XX Mediterranean mathematical olympiad,

23 april 2017

Problem 1

Determine the smallest integer n, for which there exist integers $x_1,...,x_n$ and positive integers $a_1,...,a_n$, so that

$$x_1+...+x_n=0, \qquad a_1x_1+...+a_nx_n>0, \qquad a_1^2x_1+...+a_n^2x_n<0.$$

Problem 2

Let a,b,c be positive real numbers such that $a+b+c=1$. Prove that

$$\left(x^2+y^2+z^2\right)\left(\frac{a^3}{x^2+2y^2}+\frac{b^3}{y^2+2z^2}+\frac{c^3}{z^2+2x^2}\right)\geq\frac{1}{9},$$

holds for all positive real numbers x, y, z.

Problem 3

Let ABC be an equilateral triangle, and let P be some point in its circumcircle. Determine, with reasons, all the numbers $n\in\mathbb{N}^*$ such that the sum

$$S_n(P)=|PA|^n+|PB|^n+|PC|^n,$$

is independent of the choice of the point P.

Problem 4

A set S of integers is Balearic, if there are two (not necessarily distinct) elements $s,s'\in S$ whose sum $s+s'$ is a power of two; otherwise it is called a non-Balearic set.

Find an integer n such that $\{1,2,...,n\}$ contains a 99-element non-Balearic set, whereas all the 100-element subsets are Balearic.

times: 04:30 hours **Each problem** is worth 7 points

Union of mathematicians of MACEDONIA-ARMAGANKA