Медитеранска математичка олимпијада

29.04.2018 година

Mediterranean mathematical olympiad

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Problem 1.

An integer $a \geq 1$ is called *Aegean*, if none of the numbers $a^{n+2} + 3a^n + 1$ with $n \geq 1$ is prime. Prove that there are at least 500 Aegean integers in the set $\{1, 2, \ldots, 2018\}$.

Solution

We identify two infinite families of Aegean integers a. The first family consists of the integers of the form $a \equiv 1 \pmod 5$, as then all $n \geq 1$ satisfy

$$(a^2 + 3)a^n + 1 \;\equiv\; (1^2 + 3) \cdot 1^n + 1 \;\equiv\; 5 \;\equiv\; 0 \pmod 5.$$

Consequently $a = 5b + 1$ is Aegean for $b = 1, \ldots, 403$.

The second family consists of the integers of the form $a \equiv -1 \pmod{15}$. Indeed if $n = 2k+1$ is odd, then $a \equiv -1 \pmod 3$ implies

$$(a^2 + 3)a^n + 1 \;\equiv\; ((-1)^2 + 3)(-1)^{2k+1} + 1 \;\equiv\; -4 + 1 \;\equiv\; 0 \pmod 3.$$

On the other hand if $n = 2k$ is even, then $a \equiv -1 \pmod 5$ implies

$$(a^2 + 3)a^n + 1 \;\equiv\; ((-1)^2 + 3)(-1)^{2k} + 1 \;\equiv\; 4 + 1 \;\equiv\; 0 \pmod 5.$$

This yields that $a = 15c - 1$ is Aegean for $c = 1, \ldots, 134$.

Altogether, these two (disjoint) families yield at least $403 + 134 = 537$ Aegean integers in the range $\{1, 2, \ldots, 2018\}$.

Problem 2.

Let $a_1, a_2, \ldots, a_n$ be $n \geq 2$ real numbers such that $0 \leq a_i \leq \pi/2$. Prove that

$$\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\sin a_i}\right)\left(1+\prod_{i=1}^{n}(\sin a_i)^{1/n}\right) \leq 1.$$

Solution. First, we write the inequality claimed in the equivalent form

$$\sum_{i=1}^{n}\frac{1}{1+\sin a_i} \leq \frac{n}{1+\prod_{i=1}^{n}(\sin a_i)^{1/n}},$$

and using induction, we will prove it for all $n = 2^j$, where j is a positive integer. Indeed, for $j = 1$ the inequality claimed is

$$\frac{1}{1+\sin a_1}+\frac{1}{1+\sin a_2} \leq \frac{2}{1+\sqrt{\sin a_1\,\sin a_2}}$$

or

$$2\,(1+\sin a_1)(1+\sin a_2) \geq (2+\sin a_1+\sin a_2)\,(1+\sqrt{\sin a_1\,\sin a_2}),$$

$$\sin a_1+\sin a_2+2\sin a_1\sin a_2 \geq (2+\sin a_1+\sin a_2)\,\sqrt{\sin a_1\,\sin a_2},$$

$$(\sin a_1+\sin a_2)\,(1-\sqrt{\sin a_1\,\sin a_2})-2\,\sqrt{\sin a_1\,\sin a_2}\,(1+\sqrt{\sin a_1\,\sin a_2})$$

from which

$$(\sqrt{\sin a_1}-\sqrt{\sin a_2})^2\,(1-\sqrt{\sin a_1\,\sin a_2}) \geq 0$$

follows and the inequality holds.

Assume that it holds

$$\sum_{i=1}^{2^j}\frac{1}{1+\sin a_i} \leq \frac{2^j}{1+\sqrt[2^j]{\sin a_1\,\sin a_2\cdots\sin a_{2^j}}}$$

Then, for 2^{j+1} we have

$$\begin{aligned}
\sum_{i=1}^{2^{j+1}}\frac{1}{1+\sin a_i} &= \sum_{i=1}^{2^j}\left(\frac{1}{1+\sin a_{2i-1}}+\frac{1}{1+\sin a_{2i}}\right)\\
&\leq 2\sum_{i=1}^{2^j}\frac{1}{1+\sqrt{\sin a_{2i-1}\,\sin a_{2i}}}\\
&\leq \frac{2^{j+1}}{1+\sqrt[2^j]{\sqrt{\sin a_1\,\sin a_2}\,\sqrt{\sin a_3\,\sin a_4}\ldots\sqrt{\sin a_{2^{j+1}-1}\,\sin a_{2^{j+1}}}}}\\
&= \frac{2^{j+1}}{1+\sqrt[2^{j+1}]{\sin a_1\,\sin a_2\cdots\sin a_{2^{j+1}}}}
\end{aligned}$$

Thus, by PMI the inequality holds for $n = 2^j$.

Finally, we will use Backward induction. That is, we prove $P(k) \Rightarrow P(k-1)$ for all $k \geq 3$. Putting

$$\sin a_k = \sqrt[k-1]{\sin a_1 \ \sin a_2 \cdots \sin a_{k-1}},$$

we have

$$\sum_{i=1}^{k} \frac{1}{1+\sin a_i} = \sum_{i=1}^{k-1} \frac{1}{1+\sin a_i} + \frac{1}{1+\sqrt[k-1]{\sin a_1 \ \sin a_2 \cdots \sin a_{k-1}}}$$

$$\leq \frac{k}{1+\sqrt[k]{\sin a_1 \cdots \sin a_{k-1} \cdot \sqrt[k-1]{\sin a_1 \ \sin a_2 \cdots \sin a_{k-1}}}}$$

$$= \frac{k}{1+\sqrt[k-1]{\sin a_1 \ \sin a_2 \cdots \sin a_{k-1}}}$$

from which

$$\sum_{i=1}^{k-1} \frac{1}{1+\sin a_i} \leq \frac{k-1}{1+\sqrt[k-1]{\sin a_1 \ \sin a_2 \cdots \sin a_{k-1}}}$$

follows. Equality holds when $a_1 = a_2 = \ldots = a_n$, and we are done.

Problem 3.

Determine the largest integer N, for which there exists a $6 \times N$ table T that has the following properties:

(i) Every column contains the numbers $1, 2, \ldots, 6$ in some ordering.

(ii) For any two columns $i \neq j$, there exists a row r such that $T(r, i) = T(r, j)$.

(iii) For any two columns $i \neq j$, there exists a row s such that $T(s, i) \neq T(s, j)$.

Solution

We show that $N = 5! = 120$ is the largest such integer. The lower bound construction is as follows. For every permutation of the integers $1, \ldots, 5$ create a corresponding column whose first 5 entries agree with the permutation and whose last entry (in the 6th row) equals 6.

The upper bound argument is as follows. Consider a $6 \times N$ table T with the desired properties. For each of its columns c and for every integer $x = 1, 2, \ldots, 6$ we define a new column c_x that consists of the 6entries

$$T(1, c) + x, \quad T(2, c) + x, \quad T(3, c) + x, \quad T(4, c) + x, \quad T(5, c) + x, \quad T(6, c) + x.$$

Now consider two columns i and j, and two integers x and y with $1 \leq x, y \leq 6$, and assume that the columns i_x and j_y agree componentwise modulo 6. By condition (ii) there exists a row r such that $T(r, i) = T(r, j)$. This means

$$T(r, i) + x \; = \; i_x(r) \; \equiv \; j_y(r) \; = \; T(r, j) + y \; = \; T(r, i) + y \pmod{6},$$

which implies $x \equiv y \bmod 6$ and hence $x = y$. If $i \neq j$, then by condition (iii) there exists a row s such that $T(s, i) \neq T(s, j)$. By using $x = y$ this then would imply the contradiction

$$T(s, i) + x \; = \; i_x(s) \; \equiv \; j_y(s) \; = \; T(s, j) + y \; = \; T(s, j) + x \; \not\equiv \; T(s, i) + x \pmod{6}.$$

Hence whenever two columns i_x and j_y agree componentwise modulo 6, then $i = j$ and $x = y$ must hold. This implies that the $6N$ columns c_x with $c \in T$ and $x = 1, 2, \ldots, 6$ must be pairwise distinct. By condition (i), these pairwise distinct objects correspond to pairwise distinct permutations of $1, 2, \ldots, 6$. Therefore $6N \leq 6!$, so that $N \leq 5!$.

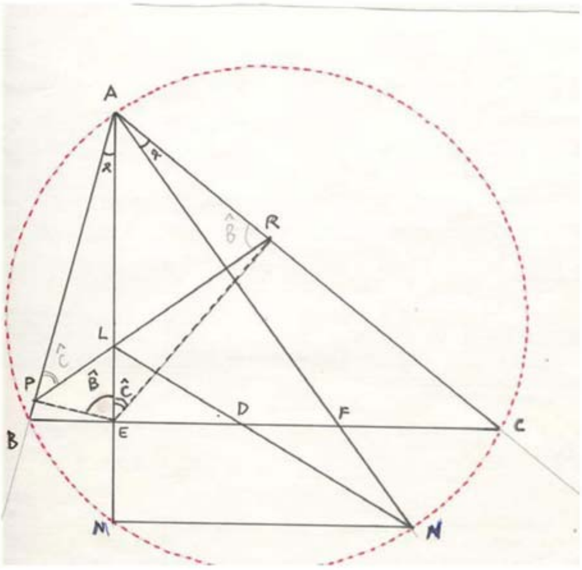
Problem 4.

ABC is an acute triangle. AE and AF are isogonal cevians, where $E \in BC$ and $F \in BC$. The straight lines AE and AF intersect again the circumcircle of ABC at points M and N, respectively. In the rays AB and AC we get points P and R such that $\measuredangle PEA = \measuredangle B$ and $\measuredangle AER = \measuredangle C$. Let $L = AE \cap PR$ and $D = BC \cap LN$. Prove, with reasons, that

$$\frac{1}{MN} + \frac{1}{EF} = \frac{1}{ED}.$$

Solution 1

Consider the following diagram:



The following couples of triangles are clearly similar:

$\triangle AEF$ and $\triangle AMN$, because a general property of isogonal show that MN and BC are parallel; then we have $\frac{AE}{AM} = \frac{EF}{MN}$ (1). By the same rason, $\triangle LED$ and $\triangle LMN$ are also similar, and so we have $\frac{LE}{LM} = \frac{ED}{MN}$ (2).

The following couples of triangles are similar, too:

$$\triangle APE \text{ and } \triangle ABE \Rightarrow AE^2 = AP \cdot AB$$

$$\triangle APL \text{ and } \triangle ABM \Rightarrow \frac{AP}{AM} = \frac{AL}{AB} \Rightarrow AM \cdot AL = AP \cdot AB = AE^2$$

And so we get

$$\frac{AM}{AE} = \frac{AE}{AL} \text{ (3)}.$$

Then using (2),

$$\frac{1}{MN}=\frac{LE}{LM\cdot ED}.$$

And using (1),

$$\frac{1}{EF}=\frac{AM}{AE\cdot MN}=\frac{AM}{AE}\cdot\frac{LE}{LM\cdot ED}.$$

Therefore,

$$\frac{1}{MN}+\frac{1}{EF}=\frac{1}{ED}\cdot\left[\frac{LE}{LM}\left(1+\frac{AM}{AE}\right)\right],$$

and so we need just to prove that the last bracket equals 1. To this, we will use (3):

$$\left[\frac{LE}{LM}\left(1+\frac{AM}{AE}\right)\right]=$$

$$\left[\frac{LE}{LM}\left(1+\frac{AM}{AE}\right)\right]=\frac{LE}{LM}\left(1+\frac{AE}{AL}\right)=\frac{(AE-AL)(AE+AL)}{(AM-AL)\cdot AL}=\frac{AE^2-AL^2}{AM\cdot AL-AL^2}=$$

$$=\frac{AE^2-AL^2}{AE^2-AL^2}$$, and we are done.■

Observation

PE and BE are antiparallel with respect to AM and AB, so P and B are homologous in the inversion of pole A and power AE^2. The same reasoning applies to PL and BM. This means that $AP\cdot AB=AL\cdot AM=AE^2$, and continue as in the featured solution.