19th Meditarranean mathematical olympiad

Problems and solutions

Problem 1

Determine all integers $n \geq 1$ for which the number $n^8 + n^6 + n^4 + 4$ is prime.

Solution. We use factorization

$$n^8 + n^6 + n^4 + 4 = (n^4 - n^3 + n^2 - 2n + 2)(n^4 + n^3 + n^2 + 2n + 2).$$

The first factor $f(n)$ satisfies

$$f(n) = n^4 - n^3 + n^2 - 2n + 2 = n^3(n-1) + (n-1)^2 + 1$$

and hence satisfies $f(n) \geq 2$ for all $n \geq 2$. The second factor $g(n) = n^4 + n^3 + n^2 + 2n + 2$ is strictly greather than 2 for all $n \geq 2$. This only leaves the case $n = 1$ as a potential candidate for a prime, and indeed $f(1)g(1) = 1 \cdot 7 = 7$ is prime.

Problem 2

Let ABC be a triangle . D is the foot of the internal bisector of the angle A. The perpendicular from D to the tangent AT (T belong to BC) to the circumscribed circle of ABC intersect the altitude AH_a at the point I (H_a belong to BC).

If P is the midpoint of AB and O is the circumcircle, TI intersect AB at M and PT intersect AD at F, prove that MF is perpendicular to AO.

Solution. Let Q be the midpoint of AC and N the intersection of AD and PQ. Then N is the midpoint of AD. As DE is perpendicular to AT, being E the intersection point of DI and AT, and as OA is perpendicular to AT, we get that DE is parallel to OA, and so the angles OAN and ADE are equal. As a consequence, triangles ADE and DAH_a are congruent.

In particular angle DAT equals to angle HAD, that is, ATD is isosceles and point I is the orthocenter of ABC.

So, TI is perpendicular to AD, and the intersection point of TI and AD is the midpoint of AD (N, say).

The four points M,N,I,T are collinear.

We will apply the Ceva theorem in the triangle APT with the cevians PN, AD and TM. We get

$$\frac{FP}{FT}\cdot 1\cdot\frac{MA}{PM}=1 \qquad \Leftrightarrow \qquad \frac{PF}{TF}=\frac{MP}{MA}.$$

(Observe that NP cut AT in its midpoint).

So, MF is parallel to AT, and form this MF is perpendicular to AO, as claimed.

Problem 3

Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that

$$\sqrt{\frac{b}{a^2+3}}+\sqrt{\frac{c}{b^2+3}}+\sqrt{\frac{a}{c^2+3}}\leq\frac{3}{2}\sqrt[4]{\frac{1}{abc}}\,.$$

Задача 3

Нека a,b,c се позитивни реални броеви такви што $a+b+c=3$. Докажи дека

$$\sqrt{\frac{b}{a^2+3}}+\sqrt{\frac{c}{b^2+3}}+\sqrt{\frac{a}{c^2+3}}\leq\frac{3}{2}\sqrt[4]{\frac{1}{abc}}\,.$$

Solution. Putting $\vec{u}=\left(\frac{1}{\sqrt{a^2+3}},\frac{1}{\sqrt{b^2+3}},\frac{1}{\sqrt{c^2+3}}\right)$ and $\vec{v}=(\sqrt{b},\sqrt{c},\sqrt{a})$ in CBS inequality, we get

$$\left(\sqrt{\frac{b}{a^2+3}}+\sqrt{\frac{c}{b^2+3}}+\sqrt{\frac{a}{c^2+3}}\right)^2\leq\left(\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}\right)(a+b+c)=$$

$$=3\left(\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}\right)$$

on account of the constain relation.

We have

$$a^2+3=a\ \ +1+1+1=\geq 4\sqrt[4]{a^2}=4\sqrt{a}\,.$$

Likewise, we get

$$b^2+3\geq 4\sqrt{b}$$

$$c^2+3\geq 4\sqrt{c}\,.$$

Therefore,

$$\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}\leq\frac{1}{4}\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)=\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{4\sqrt{abc}}\leq\frac{\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}}{4\sqrt{abc}}\leq$$

$$=\frac{a+b+c}{4\sqrt{abc}}$$

on account of AM-GM inequality.

Combining the proceeding results, we get

$$\left(\sqrt{\frac{b}{a^2+3}}+\sqrt{\frac{c}{b^2+3}}+\sqrt{\frac{a}{c^2+3}}\right)^2\leq 3\frac{a+b+c}{4\sqrt{abc}}=\frac{9}{4\sqrt{abc}}$$

from which the statement follows. Equality holds when $a=b=c=1$ and we are done.

Problem 4

Consider a 25×25 chessboard with cells $C(i,j)$ for $1\le i,j\le 25$. Find the smallest possible number n of colors with these cells can be colored subject to the following condition: For $1\le i<j\le 25$ and for $1\le s<t\le 25$, the three cells $C(i,s),C(j,s),C(j,t)$ carry at least two different colors.

Solution. We show that $n=12$ colors are necessary and sufficient.

12 colors are sufficient: We use the residual classes $0,1,2,3,4,5,6,7,8,9,10,11$ modulo 12 as collors. Cell $C(i,s)$ receives the color $\left\lfloor \frac{1}{2}(i+s)\right\rfloor$ modulo 12, where $\lfloor x\rfloor$ denotes the largest integer less or equal x. Note that the color classes form stipes.

Suppose that for some $1\le i<j\le 25$ and for $1\le s<t\le 25$ the three cells $C(i,s),C(j,s),C(j,t)$ all would receive the same color. Here are two easy observations:

- Since $\left\lfloor \frac{1}{2}(i+s)\right\rfloor=\left\lfloor \frac{1}{2}(j+s)\right\rfloor$ modulo 12, and since $0<\frac{1}{2}(j-i)\le 12$, we get $\left\lfloor \frac{1}{2}(i+s)\right\rfloor=\left\lfloor \frac{1}{2}(j+s)\right\rfloor$.

- An analogous argument yields $\left\lfloor \frac{1}{2}(j+s)\right\rfloor=\left\lfloor \frac{1}{2}(j+t)\right\rfloor$.

These two observations imply $\left\lfloor \frac{1}{2}(i+s)\right\rfloor=\left\lfloor \frac{1}{2}(j+t)\right\rfloor$. But $i+1\le j$ and $j+1\le t$ yield $\left\lfloor \frac{1}{2}(i+s)\right\rfloor<\left\lfloor \frac{1}{2}(j+t)\right\rfloor$, the desired contradiction.

12 colors are necessary: We argue that no color can occur in more than 50 cells. Then the total number of collors is at least $\frac{25^2}{50}>12$, and we are done.

Hence fix an arbitrary color(say blue) in an arbitrary coloring of the desired form, and let b denote the total number of blue cells. Consider a row r that contains at least two blue cells, and let $i_1<i_2<...<i_l$ denote the column indices of these blue cells. For $p=1,2,...,l-1$ draw a horizontal arrow from cell $C(r,i_p)$ to cell $C(r,i_{p+1})$. Similarly put vertical arrows between consecutive blue cells in columns with at least two blue cells, but orient them from larger indices towards smaller indices. Note that no blue cell has two or more out-going arrows, since otherwise the forbidden configuration would occur. Therefore the total number a of arrows satisfies $a\le b$.

Denote by r_k (respectively c_k) the number of blue cells in the k-th row (respectively k-th column). Then $b=\sum_{k=1}^{25} r_k=\sum_{k=1}^{25} c_k$. Note that the k-th row contains at least r_k-1 arrows, and hence the total number of arrows in all rows is at least $\sum_{k=1}^{25}(r_k-1)$. Similarly, the total number of arrows in all columns is at least $\sum_{k=1}^{25}(c_k-1)$. This implies

$$b\ge a\ge \sum_{k=1}^{25}(r_k-1)+\sum_{k=1}^{25}(c_k-1)=2b-50,$$

which yields the desired bound $b\le 50$.