7TH European Mathematical Cup

8th December 2018 - 16th December 2018

Junior Category

MLADI NADARENI MATEMATIČARI Marin Getaldić

Problems and Solutions

Problem 1. Let a, b, c be non-zero real numbers such that

$$a^2 + b + c = \frac{1}{a},$$
$$b^2 + c + a = \frac{1}{b},$$
$$c^2 + a + b = \frac{1}{c}.$$

Prove that at least two of a, b, c are equal.

(Daniel Paleka)

First Solution. Let's assume the opposite, i.e. a, b and c are pairwise non-equal. By subtracting the second equality from the first one, we obtain

$$\begin{aligned}(a^2 + b + c) - (b^2 + c + a) &= \frac{1}{a} - \frac{1}{b}\\ (a^2 - b^2) + (b - a) &= \frac{b - a}{ab}\\ (a - b)(a + b) - (a - b) + \frac{a - b}{ab} &= 0\\ (a - b)\left(a + b - 1 + \frac{1}{ab}\right) &= 0\end{aligned}$$

1 point.

Since $a \neq b$ we may conclude

$$a + b - 1 + \frac{1}{ab} = 0 \tag{1}$$

1 point.

Similarly, subtracting the third equality from the second one, combined with $b \neq c$, gives us

$$b + c - 1 + \frac{1}{bc} = 0 \tag{2}$$

1 point.

Expressions on the left side in (1) and (2) are both equal to 0 which specifically implies

$$\begin{aligned}a + b - 1 + \frac{1}{ab} &= b + c - 1 + \frac{1}{bc}\\ (a - c) + \frac{1}{b} \cdot \left(\frac{1}{a} - \frac{1}{c}\right) &= 0\\ (a - c) + \frac{1}{b} \cdot \frac{c - a}{ac} &= 0\\ (a - c)\left(1 - \frac{1}{b} \cdot \frac{1}{ac}\right) &= 0\\ 1 - \frac{1}{abc} &= 0\\ abc &= 1\end{aligned}$$

2 points.

Inserting that back into (1) results with

$$0 = a + b - 1 + \frac{1}{ab} = a + b - 1 + \frac{abc}{ab} = a + b - 1 + c$$

$$\Rightarrow\ b + c = 1 - a \tag{3}$$

2 points.

Finally, combining (3) with the first of the 3 given equations results with

$$\begin{aligned} a^2 + b + c &= \frac{1}{a} \\ a^2 + 1 - a &= \frac{1}{a} \\ (a^2 - a) + \left(1 - \frac{1}{a}\right) &= 0 \\ a(a-1) + \frac{a-1}{a} &= 0 \\ (a-1)\left(a + \frac{1}{a}\right) &= 0 \\ (a-1) \cdot \frac{a^2+1}{a} &= 0 \end{aligned}$$

Because of $a^2 + 1 > 0$ we obtain $a - 1 = 0$, i.e. $a = 1$.

2 points.

Analogously we also find $b = c = 1$ which is a contradiction with the assumption so we conclude that at least two of a, b, c are equal.

1 point.

Second Solution. Let's assume the contrary, i.e. a, b and c are pairwise different. After multiplying the first equation with a, the second with b, and the third with c, we get:

$$a^3 + ab + ac = b^3 + bc + ba = c^3 + ca + cb = 1.$$

0 points.

In particular, the first two expressions are equal. Subtracting them and factorizing leads to:

$$a^3 - b^3 + ac - bc = 0$$
$$(a-b)(a^2 + ab + b^2 + c) = 0$$

1 point.

Since $a \neq b$ we may conclude:

$$a^2 + ab + b^2 + c = 0$$

1 point.

Similarly, we can get the same thing for b and c:

$$b^2 + bc + c^2 + a = 0$$

1 point.

Subtracting these two equations yields:

$$a^2 - c^2 + ab - bc + c - a = 0$$
$$(a-c)(a+c+b-1) = 0$$

2 points.

Because $a \neq c$, we obtain:

$$a + b + c = 1$$

2 points.

Now we proceed to arrive to a contradiction in the same way as in the previous solution.

3 points.

Notes on marking:

- After obtaining $a = 1$, we may use that fact in (3) to conclude $b + c = 0$, i.e. $c = -b$. That gives us

$$1 = abc = 1 \cdot b \cdot (-b) = -b^2$$

  which isn't satisfied for any $b \in \mathbb{R}$. Again we reach contradiction with the assumption and conclude that at least two of a, b, c are equal. This part of the solution should be awarded with **1 point.**

Problem 2. Find all pairs (x, y) of positive integers such that

$$xy \mid x^2 + 2y - 1.$$

(Ivan Novak)

First Solution. Notice that the condition implies

$$x \mid 2y - 1.$$

1 point.

This implies that there exists a positive integer k such that $kx = 2y - 1$, so $y = \frac{kx+1}{2}$.

1 point.

Returning to the starting assertion, we get that

$$\frac{x(kx+1)}{2} \mid x^2 + kx \implies kx + 1 \mid 2(k + x).$$

2 points.

For all positive integers k, x, the following inequality is satisfied, with equality if and only if $k = 1$ or $x = 1$:

$$\frac{2(k+x)}{kx+1} \leqslant 2 \iff 2(k-1)(x-1) \geqslant 0.$$

2 points.

So as $\frac{2(k+x)}{kx+1} \in \mathbb{N}$, then we conclude that $\frac{2(k+x)}{kx+1} \in \{1, 2\}$.

1 point.

We now have two possible cases.

1. $k + x = kx + 1$. In this case, $k = 1$ or $x = 1$.
   (a) If $x = 1$, then y can be any positive integer.
   (b) If $k = 1$, then $x = 2y - 1$, where y is any positive integer.

1 point.

2. $2k + 2x = kx + 1$. Then $2x - 1 = k(x - 2) \implies x - 2 \mid 2x - 1 \implies x - 2 \mid 3$. This has only two solutions, both of which are true by an easy check: $x = 3, k = 5, y = 8$ or $x = 5, k = 3, y = 8$.

2 points.

Therefore, the set of solutions is

$$(x, y) \in \{(1, t), (2t - 1, t), (3, 8), (5, 8) \mid t \in \mathbb{N}.\}$$

Second Solution. Let (x, y) be a solution, and let $\frac{x^2+2y-1}{xy} = g$. This equation is equivalent with $x^2-(gy)x+2y-1 = 0$. We know x is one root of the equation. Let R be the other root. Using Vieta's formulas, we obtain the following system of equations:

$$x + R = gy$$
$$xR = 2y - 1.$$

3 points.

From the first equation we get that R is an integer, and from the second equation we get that it is a positive integer.

1 point.

Using the same inequality as in Solution 1, we get that $gy \leqslant 2y$, which implies $g = 1$ or $g = 2$.

3 points.

We now split into two cases:

1. If $g = 1$, then $x^2 + 2y - 1 = xy \implies x^2 - 1 = y(x - 2) \implies x - 2 \mid x^2 - 1 \implies x - 2 \mid 3$. This has only two solutions, both of which are true by an easy check: $x = 5, y = 8$ or $x = 3, y = 8$.

2 points.

2. If $g = 2$, then $x^2 + 2y - 1 = 2xy \implies x^2 - 1 = 2y(x - 1) \implies x = 1$ or $x = 2y - 1$, and y can be any positive integer.

1 point.

Therefore, the set of solutions is

$$(x, y) \in \{(1, t), (2t - 1, t), (3, 8), (5, 8) \mid t \in \mathbb{N}.\}$$

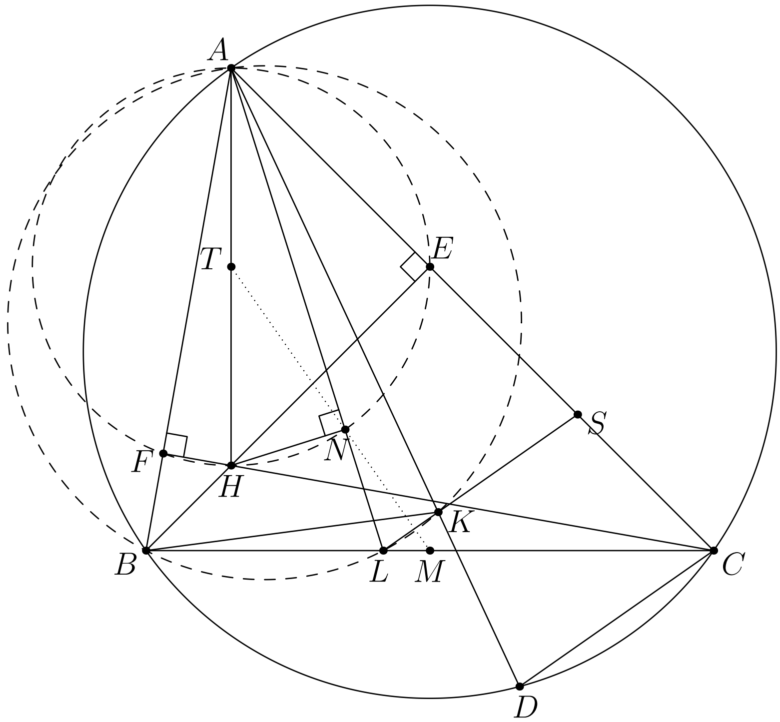
Notes on marking:

- Points from separate solutions can not be added. The competitor should be awarded the maximum of the points scored in the 2 presented solutions, or an appropriate number of points on an alternative solution.

Problem 3. Let ABC be an acute triangle with $|AB| < |AC|$ and orthocenter H. The circle with center A and radius $|AC|$ intersects the circumcircle of $\triangle ABC$ at point D and the circle with center A and radius $|AB|$ intersects the segment $\overline{AD}$ at point K. The line through K parallel to CD intersects BC at the point L. If M is the midpoint of $\overline{BC}$ and N is the foot of the perpendicular from H to AL, prove that the line MN bisects the segment $\overline{AH}$.

(Miroslav Marinov)

First Sketch.



First Solution. We start with the following:

Lemma 1. AL is the angle bisector of $\angle BAC$.

Proof: Since A, B, C and D lie on the same circle we obtain that $\angle ABC = \angle ADC = \angle ACD$.

1 point.

From that we get the following three equations:

$$\angle CAD = 180^\circ - 2\angle ADC = 180^\circ - 2\angle ABC$$

1 point.

$$\angle BCD = \angle BAD = \angle BAK = \angle BAC - (180^\circ - 2\angle ABC) = \angle ABC - \angle ACB$$

1 point.

$$\angle ABK = \angle AKB = 90^\circ - \frac{\angle BAK}{2} = 90^\circ - \frac{\angle ABC - \angle ACB}{2}$$

1 point.

Next from $LK \parallel CD$ it follows that $\angle CLK = \angle LCD = \angle BCD = \angle BAK$ so A, B, L and K are concyclic.

1 point.

Now we have

$$\angle BAL = \angle BAK - \angle LAK = \angle BAK - \angle LBK = (\angle ABC - \angle ACB) - (\angle ABC - \angle ABK) = \frac{\angle BAC}{2}$$

hence AL is the angle bisector of $\angle BAC$. □

1 point.

Let E and F be the feet of the altitudes from B and C in $\triangle ABC$. Observe that $\angle AEH = \angle AFH = \angle ANH = 90^\circ$ so A, E, H, N and F lie on the circle with diameter $\overline{AH}$.

1 point.

Since AL is the angle bisector of $\angle BAC$ its follows that $|NE| = |NF|$.

1 point.

Denote by T the midpoint of $\overline{AH}$. Since T is the circumcenter of $AEHNF$ we get $|TE| = |TF|$.

1 point.

Also since $\angle BEC = \angle CFB$, E and F lie on the circle with diameter $\overline{BC}$ from where we get $|ME| = |MF|$ so M, N and T lie on the bisector of $\overline{EF}$.

1 point.

Alternative proof of Lemma 1.

Similarly as in the first proof, we obtain that $\angle ABC = \angle ADC = \angle ACD$.

1 point.

Let S be the intersection of KL and AC. Since $LS \parallel DC$, we have $\angle ASL = \angle ACD = \angle ABC = \angle ABL$.

2 points.

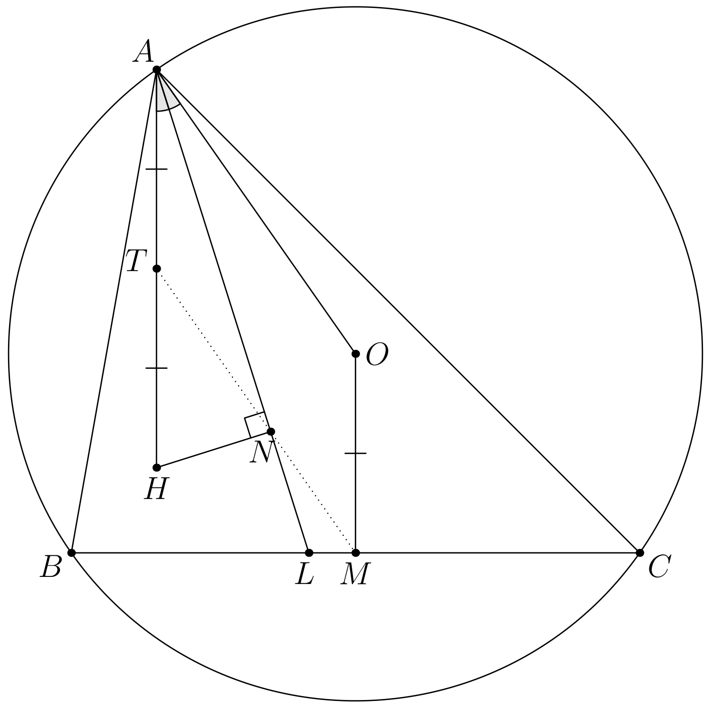
We also get $\angle AKS = \angle ADC = \angle ASK$, hence $|AS| = |AK| = |AB|$.

1 point.

Since B, L, S are not collinear (this is since SL is parallel to CD, which in turn isn't parallel to BC since $|AB| < |AC|$), we may conclude that $\triangle ABL$ and $\triangle ASL$ are congruent. The claim now follows. $\square$

2 points.

Second Sketch.



Second Solution. We get similarly as in the **First Solution** that AL is the angle bisector of $\angle BAC$.

6 points.

Denote by T the midpoint of $\overline{AH}$. As $\triangle HNA$ is right-angled, we have that $\angle NTH = 2\angle NAH$.

1 point.

Denote by O the circumcenter of $\triangle ABC$. It is known that (as a consequence of existence of Euler line)

$$|AH| = 2|OM| \implies |AT| = \frac{|AH|}{2} = |OM|$$

and as AT and OM are both orthogonal to BC, they are parallel, so $ATMO$ is an parallelogram.

1 point.

Now as $\angle HAB = 90^\circ - \angle ABC = \angle OAC$, we know that AL is the angle bisector of OAH.

1 point.

Then we have that $\angle MTH = \angle OAH = 2\angle NAH = \angle NTH$ and we conclude that T, N and M are colinear.

1 point.

Notes on marking:

- If a student has a partial solution with analytic methods, only points for proving facts that can be expressed in geometric ways and lead to a compete solution can be awarded.

Problem 4. Let n be a positive integer. Ana and Banana are playing the following game:

First, Ana arranges $2n$ cups in a row on a table, each facing upside-down. She then places a ball under a cup and makes a hole in the table under some other cup. Banana then gives a finite sequence of commands to Ana, where each command consists of swapping two adjacent cups in the row.

Her goal is to achieve that the ball has fallen into the hole during the game. Assuming Banana has no information about the position of the hole and the position of the ball at any point, what is the smallest number of commands she has to give in order to achieve her goal?

(Adrian Beker)

First Solution. We claim that the minimum number of commands is $n(3n-2)$.

Call a finite sequence of commands *valid* if it results in the ball falling into the hole while performing the commands, regardless of the inital position of the ball and the position of the hole. Also call a position an *endpoint* if it is either the first or the last position in the row.

Lemma 1. A sequence of commands is valid if and only if it results in each cup visiting both endpoints.

Proof. Suppose there exists a cup c that hasn't visited an endpoint p. Then the ball fails to fall into the hole in the case when it is under c and the hole is at p. Hence, the sequence is not valid. Conversely, if each cup has visited both endpoints, by discrete continuity it must have also visited all positions in between. Hence, the ball has certainly fallen into the hole, so the sequence is valid. $\square$

2 points.

Now consider a valid sequence of commands. We will show that it has length at least $n(3n-2)$. Let C be the set of cups. For each $c \in C$, let k_c be the total number of commands involving c. Since each command involves two cups, the total number of commands is $\frac{1}{2}\sum_{c\in C} k_c$. So it suffices to show that $\sum_{c\in C} k_c \geqslant 2n(3n-2)$.

1 point.

For each $c \in C$, let x_c be the number of commands involving c before its first visit to an endpoint. Similarly, let y_c be the number of commands involving c after its last visit to an endpoint. Since c visited both endpoints, the number of commands between its first and last visit to an endpoint must be at least $2n-1$. Hence, $k_c \geqslant x_c + y_c + 2n - 1$.

2 points.

Now for each $1 \leqslant i \leqslant 2n$, let a_i be the cup at the i-th position from the left in the initial arrangement and similarly let b_i be the cup at the i-th position in the final arrangement. Then it follows that $x_{a_i}, y_{b_i} \geqslant \min(i-1, 2n-i)$ for all $1 \leqslant i \leqslant 2n$. Hence

$$\sum_{c\in C} x_c = \sum_{i=1}^{2n} x_{a_i} \geqslant \sum_{i=1}^{2n} \min(i-1, 2n-i) = n(n-1),$$

$$\sum_{c\in C} y_c = \sum_{i=1}^{2n} y_{b_i} \geqslant \sum_{i=1}^{2n} \min(i-1, 2n-i) = n(n-1),$$

$$\sum_{c\in C} k_c \geqslant \sum_{c\in C}(x_c + y_c + 2n - 1) \geqslant n(n-1) + n(n-1) + 2n(2n-1) = 2n(3n-2),$$

as desired. It remains to exhibit a valid sequence consisting of $n(3n-2)$ commands.

2 points.

Lemma 2. Consider n cups in a row. Then there is a sequence of $\frac{n(n-1)}{2}$ commands resulting in each cup visiting the first position (and similarly for the last position).

Proof. For each $1 \leqslant i \leqslant n$ in increasing order, for each $1 \leqslant j < i$ in decreasing order, swap the cups currently at the j-th and $(j+1)$-st positions. This clearly results in each cup visiting the first position and consist of $\sum_{i=1}^{n}(i-1) = \frac{n(n-1)}{2}$ commands, as desired (the case for the last position is analogous). $\square$

Corollary. For $2n$ cups in a row, there is a sequence of $n(n-1)$ steps resulting in each of the first n cups visiting the first position and each of the last n cups visiting the last position.

2 points.

Now first apply the algorithm from the corollary. Then for each $1 \leqslant i \leqslant n$ in decreasing order, for each $0 \leqslant j < n$ in increasing order, swap the cups currently at the $(i+j)$-th and $(i+j+1)$-st positions. Finally, apply the algorithm from the corollary again. It is easy to see that the performed sequence of commands is valid and it has length $n(n-1) + n^2 + n(n-1) = n(3n-2)$, as desired.

1 point.

Second Solution. The starting lemma and the proof of the upper bound are the same as in the first solution and are worth the same number of points. In this solution we present a different way to obtain the lower bound on the answer.

Let L be the set of cups that visit the first position before the last position and similarly let R be the set of cups that visit the last position before the first position. Then C is the disjoint union of L and R, in particular $|L|+|R| = 2n$.

1 point.

Now consider two cups a and b such that a is to the left of b at the beginning. Then note that a and b have to be swapped at least once since otherwise a wouldn't visit the last position (and b wouldn't visit the first position). Moreover, if a and b are swapped exactly once, then note that we must have $a \in L$, $b \in R$.

2 points.

Hence, it follows that the total number of swaps is at least

$$\binom{2n}{2} \cdot 2 - |L| \cdot |R| \geqslant 2n(2n-1) - n^2 = n(3n-2),$$

where we used the AM-GM inequality to obtain $|L| \cdot |R| \leqslant \left(\frac{|L|+|R|}{2}\right) = n^2$.

2 points.

Notes on marking:

- Points obtained for different proofs of the lower bound are not additive, a student should be awarded the maximum of points obtained for a single approach.
- If a student states that a sequence of commands being valid is equivalent to each cup visiting each position, it should be awarded **0 points**. The reason for this is that this characterisation of valid sequences is trivial and not directly useful, whereas both solutions extensively make use of the characterisation presented in Lemma 1.

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Senior Category

MLADI NADARENI MATEMATIČARI **Marin Getaldić**

Problems and Solutions

Problem 1. A partition of a positive integer is *even* if all its elements are even numbers. Similarly, a partition is *odd* if all its elements are odd. Determine all positive integers n such that the number of even partitions of n is equal to the number of odd partitions of n.

Remark: A *partition* of a positive integer n is a non-decreasing sequence of positive integers whose sum of elements equals n. For example, $(2,3,4)$, $(1,2,2,2,2)$ and (9) are partitions of 9.

(Ivan Novak)

Solution. Answer: $n \in \{2,4\}$.

We first notice that if n is a solution, n must be even, otherwise there are no even partitions of n, and (n) is an odd partition, so the number of odd partitions is greater then the number of even partitions.

1 point.

We now construct an injection f from the set of even partitions of n of cardinality k to odd partitions of n of cardinality $2k$.

If $p = (a_1, \ldots, a_k)$, where $2 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_k$ is an even partition, let

$$f(p) = (\underbrace{1, \ldots, 1}_{k \text{ times}}, a_1 - 1, \ldots, a_k - 1).$$

5 points.

Obviously, $f(p)$ is an odd partition of n. It is easy to see that f is injective because if $f(p) = f(q)$ then the largest k elements of $f(p)$ and $f(q)$ are equal, and then p and q must be equal.

2 points.

Number of odd partitions is equal to the number of even partitions if and only if f is surjective.

1 point.

It can be checked that for $n = 2$, $n = 4$, f is a bijection. Check (no points deducted if missing):

$\underline{n=2}$	$\underline{n=4}$
$(2) \to (1,1)$	$(4) \to (1,3)$
	$(2,2) \to (1,1,1,1)$

For $n > 4$, partition $(3, n-3)$ is not in the image of f, since every element of the image contains at least one number 1, so the number of even partition is equal to the the number of odd partitions if and only if $n \in \{2,4\}$.

1 point.

Notes on marking:

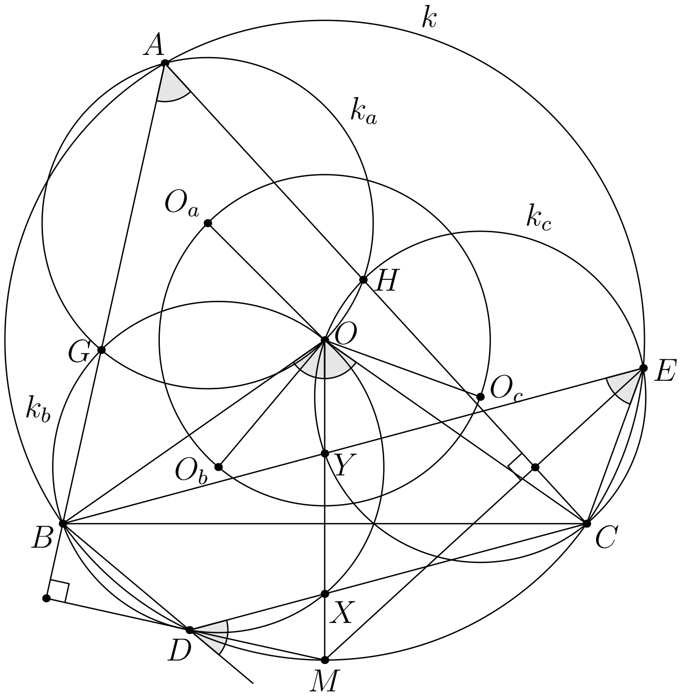
- Stating that $n = 2, 4$ are the only solutions on its own is worth **0 points**.
- Clearly attempting to construct an injection from the set of even partitions to the set of odd partitions without success is worth **1 point**.

Problem 2. Let ABC be a triangle with $|AB| < |AC|$. Let k be the circumcircle of $\triangle ABC$ and let O be the center of k. Point M is the midpoint of the arc $\widehat{BC}$ of k not containing A. Let D be the second intersection of the perpendicular line from M to AB with k and E be the second intersection of the perpendicular line from M to AC with k. Points X and Y are the intersections of CD and BE with OM respectively. Denote by k_b and k_c circumcircles of triangles BDX and CEY respectively. Let G and H be the second intersections of k_b and k_c with AB and AC respectively. Denote by k_a the circumcircle of triangle AGH.

Prove that O is the circumcenter of $\triangle O_aO_bO_c$, where O_a, O_b, O_c are the centers of k_a, k_b, k_c respectively.

(Petar Nizić-Nikolac)

First Sketch.



First Solution. We introduce standard angle notation, $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$.

As M is midpoint of arc $\widehat{BC}$, we know that $\angle MOB = \angle COM = \frac{\angle COB}{2} = \angle CAB = \alpha$, so

$$180^\circ - \angle BDX = 180^\circ - \angle BDC = \angle BAC = \angle BOM = \angle BOX$$

implying that $BDXO$ is a cyclic quadrilateral. Analogously we get that $CEOY$ is a cyclic quadrilateral.

2 points.

Another property of M being a midpoint of arc $\widehat{BC}$ is that $\angle CAM = \angle MAB = \frac{\alpha}{2}$, so

$$\angle DAB = 180^\circ - \angle ABD - \angle BDA = (\angle BDM - 90^\circ) - \angle BCA = (90^\circ - \angle MAB) - \gamma = \left(90^\circ - \frac{\alpha}{2}\right) - \gamma = \frac{\beta - \gamma}{2} \quad (1)$$

$$\angle EAC = 180^\circ - \angle CEA - \angle ACE = \angle ABC - (90^\circ - \angle CEM) = \beta - (90^\circ - \angle CAM) = \beta - \left(90^\circ - \frac{\alpha}{2}\right) = \frac{\beta - \gamma}{2} \quad (2)$$

Combining (1) and (2) we obtain that $|BD| = |EC|$.

2 points.

As B,C,D and E lie on circumcircle, $|BO| = |CO| = |DO| = |EO|$, thus $\triangle BOD \cong \triangle COD$. As k_b and k_c are circumcircles of triangles BOD and COE respectively, we conclude that $k_b \cong k_c$, thus $|OO_b| = |OO_c|$.

2 points.

Now see that

$$\angle AGO = \angle ODB = 90^\circ - \frac{\angle DOB}{2} = 90^\circ - \angle DAB \quad (3)$$

$$\angle OHA = 180^\circ - \angle OEC = 180^\circ - \left(90^\circ - \frac{\angle EOC}{2}\right) = 90^\circ + \angle EAC \quad (4)$$

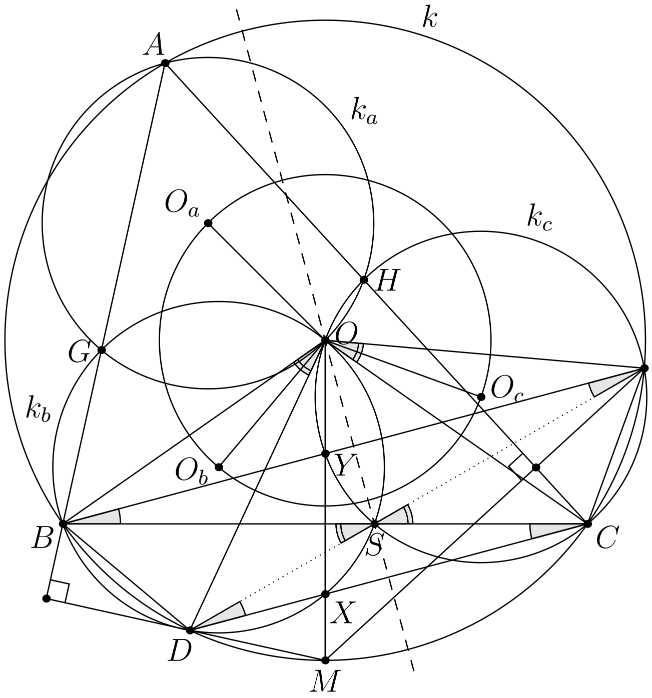
Combining (1), (2), (3) and (4) we obtain that $AGOH$ is a cyclic quadrilateral.

2 points.

Now as $|AO| = |BO|$ and $\angle AGO = \angle BDO$ we conclude that $k_a \cong k_b$, thus $|OO_a| = |OO_b| = |OO_c|$, so O is the circumcenter of $\triangle O_aO_bO_c$.

2 points.

Second Sketch.



Second Solution. We introduce standard angle notation, $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. As M is midpoint of arc $\widehat{BC}$, we know that $\angle CAM = \angle MAB = \frac{\alpha}{2}$, so

$$\angle DAB = 180^\circ - \angle ABD - \angle BDA = (\angle BDM - 90^\circ) - \angle BCA = (90^\circ - \angle MAB) - \gamma = \left(90^\circ - \frac{\alpha}{2}\right) - \gamma = \frac{\beta - \gamma}{2} \quad (1)$$

$$\angle EAC = 180^\circ - \angle CEA - \angle ACE = \angle ABC - (90^\circ - \angle CEM) = \beta - (90^\circ - \angle CAM) = \beta - \left(90^\circ - \frac{\alpha}{2}\right) = \frac{\beta - \gamma}{2} \quad (2)$$

Combining (1) and (2) we obtain that $|BD| = |EC|$, so $BDCE$ is an isoscales trapezoid.

2 points.

Let S be the intersection of diagonals of $BDCE$. Then using (1) and (2) we have

$$\angle DSB = \angle SBE + \angle SDC = 2\angle EAC = 2\angle DAB = \angle DO_bD$$

so S lies on k_b. Analogously we get that S lies on k_c as well.

2 points.

Let O' be the second intersection of k_b and k_c. Then

$$\angle EO'B = \angle EO'S + \angle SO'B = 360^\circ - \angle SCE - \angle BDS = 2(180^\circ - \angle SCE) = 2(\angle EAB) = \angle EOB$$

and as k_b is symmetric to k_c over OS (perpendicular bisector of $\overline{BE}$ and $\overline{CE}$), we conclude that O and O' lie on that line so $O \equiv O'$, and we conclude that O is the second intersection of k_b and k_c.

2 points.

As k_b is symmetric to k_c over OS, we conclude that $|OO_b| = |OO_c|$.

1 point.

As $k_a = (AGH)$, $k_b = (BSOG)$ and $k_c = (CEOS)$, due to Miquel's theorem we have that O lies on k_a.

1 point.

Now as $|AO| = |BO|$ and $\angle AGO = \angle BDO$ we conclude that $k_a \cong k_b$, thus $|OO_a| = |OO_b| = |OO_c|$, so O is the circumcenter of $\triangle O_aO_bO_c$.

2 points.

Notes on marking:

- If a student has a partial solution with analytic methods, only points for proving facts that can be expressed in geometric ways and lead to a compete solution can be awarded.

Problem 3. For which real numbers $k > 1$ does there exist a bounded set of positive real numbers S with at least 3 elements such that

$$k(a-b) \in S$$

for all $a, b \in S$ with $a > b$?

Remark: A set of positive real numbers S is *bounded* if there exists a positive real number M such that $x < M$ for all $x \in S$.

(Petar Nizić-Nikolac)

First Solution. Set of solutions:

$$k \in \left\{ \frac{1+\sqrt{5}}{2}, 2 \right\}$$

Verification:

- If $k = \phi = \frac{1+\sqrt{5}}{2}$ we can choose set $\{\phi, 1+\phi, 1+2\phi\}$. It works as $\phi(1+\phi-\phi) = \phi$, $\phi(1+2\phi-1-\phi) = \phi^2 = 1+\phi$ and $\phi(1+2\phi-\phi) = \phi+\phi^2 = 1+2\phi$ (all these properties are true as ϕ is a root of the quadratic $x^2-x-1=0$).
- If $k=2$ we can choose set $\{2,3,4\}$. It works as $2(3-2)=2$, $2(4-3)=2$ and $2(4-2)=4$.

1 point.

Now we prove that these are the only possible values of k. Suppose $k > 1$ such that all required properties are satisfied.

Lemma 1. $k(a-b) \leqslant a$ for all $a, b \in S$ with $a > b$

Proof. Assume the opposite, that there exist $a, b \in S$ with $a > b$ such that $k(a-b) > a$. Fix b and denote $f(x) = k(x-b)$. We have $f(a) > a$. Consider these two conclusion for some x such that $f(x) > x$:

$$f(x) > x \implies k(x-b) - b > x - b \implies k(k(x-b)-b) - kb > k(x-b) - kb \implies f(f(x)) > f(x) \quad (1)$$

1 point.

$$f(x) > x \implies (k-1)f(x) > (k-1)x \implies k(f(x)-b) - k(x-b) > f(x) - x \implies f(f(x)) - f(x) > f(x) - x \quad (2)$$

1 point.

By (1) we have that $f^n(a) > f^{n-1}(a) > \ldots > f(a) > a > b$ so $f^n(a) \in S, \forall n \in \mathbb{N}$. On the other hand, by (2) we have $f^k(a) - f^{k-1}(a) \geqslant f(a) - a$ for all natural k. Summing up for k from 1 to n, we obtain

$$f^n(a) - a = \sum_{k=1}^{n} (f^k(a) - f^{k-1}(a)) \geqslant n(f(a)-a)$$

However, this means that $f^n(a) \in S$ is unbounded as n grows, which is impossible. Hence, the lemma is proved. □

1 point.

Lemma 2. S has a minimum and it is greater than 0

Proof. Now, denote $m = \inf S$. Let's first settle the case $m = 0$. However, then by fixing a and taking b small enough such that $k(a-b) > a$ we contradict the lemma. Therefore, we have $m > 0$.

1 point.

Without loss of generality we can take that $m = 1$ as we can scale the whole set. Assume that $1 \notin S$, and then there exists an infinite sequence of elements of S tending to 1, i.e., for every $a \in S$ there exists $b \in S$ with $1 < b < a$. Therefore,

$$k(a-b) > 1 \implies a > b + \frac{1}{k} \implies a > 1 + \frac{1}{k}$$

However, then every a in S is larger than $1 + \frac{1}{k}$ so $1 = \inf S \geqslant 1 + \frac{1}{k} > 1$, which is a contradiction. Hence $\min S = 1$. □

1 point.

Lemma 3. For some $x \in S$, if $x > G_{n-1}$ then $x \geqslant G_n$ for all $n \in \mathbb{N}$, where $G_n = 1 + \frac{1}{k} + \ldots + \frac{1}{k^n}$

Proof. We prove by induction on n. Basis for $n = 0$ is true as

$$k(x-1) \geqslant \min S = 1 \implies x \geqslant 1 + \frac{1}{k}$$

Now we proceed with the inductive step. Take $x > G_n$. This implies that

$$k(x-1) > k(G_n - 1) = G_{n-1}$$

Obviously, $k(x-1) \in S$. However, by the induction hypothesis, it follows that $k(x-1) \geqslant G_n$ which rearranges into

$$x \geqslant \frac{1}{k}(G_n + k) = G_{n+1}$$

so the lemma is proved by mathematical induction. □

1 point.

Let $T = \{G_0, G_1, G_2, \ldots\}$. Assume that exists some $a \in S \setminus T$. Then using Lemma 3 we get that

$$a > G_n \text{ and } a \notin T \implies a \geqslant G_{n+1} \text{ and } a \notin T \implies a > G_{n+1}$$

and as $a \neq G_0 = 1 = \min S$, then $a \geqslant \sup T = \frac{k}{k-1}$.

1 point.

However, $a \leqslant \frac{k}{k-1}$ holds as a consequence of Lemma 2, so the only member of $S \setminus T$ is $\frac{k}{k-1}$. Therefore,

$$S \subseteq \left\{ \frac{k}{k-1}, G_0, G_1, G_2, \ldots \right\}$$

1 point.

However, if for some $n > 1$, $G_n \in S$, then $G_{n-1} = k(G_n - 1) \in S$, so we have that

$$k(G_n - G_{n-1}) = \frac{1}{k^{n-1}} \in S$$

which is impossible due to $k > 1$, so we in fact have

$$S \subseteq \left\{ 1, \frac{k+1}{k}, \frac{k}{k-1} \right\}$$

and due to $|S| \geqslant 3$ all three numbers must belong to the set (easy to see that they are distinct). However, then

$$k\left(\frac{k}{k-1} - \frac{k+1}{k}\right) = \frac{1}{k-1} \in \left\{1, \frac{k+1}{k}, \frac{k}{k-1}\right\}$$

which gives $k \in \left\{\frac{1+\sqrt{5}}{2}, 2\right\}$, both of which satisfy the condition by verification.

1 point.

Second Solution. Verification is the same and also worth **1 point**. For a set $A \subseteq \mathbb{R}^+$, we will write $\triangle A = \{a - b \mid a, b \in A, a > b\}$. Suppose $k > 1$ is such that there exists a set S with the required properties.

Lemma 1. If $d \in \triangle S$ is not a maximal element, then $kd \in \triangle S$.

Proof. Let $a, b \in S$ be such that $a - b = d > 0$. Since d is not maximal in $\triangle S$, either a is not maximal in S or b is not minimal in S. If the former is true, then $\exists c \in S$ with $c > a$, hence $k(c-a), k(c-b) \in S$. But then $k(c-b) - k(c-a) = k(a-b) = kd \in \triangle S$, as desired. Otherwise, $\exists c \in S$ with $c < b$, so $k(b-c), k(a-c) \in S$, hence $k(a-c) - k(b-c) = k(a-b) = kd$, so we are done. □

2 points.

Lemma 2. $\triangle S$ is a finite geometric progression with common ratio k. In particular, S is finite.

Proof. First note that $\triangle S$ must have a maximal element M. Indeed, otherwise we could take $d \in \triangle S$ and inductively obtain $k^n d \in \triangle S$ for all $n \in \mathbb{N}$, which is absurd since $\triangle S$ is bounded as S is bounded.

1 point.

Now for any $d \in \triangle S$, take the maximal $n \in \mathbb{N}_0$ such that $k^n d \leqslant M$. Then it follows inductively that $k^i d \in \triangle S$ for $0 \leqslant i \leqslant n$. By maximality of n, $k^{n+1} d > M$, so we must have $k^n d = M$ (otherwise we would have $k^{n+1} d \in \triangle S$ by the Lemma 1). It follows that $d = \frac{M}{k^n}$ and also $\frac{M}{k^i} \in \triangle S$ for all $0 \leqslant i < n$. Hence, $\triangle S$ is a (possibly infinite) geometric progression with common ratio $\frac{1}{k}$.

2 points.

Suppose that $\triangle S$ is infinite. Then S contains an infinite geometric progression with ratio $\frac{1}{k}$. Then for any $a, b \in S$ with $a > b$, one can choose c in this progression with $c < b$, so that $a - c > a - b$. This contradicts the fact that $\triangle S$ has a maximal element, so $\triangle S$ must be finite. □

1 point.

Now by scaling WLOG assume that $\triangle S = \{1, k, \ldots, k^{m-1}\}$ for some $m \in \mathbb{N}$. Then $\{k, k^2, \ldots, k^m\} \subseteq S$, hence $\triangle\{k, k^2, \ldots, k^m\} \subseteq \triangle S$. But note that $k^{i+1} - k^i < k^{i+2} - k^{i+1}$ for all $1 \leqslant i < m-1$ and $k^m - k^i > k^m - k^{i+1}$ for all $1 \leqslant i < m-1$, so it follows that $|\triangle\{k, k^2, \ldots, k^m\}| \geqslant 2m - 3$. Hence, $2m - 3 \leqslant m$, i.e. $m \leqslant 3$.

1 point.

Now $m \geqslant |S| - 1$, so $|S| \leqslant 4$. If $|S| = 4$, then $m = 3$ and it can easily be checked that S is an arithmetic progression, say with difference $d > 0$. But then $\triangle S = \{d, 2d, 3d\}$, which is not a geometric progression. Hence, $|S| = 3$.

1 point.

Now we can write $S = \{a, b, c\}$, with $a < b < c$. As $k(b-a), k(c-b) < k(c-a)$ and $k\triangle S \subseteq \{a, b, c\}$, five cases arise:

- If $k(b-a) = a$, $k(c-b) = a$ and $k(c-a) = b$. Then $\frac{k+1}{k}a = b = k(c-a) = k(c-b) + k(b-a) = 2a$, so $k = 1$. ✗
- If $k(b-a) = a$, $k(c-b) = a$ and $k(c-a) = c$. Then $\frac{k}{k-1}a = c = k(c-a) = k(c-b) + k(b-a) = 2a$, so $k = 2$. ✓
- If $k(b-a) = a$, $k(c-b) = b$ and $k(c-a) = c$. Then $\frac{k+1}{k}a = b = \frac{k}{k+1}c = \frac{k^2}{(k+1)(k-1)}a$, so $k = \frac{1+\sqrt{5}}{2}$ or $\frac{1-\sqrt{5}}{2}$. ✓ or ✗
- If $k(b-a) = b$, $k(c-b) = a$ and $k(c-a) = c$. Then $b = \frac{k}{k-1}a = c$, which is impossible. ✗
- If $k(b-a) = b$, $k(c-b) = b$ and $k(c-a) = c$. Then $\frac{k+1}{k}b = c = k(c-a) = k(c-b) + k(b-a) = 2b$, so $k = 1$. ✗

1 point.

Third Solution. Verification is the same and also worth **1 point**. We use the same notation as in the **Second Solution.**

Lemma 1. S is finite.

Proof. Let $m = \inf S$, $M = \sup S$ (these exist since S is bounded both below and above as a subset of $\mathbb{R}$). Then note that $\sup \triangle S = M - m$. This holds since for any $a, b \in S$ we have $a - b \leqslant M - m$ and moreover given any $\varepsilon > 0$, there exist $a, b \in S$ such that $a > M - \frac{\varepsilon}{2}$, $b < m + \frac{\varepsilon}{2}$, so that $a - b > M - m - \varepsilon$.

1 point.

Since $k\triangle S \subseteq S$, we have $\sup(k\triangle S) \leqslant M$, i.e. $\sup \triangle S \leqslant \frac{M}{k}$, $M - m \leqslant \frac{M}{k}$, $m \geqslant \frac{k-1}{k}M$.

Again since $k\triangle S \subseteq S$, we have $\inf(k\triangle S) \geqslant m$, i.e. $\inf \triangle S \geqslant \frac{m}{k} \geqslant \frac{k-1}{k^2}M$.

1 point.

So if $a_1, a_2, \ldots, a_n$ are some elements of S with $m \leqslant a_1 < a_2 < \ldots < a_n \leqslant M$, we have $a_{i+1} - a_i \geqslant \frac{k-1}{k^2}M$ for all $1 \leqslant i < n$, so we get

$$\frac{M}{k} \geqslant M - m \geqslant a_n - a_1 = \sum_{i=1}^{n-1} a_{i+1} - a_i \geqslant (n-1) \cdot \frac{k-1}{k^2}M,$$

hence $n \leqslant \frac{2k-1}{k-1}$. In particular, S is finite. □

1 point.

Lemma 2. $|S| = 3$.

Proof. Let $a_1 < a_2 < \ldots < a_n$ be the elements of S, and assume for the sake of contradiction that $|S| \geqslant 4$.

We know $k(a_n - a_1) > k(a_{n-1} - a_1) > \ldots > k(a_2 - a_1)$ are elements of S, and there are at least $n - 2$ elements of S greater than $k(a_2 - a_1)$. This implies $k(a_2 - a_1) \in \{a_1, a_2\}$. Using a similar argument, $k(a_3 - a_1) \in \{a_2, a_3\}$, $k(a_3 - a_2) \in \{a_1, a_2\}$ and $k(a_4 - a_1) \in \{a_3, a_4\}$.

2 points.

If $k(a_2 - a_1) = a_2$, then $k(a_3 - a_1) = a_3$, so $a_2 = a_1\frac{k}{k-1} = a_3$, which is impossible, therefore $k(a_2 - a_1) = a_1$ which implies that $a_2 = a_1(1 + \frac{1}{k})$.

1 point.

If $k(a_3 - a_1) = a_2$, then $a_3 = a_1 + \frac{a_2}{k} = a_1(1 + \frac{1}{k} + \frac{1}{k^2})$, so $k(a_3 - a_2) = ka_1(1 + \frac{1}{k} + \frac{1}{k^2} - 1 - \frac{1}{k}) = \frac{a_1}{k} < a_1$, which is impossible. Therefore, $k(a_3 - a_1) = a_3$ which implies that $a_3 = a_1\frac{k}{k-1}$.

1 point.

Now, because $k(a_4 - a_1) > k(a_3 - a_1) = a_3$, we know that $k(a_4 - a_1) = a_4$ as there are $n - 4$ differences greater than this, but this implies $a_4 = a_1\frac{k}{k-1} = a_3$, a contradiction. Therefore, $|S| = 3$. □

1 point.

Similar finish as in the **Second Solution** which is also worth **1 point.**

Alternative proof of Lemma 2.

Fact. Let $A = \{a_1, a_2, \dots, a_n\}$ with $a_1 < a_2 < \dots < a_n$ and $n \geqslant 3$ be a finite set of real numbers such that $|\triangle A| \leqslant |A|$. Then either

- there exist $j \in \{1, \dots, n-1\}$ and $0 < d \leqslant a_{j+1} - a_j$ such that $a_{i+1} - a_i = d$ for all $1 \leqslant i < n$ with $i \neq j$ or
- $a_2 - a_1 = a_n - a_{n-1}$ and there exists $0 < d < a_2 - a_1$ such that $a_{i+1} - a_i = d$ for all $1 < i < n-1$.

Proof. Take $j \in \{1, \dots, n-1\}$ that maximizes $a_{j+1} - a_j$. Suppose first that j can be taken so that $1 < j < n-1$. If $a_{t+1} - a_t = a_{j+1} - a_j$ for all $1 \leqslant t < n$, then we are done, so suppose $\exists t \in \{1, \dots, n-1\}$ such that $a_{t+1} - a_t < a_{j+1} - a_j$.

Now call a sequence of pairs of indices $(l_1, r_1), (l_2, r_2), \dots, (l_{n-1}, r_{n-1})$ a *path* if $(l_1, r_1) = (j, j+1)$ and $(l_{i+1}, r_{i+1}) \in \{(l_i, r_i + 1), (l_i - 1, r_i)\}$ for all $1 \leqslant i < n-1$. Define the *signature* of a path to be the sequence $(a_{r_i} - a_{l_i})_{1 \leqslant i \leqslant n-1}$.

We claim that any two paths have the same signature. Indeed, note that for any path, $a_{t+1} - a_t, a_{r_1} - a_{l_1}, a_{r_2} - a_{l_2}, \dots, a_{r_{n-1}} - a_{l_{n-1}}$ is a strictly increasing sequence of n elements of $\triangle A$, so the elements of the signature are fixed since $|\triangle A| \leqslant n$.

Now given any $p < j, q > j$, we can choose two paths (l_i, r_i) and (l'_i, r'_i) such that $(l_{q-p}, r_{q-p}) = (p, q)$ and $(l'_{q-p}, r'_{q-p}) = (p+1, q+1)$. By the previous observation, it follows that $a_q - a_p = a_{q+1} - a_{p+1}$, i.e. $a_{p+1} - a_p = a_{q+1} - a_q$. Since $1 < j < n-1$, it follows that $a_{q+1} - a_q = a_2 - a_1$ for all $q > j$ and also $a_{p+1} - a_p = a_n - a_{n-1}$ for all $p < j$. Since $a_2 - a_1 = a_n - a_{n-1}$, we have $a_{i+1} - a_i = a_2 - a_1$ for all $i \neq j$, as desired.

It remains to deal with the case when $a_{i+1} - a_i < a_{j+1} - a_j$ for $1 < i < n-1$. Note that $|\{a_{i+1} - a_i \mid 1 \leqslant i < n\}| \leqslant 2$ since otherwise we could choose $1 \leqslant s, t < n$ and a path (l_i, r_i) such that $a_{s+1} - a_s, a_{t+1} - a_t, a_{r_1} - a_{l_1}, a_{r_2} - a_{l_2}, \dots, a_{r_{n-1}} - a_{l_{n-1}}$ is a strictly increasing sequence of $n+1$ elements of $\triangle A$, which is absurd. The claim now follows. $\square$

3 points.

Now we proceed by proving $|S| = 3$. Suppose for the sake of contradiction that $|S| \geqslant 4$. Enumerate S as $x_1 < x_2 < \dots < x_n$, where $n \geqslant 4$, Since S satisfies the hypothesis of the lemma, we may consider the following cases:

Case 1. (x_i) is an arithmetic sequence
Let d be the difference of (x_i). Then the enumeration of $k\triangle S$ is an arithmetic subsequence of (x_i) of length $n-1$, with difference kd. Since $n \geqslant 4$, it is either $x_1, \dots, x_{n-1}$ or $x_2, \dots, x_n$, so it must have difference d, contradiction.

Case 2. $\exists a, b > 0, j \in \{1, \dots, n-1\}$ such that $a < b, x_{j+1} - x_j = b$ and $x_{i+1} - x_i = a$ for $1 \leqslant i < n, i \neq j$
Then $k\triangle S = \{ka, kb, k(b+a), \dots, k(b + (n-2)a)\}$, where $ka < kb < k(b+a) < \dots < k(b + (n-2)a)$. Hence, $x_2 - x_1 = k(b-a)$ and $x_{i+1} - x_i = ka$ for $1 < i < n$. It follows that $j = 1$, $k(b-a) = b$ and $ka = a$, which is absurd since $k > 1$.

Case 3. $\exists a, b > 0$ such that $a < b, x_2 - x_1 = x_n - x_{n-1} = b$ and $x_{i+1} - x_i = a$ for $1 < i < n-1$
Then $k\triangle S = \{ka, kb, k(b+a), \dots, k(b+(n-3)a), k(2b+(n-3)a)\}$, where $ka < kb < \dots < k(b+(n-3)a) < k(2b+(n-3)a)$. Hence, $x_n - x_{n-1} = kb$, which is absurd since $k > 1$. $\square$

2 points.

Notes on marking:

- A student cannot be awarded with points from two different solutions.
- In all solutions, if a student states that verification "is trivial" it should be awarded **0 points.** However, it is enough to give examples of sets for two possible values of k and then the student should be awarded **1 point.** This point can be awarded even if student hasn't solved the problem completely.
- In **First Solution**, if a student writes explicitly that $a \leqslant \frac{k}{k-1}$ without showing that $S \subseteq \left\{\frac{k}{k-1}, G_0, G_1, G_2, \dots\right\}$ it should also be awarded **1 point.**
- In **Second Solution**, if a student states that deduction from $|S| = 3$ to $k = \frac{1+\sqrt{5}}{2}$ or 2 "is trivial" it should be awarded **0 points.**
- In **Third Solution**, if a student states that $|\triangle A| = |A| - 1$ iff A is an aritmetic sequence, it should be awarded **1 point.** However, if a student states just that $|\triangle A| \geqslant |A| - 1$ for all sequences, it should be awarded **0 points.**
- In **Alternative proof of Lemma 2**, if a student states correctly the whole class of sequences satisfying $|\triangle A| = |A|$, it should be awarded **1 point**.
- If student's solution is true with fact that S is finite, it should be awarded at most **7 points.**
- If student proves that $|S| \leqslant c$ for some $c \in \mathbb{N}$ independent of k, it should be awarded **5 points**) (**1 point** for verification is not included and can also be awarded separetly).

Problem 4. Let x, y, m, n be integers greater than 1 such that

$$\underbrace{x^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}}_{m \text{ times}} = \underbrace{y^{y^{y^{\cdot^{\cdot^{\cdot^{y}}}}}}}_{n \text{ times}}.$$

Does it follow that $m = n$?

Remark: This is a tetration operation, so we can also write ${}^m x = {}^n y$ for the initial condition.

(Petar Nizić-Nikolac)

Solution. Yes, it does. Assume for the sake of contradiction that $x < y$. Then $m > n$. Define function f recursively

$$f(r) = \begin{cases} f(\log_x(r)) + 1 & \text{if } \log_x(r) \in \mathbb{N} \\ 0 & \text{else} \end{cases}$$

for example, if $x = 2$, then $f(256) = f\left(2^{2^3}\right) = 2$. Essentially it is the least possible height of an exponent different from x.

Lemma 1. $f(y) \geqslant 1$.

Proof. Let p be a prime number such that $p \mid x$ (it exists as $x > 1$). Then $p \mid y$, so write $x = p^a \cdot x'$ and $y = p^b \cdot y'$, where $p \nmid x', y'$. Let $a' = \frac{a}{(a,b)}$ and $b' = \frac{b}{(a,b)}$. Let $v_p(r)$ denote the largest integer such that $p^{v_p(r)} \mid r$. Then

$$v_p\left({}^m x\right) = v_p\left({}^n y\right) \implies v_p\left(\left(p^a\right)^{{}^{m-1}x}\right) = v_p\left(\left(p^b\right)^{{}^{n-1}y}\right) \implies a \cdot {}^{m-1}x = b \cdot {}^{n-1}y \implies x^{a \cdot {}^{m-1}x} = x^{b \cdot {}^{n-1}y} \implies$$

$$\implies \left({}^m x\right)^a = x^{b \cdot {}^{n-1}y} \implies \left({}^n y\right)^a = x^{b \cdot {}^{n-1}y} \implies y^{a \cdot {}^{n-1}y} = x^{b \cdot {}^{n-1}y} \implies y^a = x^b \implies y^{a'} = x^{b'}$$

so there exists z such that $x = z^{a'}$ and $y = z^{b'}$.

2 points.

As $1 \leqslant a' < b'$ and $(a', b') = 1$, then

$$a \cdot {}^{m-1}x = b \cdot {}^{n-1}y \implies \frac{b'}{a'} = \frac{b}{a} = \frac{{}^{n-1}y}{{}^{m-1}x} = \frac{\left(z^{b'}\right)^{{}^{n-2}y}}{\left(z^{a'}\right)^{{}^{m-2}x}} = z^{b' \cdot {}^{n-2}y - a' \cdot {}^{m-2}x} \implies a' \mid b' \implies a' = 1 \implies y = x^{b'}$$

so we conclude that $f(y) \geqslant 1$. $\square$

1 point.

Lemma 2. $f\left({}^n y\right) \leqslant 2$.

Proof. We have two cases depending on $f(y)$.

Case 1. $f(y) = 1$

Write $y = x^k$ where $f(k) = 0$. Then

$$f\left({}^n y\right) = f\left(\left(x^k\right)^{{}^{n-1}y}\right) = f\left(x^{k \cdot {}^{n-1}y}\right) = f\left(k \cdot {}^{n-1}y\right) + 1 = f\left(k \cdot x^{k \cdot {}^{n-2}y}\right) + 1 = 1$$

as if $k \cdot x^{k \cdot {}^{n-2}y} = x^l \implies f(k) = 1$ or $k = 1$ which is impossible, so $f\left(k \cdot x^{k \cdot {}^{n-2}y}\right) = 0$.

3 points.

Case 2. $f(y) \geqslant 2$

Write $y = x^{x^k}$. Then

$$f\left({}^n y\right) = f\left(\left(x^{x^k}\right)^{{}^{n-1}y}\right) = f\left(x^{x^k \cdot {}^{n-1}y}\right) = f\left(x^k \cdot {}^{n-1}y\right) + 1 = f\left(x^k \cdot x^{x^k \cdot {}^{n-2}y}\right) + 1 = f\left(k + x^k \cdot {}^{n-2}y\right) + 2 = 2$$

as if $k + x^k \cdot {}^{n-2}y = x^l \implies x^k \mid k$ which is impossible, so $f\left(k + x^k \cdot {}^{n-2}y\right) = 0$. $\square$

3 points.

Using this conlusion we have that

$$2 \leqslant m = f\left({}^m x\right) = f(LHS) = f(RHS) = f\left({}^n y\right) \leqslant 2 \implies m = 2 \implies x^x < y^y = {}^2 y \leqslant {}^n y = x^x$$

which is impossible, so we conclude $m = n$.

1 point.