only if the numbers in the sequence $f^m(b_1,b_2,...,b_n)$ are equal between themselves for m big enough.

We will prove that $S_{r+1}=S_r\Rightarrow d_1=d_2=...=d_n$, where $f^m(a_1,a_2...,a_n)=(d_1,...,d_n)$ for m big enough. Really, in order to have the equality sign, it is necessary that $n-1|S_r-b_i, 0\leq i\leq n$. So we get that

$$b_1\equiv b_2\equiv...\equiv b_n(\mathrm{mod}\, n-1)...(2).$$

From (2) we get that $\frac{S_r-b_1}{n-1},\frac{S_r-b_2}{n-1},...,\frac{S_r-b_n}{n-1}$, $1\leq i\leq n$, are positive integers. Also,

$$|c_i-c_j|=\left|\left[\frac{S_r-b_i}{n-1}\right]-\left[\frac{S_r-b_j}{n-1}\right]\right|=\left|\frac{S_r-b_i}{n-1}-\frac{S_r-b_j}{n-1}\right|=\left|\frac{b_i-b_j}{n-1}\right|<|b_i-b_j|...(3)$$

From (1) and (2) we have (3) and that in every step we get smaller and smaller positive integer, greater or equal to 0. After a finite number of steps, we get that $d_i = d_j$. There is a finite number of combinations i, j where from it follows that there is k_0 greater then the maximum of the number of steps for every pair i, j, where from we have the statement. So, there is a positive integer k_0 such that for $m \geq k_0$, the elements of the sequence $f^m(a_1, a_2, ..., a_n)$ are equal between themselves where from we have that $g(m) = 1$.

Let $n = 2$. It is clear that $(a_1, a_2) = f(a_1, a_2)$, so $g(m) \leq 2$ for every positive integer m.

b) For $n > 2$ and for an arbitrary positive integer k,

$$\sum_{m=1}^{k} \frac{g(m)}{m(m+1)} < \sum_{m=1}^{\infty} \frac{g(m)}{m(m+1)} = \sum_{m=1}^{k_0} \frac{g(m)}{m(m+1)} + \sum_{m=k_0+1}^{\infty} \frac{g(m)}{m(m+1)}$$

where k_0 is the positive integer from a). Then

$$\sum_{m=1}^{k} \frac{g(m)}{m(m+1)} < \sum_{m=1}^{\infty} \frac{g(m)}{m(m+1)} = \sum_{m=1}^{k_0} \frac{g(m)}{m(m+1)} + \sum_{m=k_0+1}^{\infty} \frac{1}{m(m+1)} \leq n \sum_{m=1}^{k_0} \frac{1}{m(m+1)} + \frac{1}{k_0+1} = C.$$

For $n = 2$, $\sum_{m=1}^{k} \frac{g(m)}{m(m+1)} < 2 \sum_{m=1}^{\infty} \frac{1}{m(m+1)} = 2.$

21-th Junior Balkan Mathematical Olympiad, JBMO 2017 Varna, Bulgaria, Monday, June 26, 2017

Problem 1. Determine all the sets of six consecutive positive integers such that the product of some two of them, added to the product of some other two of them is equal to the product of the remaining two numbers.

Solution. Exactly two of the six numbers are multiples of 3 and these two need to be multiplied together, otherwise two of the three terms of the equality are multiples of 3 but the third one is not.

Let n and $n+3$ denote these multiples of 3. Two of the four remaining numbers give remainder $1$when divided by 3, while the other two give remainder 2, so the two other products are either $\equiv 1 \cdot 1 = 1 (\text{mod} 3)$ and $\equiv 2 \cdot 2 = 1 (\text{mod} 3)$, or they are both $\equiv 1 \cdot 2 = 2 (\text{mod} 3)$. In conclusion, the term $n(n+3)$needs to be on the right hand side of the equality.

Looking at parity, three of the numbers are odd, and three are even. One of n and $n+3$ is odd, the other even, so exactly two of the other numbers are odd. As $n(n+3)$is even, the two remaining odd numbers need to appear in different terms.

We distinguish the following cases:

I. The numbers are $n-2, n-1, n, n+1, n+2, n+3$.

The product of the two numbers on the RHS needs to be larger than $n(n+3)$. The only possibility is $(n-2)(n-1)+n(n+3)=(n+1)(n+2)$ which leads to $n=3$. Indeed, $1\cdot 2+3\cdot 6=4\cdot 5$.

II. The numbers are $n-1, n, n+1, n+2, n+3, n+4$.

As $(n+4)(n-1)+n(n+3)=(n+1)(n+2)$ has no solutions, $n+4$ needs to be on the RHS, multiplied with a number having a different parity, so $n-1$ on $n+1$.

$(n+2)(n-1)+n(n+3)=(n+1)(n+4)$ leads to $n=3$. Indeed, $2\cdot 5+3\cdot 6=4\cdot 7$.

$(n+2)(n+1)+n(n+3)=(n-1)(n+4)$ has no solution.

III. The numbers are $n, n+1, n+2, n+3, n+4, n+5$.

We need to consider the following situations:

$(n+1)(n+2)+n(n+3)=(n+4)(n+5)$ which leads to $n=6$; indeed $7\cdot 8+6\cdot 9=10\cdot 11$;

$(n+2)(n+5)+n(n+3)=(n+1)(n+4)$ obviously without solutions, and

$(n+1)(n+4)+n(n+3)=(n+2)(n+5)$ which leads to $n=2$ (not a multiple of 3).

In conclusion, the problem has three solutions:

$$1\cdot 2+3\cdot 6=4\cdot 5, \qquad 2\cdot 5+3\cdot 6=4\cdot 7 \qquad \text{and} \qquad 7\cdot 8+6\cdot 9=10\cdot 11.$$

Problem 2. Let x, y, z be positive integers such that $x\neq y\neq z\neq x$. Prove that

$$(x+y+z)(xy+yz+zx-2)\geq 9xyz.$$

When does the equality hold?

Solution. Since x, y, z are distinct positive integers, the required inequality is symmetric and WLOG we can suppose that $x\geq y+1\geq z+2$. We consider 2 possible cases:

Case 1. $y\geq z+2$. Since $x\geq y+1\geq z+3$ it follows that

$$(x-y)^2\geq 1,\quad (y-z)^2\geq 4,\quad (x-z)^2\geq 9$$

which are equivalent to

$$x^2+y^2\geq 2xy+1,\quad y^2+z^2\geq 2yz+4,\quad x^2+z^2\geq 2xz+9$$

or otherwise

$$zx^2+zy^2\geq 2xyz+z,\quad xy^2+xz^2\geq 2xyz+4x,\quad yx^2+yz^2\geq 2xyz+9y$$

Adding up the last three inequalities we have

$$xy(x+y)+yz(y+z)+zx(z+x)\geq 6xyz+4x+9y+z$$

which implies that $(x+y+z)(xy+yz+zx-2)\geq 9xyz+2x+7y-z$.

Since $x\geq z+3$ it follows that $2x+7y-z\geq 0$ and our inequality follows.

Case 2. $y=z+1$. Since $x\geq y+1=z+2$ it follows that $x\geq z+2$, and replacing $y=z+1$ in the required inequality we have to prove

$$(x+z+1+z)(x(z+1)+(z+1)z+zx-2)\geq 9x(z+1)z$$

which is equivalent to

$$(x+2z+1)(z^2+2zx+z+x-2)-9x(z+1)z\geq 0$$

Doing easy algebraic manipulations, this is equivalent to prove

$$(x-z-2)(x-z+1)(2z+1)\geq 0$$

which is satisfied since $x\geq z+2$.

The equality is achieved only in the Case 2 for $x=z+2$, so we have equality when $(x,y,z)=(k+2,k+1,k)$ and all the permutations for any positive integer k.

Problem 3. Let ABC be an acute triangle such that $AB \neq AC$, with circumcircle Γ and circumcenter O. Let M be the midpoint of BC and D be a point on Γ such that $AD \perp BC$. Let T be a point such that $BDCT$ is a parallelogram and Q a point on the same side of BC as A, such that

$$\sphericalangle BQM = \sphericalangle BCA \text{ and } \sphericalangle CQM = \sphericalangle CBA.$$

Let the line AO intersect Γ at E, ($E \neq A$) and let the circumcircle of $\triangle ETQ$ intersect Γ at point $X \neq E$. Prove that the points A, M and X are collinear.

Solution. Let X' be symmetric point to Q in line BC. Now since $\sphericalangle CBA = \sphericalangle CQM = \sphericalangle CX'M$, $\sphericalangle BCA = \sphericalangle BQM = \sphericalangle BX'M$, we have

$$\sphericalangle BX'C = \sphericalangle BX'M + \sphericalangle CX'M = \sphericalangle CBA + \sphericalangle BCA = 180^\circ - \sphericalangle BAC$$

we have that $X' \in \Gamma$. Now since $\sphericalangle AX'B = \sphericalangle ACB = \sphericalangle MX'B$ we have that A, M, X' are collinear. Note that since

$$\sphericalangle DCB = \sphericalangle DAB = 90^\circ - \sphericalangle ABC = \sphericalangle OAC = \sphericalangle EAC$$

we get that $DBCE$ is an isosceles trapezoid.

CRTEZ

Since $BDCT$ is a parallelogram we have $MT = MD$, with M, D, T being collinear, $BD = CT$, and since $BDEC$ is an isosceles trapezoid we have $BD = CE$ and $ME = MD$. Since

$$\sphericalangle BTC = \sphericalangle BDC = \sphericalangle BED, \quad CE = BD = CT \text{ and } ME = MT$$

we have that E and T are symmetric with respect to the line BC. Now since Q and X' are symmetric with respect to the line BC as well, this means that $QX'ET$ is an isosceles trapezoid which means that Q, X', E, T are concyclic. Since $X' \in \Gamma$ this means that $X \equiv X'$ and therefore A, M, X are collinear.

Alternative solution. Denote by H the orthocenter of $\triangle ABC$. We use the following well known properties:

(i) Point D is the symmetric point of H with respect to BC. Indeed, if H_1 is the symmetric point of H with respect to BC then $\sphericalangle BH_1C + \sphericalangle BAC = 180^\circ$ and therefore $H_1 \equiv D$.

(ii) The symmetric point of H with respect to M is the point E. Indeed, if H_2 is the symmetric point of H with respect to M then BH_2CH is parallelogram, $\sphericalangle BH_2C + \sphericalangle BAC = 180^\circ$ and since $EB \| CH$ we have $\sphericalangle EBA = 90^\circ$.

Since $DETH$ is a parallelogram and $MH = MD$ we have that $DETH$ is a rectangle. Therefore $MT = ME$ and $TE \perp BC$ implying that T and E are symmetric with respect to BC. Denote by Q' the symmetric point of Q with respect to BC. Then $Q'ETQ$ is isosceles trapezoid, so Q' is a point on the circumcircle of $\triangle ETQ$. Moreover $\sphericalangle BQ'C + \sphericalangle BAC = 180^\circ$ and we conclude that $Q' \in \Gamma$. Therefore $Q' \equiv X$.

It remains to observe that $\sphericalangle CXM = \sphericalangle CQM = \sphericalangle CBA$ and $\sphericalangle CXA = \sphericalangle CBA$ and we infer that X, M and A are collinear.

Problem 4. Consider a regular $2n$-gon P, $A_1A_2 \ldots A_{2n}$ in the plane, where n is a positive integer. We say that a point S on one of the sides of P can be seen from a point E that is external to P, if the line segment SE contains no other points that lie on the sides of P except S. We color the sides of P in 3 different colors (ignore the vertices of P, we consider them colorless), such that every side is colored in exactly one color, and each color is used at least once. Moreover, from every point in the plane external to P, points of at most 2 different colors on P can be seen. Find the number of distinct such colorings of P (two colorings are considered distinct if at least one of the sides is colored differently).

Solution. Answer: For $n=2$, the answer is 36; for $n=3$, the answer is 30 and for $n\geq 4$, the answer is $6n$.

Lemma 1. Given a regular $2n-$gon in the plane and a sequence of n consecutive sides $s_1,s_2,\dots,s_n$ there is an external point Q in the plane, such that the color of each s_i can be seen from Q, for $i=1,2,\dots,n.$.

Proof. It is obvious that for a semi-circle S, there is a point R in the plane far enough on the bisector of its diameter such that almost the entire semi-circle can be seen from R.

Now, it is clear that looking at the circumscribed circle around the $2n-$gon, there is a semi-circle S such that each s_i either has both endpoints on it, or has an endpoint that's on the semi-circle, and is not on the semi-circle's end. So, take Q to be a point in the plane from which almost all of S can be seen, clearly, the color of each s_i can be seen from Q.

Lemma 2. Given a regular $2n-$gon in the plane, and a sequence of $n+1$ consecutive sides $s_1,s_2,\dots,s_{n+1}$ there is no external point Q in the plane, such that the color of each s_i can be seen from Q, for $i=1,2,\dots,n+1.$.

Proof. Since s_1 and s_{n+1} are parallel opposite sides of the $2n-$gon, they cannot be seen at the same time from an external point.

For $n=2$, we have a square, so all we have to do is make sure each color is used. Two sides will be of the same color, and we have to choose which are these 2 sides, and then assign colors according to this choice, so the answer is $\binom{4}{2}\cdot 3\cdot 2=36$.

For $n=3$, we have a hexagon. Denote the sides as $a_1,a_2,\dots,a_6$, in that order. There must be 2 consecutive sides of different colors, say a_1 is red, a_2 is blue. We must have a green side, and only a_4 and a_5 can be green. We have 3 possibilities:

1) a_4 is green, a_5 is not. So, a_3 must be blue and a_5 must be blue (by elimination) and a_6 must be blue, so we get a valid coloring.

2) Both a_4 and a_5 are green, thus a_6 must be red and a_5 must be blue, and we get the coloring rbbggr.

3) a_5 is green, a_4 is not. Then a_6 must be red. Subsequently, a_4 must be red (we assume it is not green). It remains that a_3 must be red, and the coloring is rbrrgr.

Thus, we have 2 kinds of configurations:

i) 2 opposite sides have 2 opposite colors and all other sides are of the third color. This can happen in $3\cdot(3\cdot 2\cdot 1)=18$ ways (first choosing the pair of opposite sides, then assigning colors),

ii) 3 pairs of consecutive sides, each pair in one of the 3 colors. This can happen in $3\cdot 6=12$ ways (2 partitioning into pairs of consecutive sides, for each partitioning, 6 ways to assign the colors).

Thus, for $n=3$, the answer is $18+12=30$.

Finally, let's address the case $n \geq 4$. The important thing now is that any 4 consecutive sides can be seen from an external point, by Lemma 1.

Denote the sides as $a_1, a_2, \ldots, a_{2n}$. Again, there must be 2 adjacent sides that are of different colors, say a_1 is blue and a_2 is red. We must have a green side, and by Lemma 1, that can only be a_{n+1} or a_{n+2}. So, we have 2
cases:

Case 1: a_{n+1} is green, so an must be red (cannot be green due to Lemma 1 applied to $a_1, a_2, \ldots, a_n$, cannot be blue for the sake of $a_2, \ldots, a_{n+1}$. If a_{n+2} is red, so are $a_{n+3}, \ldots, a_{2n}$, and we get a valid coloring: a_1 is blue, a_{n+1} is green, and all the others are red.

If a_{n+2} is green:

a) a_{n+3} cannot be green, because of $a_2, a_1, a_{2n}, \ldots, a_{n+3}$.

b) a_{n+3} cannot be blue, because the 4 adjacent sides $a_n, \ldots, a_{n+3}$ can be seen (this is the case that makes the separate treatment of $n \geq 4$ necessary)

c) a_{n+3} cannot be red, because of $a_1, a_{2n}, \ldots, a_{n+2}$.

So, in the case that a_{n+2} is also green, we cannot get a valid coloring.

Case 2: a_{n+2} is green is treated the same way as Case 1.

This means that the only valid configuration for $n \geq 4$ is having 2 opposite sides colored in 2 different colors, and all other sides colored in the third color. This can be done in $n \cdot 3 \cdot 2 = 6n$ ways.

BALKAN MATHEMATICAL OLYMPIAD

Ohrid, 04.05.2017, Republic of MACEDONIA

Problem 1. Find all pairs (x, y) of positive integers such that

$$x^3 + y^3 = x^2 + 42xy + y^2.$$

Solution. Let $d = (x, y)$ be the greatest common divisor of positive integers x and y. So, $x = ad$, $y = bd$, where $d \in \mathbb{N}$, $(a,b) = 1$, $a, b \in \mathbb{N}$. We have

$$\begin{aligned} x^3 + y^3 = x^2 + 42xy + y^2 \quad &\Leftrightarrow \quad d^3(a^3 + b^3) = d^2(a^2 + 42ab + b^2) \\ &\Leftrightarrow \quad d(a+b)(a^2 - ab + b^2) = a^2 + 42ab + b^2 \\ &\Leftrightarrow \quad (da + db - 1)(a^2 - ab + b^2) = 43ab. \end{aligned}$$

If we denote $c = da + db - 1 \in \mathbb{N}$, then the equality $a^2c - abc + b^2c = 43ab$ implies the relations

$$\left.\begin{array}{l} b | ca^2 \;\Rightarrow\; b | c \\ a | cb^2 \;\Rightarrow\; a | c \end{array}\right\} \Rightarrow\; (ab) | c$$

$$\begin{aligned} &\Leftrightarrow \quad c = mab, \; m \in \mathbb{N}^+ \\ &\Rightarrow \quad m(a^2 - ab + b^2) = 43 \\ &\Rightarrow \quad (a^2 - ab + b^2) \,|\, 43 \\ &\Leftrightarrow \quad a^2 - ab + b^2 = 1 \quad \text{or} \quad a^2 - ab + b^2 = 43. \end{aligned}$$

If $a^2-ab+b^2=1$, then $(a-b)^2=1-ab\geq 0 \Rightarrow a=b=1,\ 2d=44,\ (x,y)=(22,22)$.

If $a^2-ab+b^2=43$, then, by virtue of simmetry, we suppose that $x\geq y \Rightarrow a\geq b$. We obtain that $43=a^2-ab+b^2\geq ab\geq b^2 \Rightarrow b\in\{1,2,3,4,5,6\}$.

If $b=1$, then $a=7$, $d=1$, $(x,y)=(7,1)$ or $(x,y)=(1,7)$.

If $b=6$, then $a=7$, $d=\frac{43}{13}\notin\mathbb{N}$.

For $b\in\{2,3,4,5\}$ there no positive integer solutions for a.

Finally, we have $(x,\ y)\in\{(1,7),(7,1),(22,22)\}$.

Problem 2. Let ABC be a triangle with $AB<AC$ inscribed into a circle c. The tangent of c at the point C meets the parallel from B to AC at the point D. The tangent of c at the point B meets the parallel from C to AB at the point E and the tangent of c at the point C at the point L. Suppose that the circumcircle c_1 of the triangle BDC meets AC at the point T and the circumcircle c_2 of the triangle BEC meets AB at the point S. Prove that the lines ST, BC, AL are concurrent.

Solution. We will prove first that the circle c_1 is tangent to AB at the point B. In order to prove this, we have to prove that $\measuredangle BDC=\measuredangle ABC$. Indeed, since $BD\,||\,AC$, we have that $\measuredangle DBC=\measuredangle ACB$. Additionally, $\measuredangle BCD=\measuredangle BAC$ (by chord and tangent), which means that the triangles ABC, BDC have two equal angles and so the third ones are also equal. It follows that $\measuredangle BDC=\measuredangle ABC$, so c_1 is tangent to AB at the point B.

Similarly, the circle c_2 is tangent to AC at the point C.

As a consequence, $\measuredangle ABT=\measuredangle ACB$ (by chord and tangent) and also $\measuredangle BSC=\measuredangle ACB$.

By the above, we have that $\measuredangle ABT=\measuredangle BSC$, so the lines BT, SC are parallel.

Now, let ST intersect BC at the point K. It suffice to prove that K belongs to AL.

From the trapezoid $BTCS$ we get that

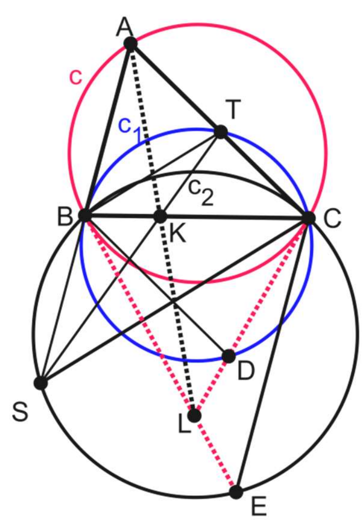
$$\frac{BK}{KC}=\frac{BT}{SC} \qquad (1)$$

and from the similar triangles ABT, ASC, we have that

$$\frac{BT}{SC}=\frac{AB}{AS}. \qquad (2)$$

By (1), (2) we get that

$$\frac{BK}{KC}=\frac{AB}{AS}. \qquad (3)$$



From the power of point theorem, we have that

$AC^2 = AB \cdot AS \Rightarrow AS = \frac{AC^2}{AB}$.

Going back into (3), it gives that

$$\frac{BK}{KC} = \frac{AB^2}{AC^2}.$$

From the last one, it follows that K belongs to the symmedian of the triangle ABC.

Finally, recall that the well known fact that since LB and LC are tangents, it follows that AL is the symmedian of the triangle ABC, so K belongs to AL, as needed.

Problem 3. Find all the functions $f:\mathbb{N}\to\mathbb{N}$ such that:

$$n+f(m) \mid f(n)+nf(m) \qquad (1)$$

for any $m,n\in\mathbb{N}$

Solution. We will consider 2 cases, whether the range of the functions is infinite or finite or in other words the function take infinite or finite values.

Case 1. The Function has an infinite range. Let's fix a random natural number n and let m be any natural number. Then using (1) we have

$$n+f(m) \mid f(n)+nf(m) = f(n)-n^2+n(f(m)+n) \quad \Rightarrow \quad n+f(m) \mid f(n)-n^2.$$

Since n is a fixed natural number, then $f(n)-n^2$ is as well a fixed natural number, and since the above results is true for any m and the function f has an infinite range, we can choose m such that $n+f(m) > |f(n)-n^2|$. This implies that $f(n)=n^2$ for any natural number n. We now check that it is a solution. Since

$$n+f(m)=n+m^2$$

and

$$f(n)+nf(m)=n^2+nm^2=n(n+m^2)$$

it is straightforward that $n+f(m) \mid f(n)+nf(m)$.

Case 2. The Function has a finite range. Since the function takes finite values, then it exists a natural number k such that $1 \le f(n) \le k$ for any natural number n. It is clear that it exists at least one natural number s (where $1 \le s \le k$) such that $f(n)=s$ for infinite natural numbers n. Let m,n be any natural numbers such that $f(m)=f(n)=s$. Using (1) we have

$$n+s \mid s+ns = s-s^2+s(n+s) \quad \Rightarrow \quad n+s \mid s^2-s.$$

Since this is true for any natural number n such that $f(n)=s$ and since exist infinite natural numbers n such that $f(n)=s$, we can choose the natural number n such that $n+s > s^2-s$, which implies that $s^2=s \Rightarrow s=1$, or in other words $f(n)=1$ for an infinite natural number n.

Let's fix a random natural number m and let n be any natural number $f(n)=1$. Then using (1) we have

$$n+f(m)\,|\,1+nf(m)=1-(f(m))^2+f(m)(n+f(m)) \quad \Rightarrow \quad n+f(m)\,|\,(f(m))^2-1$$

Since m is a fixed a random natural number, then $(f(m))^2-1$ is a fixed non-negative integer and since n is any natural nummber such that $f(n)=1$ and since exist infinite numbers n such that $f(n)=1$, we can choose the the natural number n such that $n+f(m)>(f(m))^2-1$. This implies $f(m)=1$ for any natural number m. We now check that it is a solution. Since

$$n+f(m)=n+1$$

and

$$f(n)+nf(m)=1+n$$

it is straightforward that $n+f(m)\,|\,f(n)+nf(m)$.

So, all the functions that satisfy the given condition are $f(n)=n^2$ for any $n\in\mathbb{N}$ or $f(n)=1$ for any $n\in\mathbb{N}$.

Problem 4. We have n students sitting at a round table. Initially each student is given one candy. At each step each student having candies either picks one of its candies and gives it to one of its neighbouring students, or distributes all of its candies to its neighbouring students in any way he wishes. A distribution of candies is called legal if it can be reached from the initial distribution via a sequence of steps.

Determine the number of legal distributions. (All the candies are udentical.)

Solution. The answer turns out to be $\binom{2n-1}{n}$ if n is odd and $\binom{2n-1}{n}-2\binom{\frac{3n}{2}-1}{n}$ if n is even.

***Case* 1.** Suppose n is odd, say $n=2m+1$. In this case we will show that any distribution of candies is legal. Thus the number of legal distributions is indeed $\binom{2n-1}{n}$.

In this case we can achieve the above claim by letting each student to always distribute all of its candies to its two neighbouring students in some way. Thus at each step each candy will move either one position clockwise or one anticlockwise.

We now look at the initial distribution of candies and the required final distribution. We specify arbitrarily for each candy in the initial distribution, the position we wish this candy to end up in the required final distribution. Because n is odd, either the clockwise distance or the anticlockwise distance between the initial position of the candy and the required final position is even and at most m.

Thus after an even number of steps (at most m) we can move each candy to its required final position. (Note that if the candy reaches the required position earlier, we can move it back and forth until all candies reach their required position.) This completes the proof of our claim in this case.

***Case* 2.** Suppose n is even, say $n=2m$. Let $x_1,\dots,x_{2m}$ be the students in this cyclic order. Observe that initially the students with even indices (even students) have at least one candy in total, and so do the students with odd indices (odd students). This property is preserved after each step.

We will show that every distribution in which the even students have at least one candy in total and the odd students also have at least one candy in total is legal.

Let us suppose that the required final distribution has a candies in odd positions and b candies in even positions. (Where $a,b\geq 1$.) It will be enough to reach any position with a candies in even positions and b candies in odd positions as then we can follow the same approach as in Case 1.

To achieve this we will first move all candies to students x_1 and x_2. This is easy by specifying that at each step x_1 moves all of its candies to x_2 while for $1\leq r\leq 2m-1$ student x_{r+1} moves all of its candies to x_r.

Suppose that we now have $a+k$ candies at x_1 and $b-k$ candies at x_2 where without loss of generality $k\geq 0$. If $k=0$ we have reached our target. If not, in the next step x_1 moves a candy to x_2 and x_2 moves a candy to x_3. In the next step x_1 (it still has $a+k-1\geq a>0$ candies) moves a candy to x_2, x_2 moves a candy to x_1 and x_3 moves a candy to x_2. We now have $a+k-1$ candies in x_1 and $b+1-k$ in x_2. Repeating this process another $k-1$ times we end up with a candies in x_1 and b candies in x_2 as required.

It remains to count the total number of legal configurations in this case. This is indeed equal to

$$\binom{2n-1}{n}-2\binom{\frac{3n}{2}-1}{n}$$

as $\binom{2n-1}{n}$ counts the total number of configurations while $\binom{\frac{3n}{2}-1}{n}$ counts the number of illegal configurations where either all n candies belong to the $\frac{n}{2}$ odd positions or all n candies belong to the $\frac{n}{2}$ even positions.

XX Mediterranean mathematical olympiad, 23 april 2017, Faculty of mechanical ingineering

Problem 1. Determine the smallest integer n, for which there exist integers $x_1,\ldots,x_n$ and positive integers $a_1,\ldots,a_n$, so that

$$x_1+\ldots+x_n=0, \qquad a_1x_1+\ldots+a_nx_n>0, \qquad a_1^2x_1+\ldots+a_n^2x_n<0.$$

Solution. The answer is $n=3$. One possible example for $n=3$ is $x_1=2$ and $x_2=x_3=-1$, with $a_1=4, a_2=1, a_3=6$.

For $n=1$, the fiorst constraint enforces $x_1=0$; this is in contradiction with the other two constrains. For $n=2$, the first constraint enforces $x_2=-x_1$. Then the second constraint is equivalent to $a_1x_1-a_2x_1>0$. If we multiply this inequality by the positive value a_1+a_2, we get $a_1^2x_1-a_2^2x_1>0$; this is equivalent to $a_1^2x_1+a_2^2x_1>0$ and contradicts the third constraint.

Problem 2. Let a,b,c be positive real numbers such that $a+b+c=1$. Prove that

$$(x^2+y^2+z^2)\left(\frac{a^3}{x^2+2y^2}+\frac{b^3}{y^2+2z^2}+\frac{c^3}{z^2+2x^2}\right)\geq\frac{1}{9},$$

holds for all positive real numbers x,y,z.

Solution. On account of the constrain $a+b+c=1$ we will prove that it holds the equivalent inequality

$$(x^2+y^2+z^2)\left(\frac{a^3}{x^2+2y^2}+\frac{b^3}{y^2+2z^2}+\frac{c^3}{z^2+2x^2}\right)\geq\frac{(a+b+c)^3}{9}.$$

Indeed, Holder's inequality claims that

$$\prod_{i=1}^{3}(a_i^3+b_i^3+c_i^3)^{1/3}\geq a_1a_2a_3+b_1b_2b_3+c_1c_2c_3$$

for all positive reals a_i,b_i,c_i, $1\leq i\leq 3$. Putting in the preceding

$$(a_1,a_2,a_3)=\left(\frac{a}{\sqrt[3]{x^2+2y^2}},1,\sqrt[3]{x^2+2y^2}\right),$$

$$(b_1,b_2,b_3)=\left(\frac{b}{\sqrt[3]{y^2+2z^2}},1,\sqrt[3]{y^2+2z^2}\right),$$

and

$$(c_1,c_2,c_3)=\left(\frac{c}{\sqrt[3]{z^2+2x^2}},1,\sqrt[3]{z^2+2x^2}\right)$$

yields

$$\sqrt[3]{3}\left(\frac{a^3}{x^2+2y^2}+\frac{b^3}{y^2+2z^2}+\frac{c^3}{z^2+2x^2}\right)^{\frac{1}{3}}(3x^2+3y^2+3z^2)^{\frac{1}{3}}\geq(a+b+c).$$

Cubing both sides and dividing both sides by $9(x^2+y^2+z^2)$ we obtain

$$\frac{a^3}{x^2+2y^2}+\frac{b^3}{y^2+2z^2}+\frac{c^3}{z^2+2x^2}\geq\frac{(a+b+c)^3}{9(x^2+y^2+z^2)}$$

from which claimed inequality follows. Equality holds when $a=b=c=x=y=z=\frac{1}{3}$, and the proof is complete.

Problem 3. Let ABC be an equilateral triangle, and let P be some point in its circumcircle . Determine, with reasons, all the numbers $n\in\mathbb{N}^*$ such that the sum

$$S_n(P)=|PA|^n+|PB|^n+|PC|^n,$$

is independent of the choice of the point P.

Solution. We will take an orthonormal coordinate system, with origin in the point O (center of the circumcircle of ABC), taking moreover the point A on the Ox axis, and

$|OA|=1$. In the complex numers z_A, z_B, z_C and z are respectively the affixes of the points A, B, C, P we have

$$|z_A|=|z_B|=|z_C|=|z|=1,$$

and therefore the first three are the rots of $z^3=1$, that is

$$z_A=1, z_B=-\frac{1}{2}+i\frac{\sqrt{3}}{2}, z_C=-\frac{1}{2}-i\frac{\sqrt{3}}{2}.$$

For another hand, $z=a+ib$, with $a^2+b^2=1$. Then we have

$$S_n(P)=|PA|^n+|PB|^n+|PC|^n=|z-z_A|^n+|z-z_B|^n+|z-z_C|^n. \quad (*)$$

But as

$$|z-z_A|=\sqrt{2}(1-a)^{\frac{n}{2}}; \quad |z-z_B|=\sqrt{2+a-b\sqrt{3}}; \; |z-z_C|=\sqrt{2+a+b\sqrt{3}},$$

we get from (*)

$$S_n(P)=2^{\frac{n}{2}}(1-a)^{\frac{n}{2}}+(2+a-b\sqrt{3})^{\frac{n}{2}}+(2+a+b\sqrt{3})^{\frac{n}{2}}. \quad (**)$$

If $P=A$, then $S_n(A)=3^{\frac{n}{2}}+3^{\frac{n}{2}}=2\cdot 3^{\frac{n}{2}}$. If $P_1\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, entonces $z=\frac{1}{2}+i\frac{\sqrt{3}}{2}$ that is, $a=\frac{1}{2}, b=\frac{\sqrt{3}}{2}$ and from (**) we get

$$S_n(P_1)=2^{\frac{n}{2}}\cdot 2^{\frac{n}{2}}+\left(2+\frac{1}{2}-\frac{3}{2}\right)^{\frac{n}{2}}+\left(2+\frac{1}{2}+\frac{3}{2}\right)^{\frac{n}{2}}=2+2^n.$$

Then, if $S_n(P)$ must to be independent of P, we get $S_n(A)=S_n(P_1) \Leftrightarrow 2\cdot 3^{\frac{n}{2}}=2+2^n \Leftrightarrow n=2$ or $n=4$.

Problem 4. A set S of integers is Balearic, if there are two (not necessarily distinct) elements $s, s' \in S$ whose sum $s+s'$ is a power of two; otherwise it is called a non-Balearic set.

Find an integer n such that $\{1,2,...,n\}$ contains a 99-element non-Balearic set, whereas all the 100-element subsets are Balearic.

Solution. Let $f(n)$ denote the largest cardinality of a non_Balearic set in $\{1,2,...,n\}$. One easily verifies that $f(0)=f(1)=0$. Now consider an integer $n\geq 2$ and write it in the form $n=2^a+b$ with $0\leq b\leq 2^a-1$. We want to show

$$f(n)=f(2^a+b)=f(2^a-b-1)+b.$$

Partition $\{1,2,...,n\}$ into $X=\{1,2,...,2^a-b-1\}$ and $Y=\{2^a-b,....,2^a+b\}$. A non-Balearic-subeset S of $\{1,2,...,n\}$ contains at most $f(2^a-b-1)$ elements from X (by definition of f) and at most b elements from Y (as it cannot contain 2^a altogether, and as it contains at most one of the two numbers 2^a-x and 2^a+x). This establishes the first inequality $f(n)\leq f(2^a-b-1)+b$).

Next consider a non-Balearic set $T\subseteq X$ of caridnality $f(2^a-b-1)$. We claim that also $S=T\cup\{2^a+1,...,2^a+b\}$ is a non-Balearic set. Suppose for the sake of contradiction that the sum $s+s'$ of some $s,s'\in S$ is a power of two. Then $s,s'\in T$ is impossible, as T itself is a non-Balearic set. Also $s,s'\in\{2^a+1,...,2^a+b\}$ is impossible, as

$$2^{a+1}<(2^a+1)+(2^a+1)\leq s+s'\leq(2^a+b)+(2^a+b)<2^a+2.$$

Hence one of s and s' must be in T and the other one in $\{2^a+1,....,2^a+b\}$, which yields the final contradiction

$$2^a<s+s'\leq(2^a-b-1)+(2^a+b)<2^{a+1}.$$

Since the constructed non-Balearic set S is of cardinality $f(2^a-b-1)+b$, we have established the second inequality $f(n)\geq f(2^a-b-1)+b$. The two established inequalities together imply the desired recursive equation $f(n)=f(2^a-b-1)+b$ displayed above.

The rest is computation.

It is easy to see (or to determine through the recursive equation) that $f(4)=1$.

For $2^a=8$ and $b=3$, the recursion yields $f(11)=f(4)+3=4$.

For $2^a=32$ and $b=20$, the recursion yields $f(52)=f(11)+20=24$.

For $2^a=128$ and $b=75$, the recursion yields $f(203)=f(52)+75=99$.

Hence an answer to the problem is $n=203$ with $f(203)=99$.

(Similar computations yield $f(202)=98$ and $f(204)=100$. Hence $n=203$ constitutes the unique possible answer for the problem).

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