Sixth South Eastern European Mathematical Olympiad for University Students

Blagoevgrad, Bulgaria

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Problem 1. Let $A = (a_{ij})$ be the $n \times n$ matrix, where a_{ij} is the remainder of the division of $i^j + j^i$ by 3 for $i, j = 1, 2, \ldots, n$. Find the greatest n for which $\det A \neq 0$.

Solution. We show that $a_{i+6,j} = a_{ij}$ for all $i, j = 1, 2, \ldots, n$. First note that if $j \equiv 0 \pmod 3$ then $j^i \equiv 0 \pmod 3$, and if $j \equiv 1$ or $2 \pmod 3$ then $j^6 \equiv 1 \pmod 3$. Hence, $j^i(j^6 - 1) \equiv 0 \pmod 3$ for $j = 1, 2, \ldots, n$, and

$$a_{i+6,j} \equiv (i+6)^j + j^{i+6} \equiv i^j + j^i \equiv a_{ij} \pmod 3,$$

or $a_{i+6,j} = a_{ij}$. Consequently, $\det A = 0$ for $n \geq 7$. By straightforward calculation, we see that $\det A = 0$ for $n = 6$ but $\det A \neq 0$ for $n = 5$, so the answer is $n = 5$.

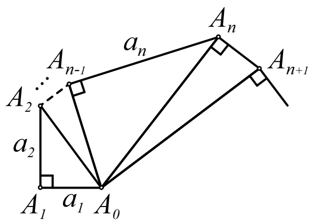
Grading of Problem 1.

5p: Concluding that $\Delta_n = 0$ for each $n \geq 7$

5p: Computing $\Delta_5 = 12,\ \Delta_6 = 0$

2p: Computing $\Delta_3 = -10,\ \Delta_4 = 4$ (in case none of the above is done)

Problem 2. Let $a_n > 0$, $n \geq 1$. Consider the right triangles $\triangle A_0A_1A_2$, $\triangle A_0A_2A_3, \ldots$, $\triangle A_0A_{n-1}A_n, \ldots$, as in the figure. (More precisely, for every $n \geq 2$ the hypotenuse A_0A_n of $\triangle A_0A_{n-1}A_n$ is a leg of $\triangle A_0A_nA_{n+1}$ with right angle $\angle A_0A_nA_{n+1}$, and the vertices A_{n-1} and A_{n+1} lie on the opposite sides of the straight line A_0A_n; also, $|A_{n-1}A_n| = a_n$ for every $n \geq 1$.)



Is it possible for the set of points $\{A_n \mid n \geq 0\}$ to be unbounded but the series $\sum\limits_{n=2}^{\infty} m(\angle A_{n-1}A_0A_n)$ to be convergent? Here $m(\angle ABC)$ denotes the measure of $\angle ABC$.

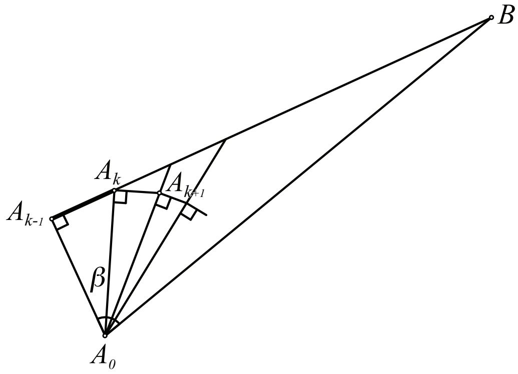
Note. A subset B of the plane is bounded if there is a disk D such that $B \subseteq D$.

Solution. We have $|A_0A_n| = \sqrt{\sum\limits_{i=1}^{n} a_i^2}$ and $\sum\limits_{n=2}^{k} m(\angle A_{n-1}A_0A_n) = \sum\limits_{n=2}^{k} \arctan \dfrac{a_n}{\sqrt{a_1^2 + \cdots + a_{n-1}^2}}$.
The set of points $\{A_n \mid n \geq 0\}$ will be unbounded if and only if the sequence of the lengths of the segments A_0A_n is unbounded. Put $a_i^2 = b_i$. Then the question can be reformulated as follows: Is it possible for a series with positive terms to be such that $\sum_{i=1}^{\infty} b_i = \infty$ and

$$\sum_{n=2}^{\infty} \arctan \sqrt{\frac{b_n}{b_1 + \cdots + b_{n-1}}} < \infty.$$

Denote $s_n = \sum_{i=1}^{n} b_i$. Since $\arctan x \sim x$ as $x \to 0$, the question we need to ask is whether one can have $s_n \to \infty$ as $n \to \infty$ and $\sum\limits_{n=2}^{\infty} \sqrt{\frac{s_n - s_{n-1}}{s_{n-1}}} < \infty$. Put $\sqrt{\frac{s_n - s_{n-1}}{s_{n-1}}} = u_n > 0$. Then $\frac{s_n}{s_{n-1}} = 1 + u_n^2$, $\ln s_n - \ln s_{n-1} = \ln(1+u_n^2)$, $\ln s_k = \ln s_1 + \sum_{n=2}^{k} \ln(1+u_n^2)$. Finally, the question is whether it is possible to have $\sum_{n=2}^{\infty} \ln(1+u_n^2) = \infty$ and $\sum_{n=2}^{\infty} u_n < \infty$. The answer is negative, since $\ln(1+x) \sim x$ as $x \to 0$ and $u_n^2 \le u_n \le 1$ for large enough n.

Different solution. Since $\sum_{n=2}^{\infty} m(\angle A_{n-1}A_0A_n) < \infty$, there exists some large enough k for which $\sum_{n=k}^{\infty} m(\angle A_{n-1}A_0A_n) \le \beta < \frac{\pi}{2}$. Then all the vertices A_n, $n \ge k-1$, lie inside the triangle $\triangle A_0A_{k-1}B$, where the side $A_{k-1}B$ of $\triangle A_0A_{k-1}B$ is a continuation of the side $A_{k-1}A_k$ of $\triangle A_0A_{k-1}A_k$ and $\angle A_{k-1}A_0B = \beta$. Consequently, the set $\{A_n \,|\, n \ge 0\}$ is bounded which is a contradiction.



Grading of Problem 2.

1p: Noting that $\{A_n \,|\, n \ge 0\}$ is unbounded $\Leftrightarrow$ $|A_0A_n|$ is unbounded **OR** expressing $|A_0A_n|$

1p: Observing that $\sum_{n=2}^{\infty} m(\angle A_{n-1}A_0A_n)$ is convergent $\Leftrightarrow$ A_0A_n tends to A_0B **OR** expressing the angles by arctan

8p: Proving the assertion

Problem 3.

a) Prove that if k is an even positive integer and A is a real symmetric $n \times n$ matrix such that $(\mathrm{Tr}(A^k))^{k+1} = (\mathrm{Tr}(A^{k+1}))^k$, then

$$A^n = \mathrm{Tr}(A)\, A^{n-1}.$$

b) Does the assertion from *a)* also hold for odd positive integers k?

Solution. *a)* Let $k = 2l$, $l \ge 1$. Since A is a symmetric matrix all its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real numbers. We have,

$$\mathrm{Tr}(A^{2l}) = \lambda_1^{2l} + \lambda_2^{2l} + \cdots + \lambda_n^{2l} = a \tag{1}$$

and

$$\mathrm{Tr}(A^{2l+1}) = \lambda_1^{2l+1} + \lambda_2^{2l+1} + \cdots + \lambda_n^{2l+1} = b. \tag{2}$$

By (1) we get that $a \ge 0$, so there is some $a_1 \ge 0$ such that $a = a_1^{2l}$. On the other hand, the equality $a^{2l+1} = b^{2l}$ implies that $(a_1^{2l+1})^{2l} = b^{2l}$ and hence

$$b = \pm a_1^{2l+1} = (\pm a_1)^{2l+1} \quad \text{and} \quad a = a_1^{2l} = (\pm a_1)^{2l}.$$

Then equalities (1) and (2) become

$$\lambda_1^{2l} + \lambda_2^{2l} + \cdots + \lambda_n^{2l} = c^{2l} \tag{3}$$

and

$$\lambda_1^{2l+1} + \lambda_2^{2l+1} + \cdots + \lambda_n^{2l+1} = c^{2l+1}, \tag{4}$$

where $c = \pm a_1$. We consider the following cases.

Case 1. If $c = 0$ then $\lambda_1 = \cdots = \lambda_n = 0$, so $\mathrm{Tr}(A) = 0$ and we note that the characteristic polynomial of A is $f_A(x) = x^n$. We have, based on the Cayley-Hamilton Theorem, that

$$A^n = 0 = \mathrm{Tr}(A)\, A^{n-1}.$$

Case 2. If $c \neq 0$ then let $x_i = \lambda_i / c$, $i = 1, 2, \ldots, n$. In this case equalities (3) and (4) become

$$x_1^{2l} + x_2^{2l} + \cdots + x_n^{2l} = 1 \tag{5}$$

and

$$x_1^{2l+1} + x_2^{2l+1} + \cdots + x_n^{2l+1} = 1. \tag{6}$$

The equality (5) implies that $|x_i| \leq 1$ for all $i = 1, 2, \ldots, n$. We have $x^{2l} \geq x^{2l+1}$ for $|x| \leq 1$ with equality reached when $x = 0$ or $x = 1$. Then, by (5), (6), and the previous observation, we find without loss of generality that $x_1 = 1$, $x_2 = x_3 = \cdots = x_n = 0$. Hence $\lambda_1 = c$, $\lambda_2 = \cdots = \lambda_n = 0$, and this implies that $f_A(x) = x^{n-1}(x - c)$ and $\mathrm{Tr}(A) = c$. It follows, based on the Cayley-Hamilton Theorem, that

$$f_A(A) = A^{n-1}(A - cI_n) = 0 \quad \Leftrightarrow \quad A^n = \mathrm{Tr}(A)\, A^{n-1}.$$

b) The answer to the question is negative. We give the following counterexample:

$$k = 1, \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

Grading of Problem 3.

3p: Reformulating the problem through eigenvalues:

$$\left(\sum \lambda_i^{2l}\right)^{2l+1} = \left(\sum \lambda_i^{2l+1}\right)^{2l} \Rightarrow \forall i:\ \lambda_i^n = (\lambda_1 + \cdots + \lambda_n)\lambda_i^{n-1}$$

4p: Only $(\lambda_i) = (0, \ldots, 0, c, 0, \ldots, 0)$ or $(0, \ldots, 0)$ are possible

3p: Finding a counterexample

Problem 4.

a) Compute

$$\lim_{n\to\infty} n \int_0^1 \left(\frac{1-x}{1+x}\right)^n dx.$$

b) Let $k \geq 1$ be an integer. Compute

$$\lim_{n\to\infty} n^{k+1} \int_0^1 \left(\frac{1-x}{1+x}\right)^n x^k\, dx.$$

Solution. *a*) The limit equals $\frac{1}{2}$. The result follows immediately from *b*) for $k = 0$.

b) The limit equals $\frac{k!}{2^{k+1}}$. We have, by the substitution $\frac{1-x}{1+x} = y$, that

$$\begin{aligned} n^{k+1} \int_0^1 \left(\frac{1-x}{1+x}\right)^n x^k \, dx &= 2n^{k+1} \int_0^1 y^n (1-y)^k \frac{dy}{(1+y)^{k+2}} \\ &= 2n^{k+1} \int_0^1 y^n f(y) dy, \end{aligned}$$

where

$$f(y) = \frac{(1-y)^k}{(1+y)^{k+2}}.$$

We observe that

$$f(1) = f'(1) = \cdots = f^{(k-1)}(1) = 0. \tag{7}$$

We integrate k times by parts $\int_0^1 y^n f(y) dy$, and by (7) we get

$$\int_0^1 y^n f(y) dy = \frac{(-1)^k}{(n+1)(n+2)\dots(n+k)} \int_0^1 y^{n+k} f^{(k)}(y) dy.$$

One more integration implies that

$$\begin{aligned} \int_0^1 y^n f(y) dy &= \frac{(-1)^k}{(n+1)(n+2)\dots(n+k)(n+k+1)} \\ &\quad \times \left(f^{(k)}(y) y^{n+k+1} \Big|_0^1 - \int_0^1 y^{n+k+1} f^{(k+1)}(y) dy \right) \\ &= \frac{(-1)^k f^{(k)}(1)}{(n+1)(n+2)\dots(n+k+1)} \\ &\quad + \frac{(-1)^{k+1}}{(n+1)(n+2)\dots(n+k+1)} \int_0^1 y^{n+k+1} f^{(k+1)}(y) dy. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} 2n^{k+1} \int_0^1 y^n f(y) dy = 2(-1)^k f^{(k)}(1),$$

since

$$\lim_{n\to\infty} \int_0^1 y^{n+k+1} f^{(k+1)}(y) dy = 0,$$

$f^{(k+1)}$ being continuous and hence bounded. Using Leibniz's formula we get that

$$f^{(k)}(1) = (-1)^k \frac{k!}{2^{k+2}},$$

and the problem is solved.

Grading of Problem 4.

3p: For computing *a*)

7p: For computing *b*)