24th Junior Balkan Mathematical Olympiad (JBMO) 9 - 13 September 2020
Mathematical Society of South Eastern Europe (MASSEE)

Language: **English**

Friday, September 11, 2020

Problem 1. Find all triples (a, b, c) of real numbers that satisfy the system of equations:

$$a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \text{ and } a^2+b^2+c^2=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

Problem 2. Let $\triangle ABC$ be a right-angled triangle with $\angle BAC = 90°$ and let E be the foot of the perpendicular from A on BC. Let $Z \neq A$ be a point on the line AB with $AB = BZ$. Let (c) be the circumcircle of the triangle $\triangle AEZ$. Let D be the second point of intersection of (c) with ZC and let F be the antidiametric point of D with respect to (c). Let P be the point of intersection of the lines FE and CZ. If the tangent to (c) at Z meets PA at T, prove that the points T, E, B, Z are concyclic.

Problem 3. Alice and Bob play the following game: Alice picks a set $A = \{1, 2, \ldots, n\}$ for some natural number $n \geqslant 2$. Then starting with Bob, they alternatively choose one number from the set A, according to the following conditions: initially Bob chooses any number he wants, afterwards the number chosen at each step should be distinct from all the already chosen numbers and should differ by 1 from an already chosen number. The game ends when all numbers from the set A are chosen. Alice wins if the sum of all the numbers that she has chosen is composite. Otherwise Bob wins. Decide which player has a winning strategy.

Problem 4. Find all prime numbers p and q such that

$$1+\frac{p^q-q^p}{p+q}$$

is a prime number.

Time: 4 hours and 30 minutes
Each problem is worth 10 points

Problem 1. Find all triples (a, b, c) of real numbers such that the following system holds:

$$\begin{cases} a+b+c = \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} \\ a^2+b^2+c^2 = \dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2} \end{cases}$$

Solution. First of all if (a, b, c) is a solution of the system then also $(-a, -b, -c)$ is a solution. Hence we can suppose that $abc > 0$. From the first condition we have

$$a+b+c = \frac{ab+bc+ca}{abc}. \tag{1}$$

Now, from the first condition and the second condition we get

$$(a+b+c)^2 - (a^2+b^2+c^2) = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 - \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

The last one simplifies to

$$ab+bc+ca = \frac{a+b+c}{abc}. \tag{2}$$

First we show that $a+b+c$ and $ab+bc+ca$ are different from 0. Suppose on contrary then from relation (1) or (2) we have $a+b+c = ab+bc+ca = 0$. But then we would have

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 0,$$

which means that $a = b = c = 0$. This is not possible since a, b, c should be different from 0.
Now multiplying (1) and (2) we have

$$(a+b+c)(ab+bc+ca) = \frac{(a+b+c)(ab+bc+ca)}{(abc)^2}.$$

Since $a+b+c$ and $ab+bc+ca$ are different from 0, we get $(abc)^2 = 1$ and using the fact that $abc > 0$ we obtain that $abc = 1$. So relations (1) and (2) transform to

$$a+b+c = ab+bc+ca.$$

Therefore,

$$(a-1)(b-1)(c-1) = abc - ab - bc - ca + a + b + c - 1 = 0.$$

This means that at least one of the numbers a, b, c is equal to 1. Suppose that $c = 1$ then relations (1) and (2) transform to $a+b+1 = ab+a+b \Rightarrow ab = 1$. Taking $a = t$ then we have $b = \frac{1}{t}$. We can now verify that any triple $(a, b, c) = \left(t, \frac{1}{t}, 1\right)$ satisfies both conditions. $t \in \mathbb{R} \setminus \{0\}$. From the initial observation any triple $(a, b, c) = \left(t, \frac{1}{t}, -1\right)$ satisfies both conditions. $t \in \mathbb{R} \setminus \{0\}$. So, all triples that satisfy both conditions are $(a, b, c) = \left(t, \frac{1}{t}, 1\right), \left(t, \frac{1}{t}, -1\right)$ and all permutations for any $t \in \mathbb{R} \setminus \{0\}$. $\square$

Comment by PSC. After finding that $abc = 1$ and

$$a+b+c = ab+bc+ca,$$

we can avoid the trick considering $(a-1)(b-1)(c-1)$ as follows. By the Vieta's relations we have that a, b, c are roots of the polynomial

$$P(x) = x^3 - sx^2 + sx - 1$$

which has one root equal to 1. Then, we can conclude as in the above solution.

Problem 2. Let $\triangle ABC$ be a right-angled triangle with $\angle BAC = 90^\circ$ and let E be the foot of the perpendicular from A on BC. Let $Z \neq A$ be a point on the line AB with $AB = BZ$. Let (c) be the circumcircle of the triangle $\triangle AEZ$. Let D be the second point of intersection of (c) with ZC and let F be the antidiametric point of D with respect to (c). Let P be the point of intersection of the lines FE and CZ. If the tangent to (c) at Z meets PA at T, prove that the points T, E, B, Z are concyclic.

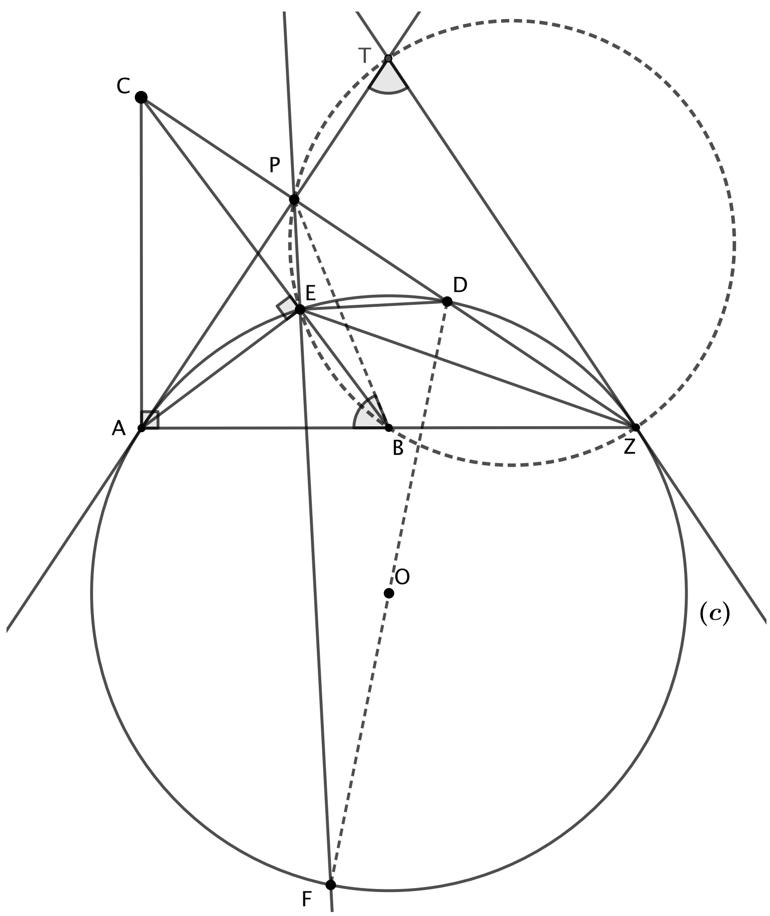
Solution. We will first show that PA is tangent to (c) at A.

Since E, D, Z, A are concyclic, then $\angle EDC = \angle EAZ = \angle EAB$. Since also the triangles $\triangle ABC$ and $\triangle EBA$ are similar, then $\angle EAB = \angle BCA$, therefore $\angle EDC = \angle BCA$.

Since $\angle FED = 90^\circ$, then $\angle PED = 90^\circ$ and so

$$\angle EPD = 90^\circ - \angle EDC = 90^\circ - \angle BCA = \angle EAC.$$

Therefore the points E, A, C, P are concyclic. It follows that $\angle CPA = 90^\circ$ and therefore the triangle $\angle PAZ$ is right-angled. Since also B is the midpoint of AZ, then $PB = AB = BZ$ and so $\angle ZPB = \angle PZB$.



Furthermore, $\angle EPD = \angle EAC = \angle CBA = \angle EBA$ from which it follows that the points P, E, B, Z are also concyclic.

Now observe that

$$\angle PAE = \angle PCE = \angle ZPB - \angle PBE = \angle PZB - \angle PZE = \angle EZB.$$

Therefore PA is tangent to (c) at A as claimed.

It now follows that $TA = TZ$. Therefore

$$\begin{aligned}\angle PTZ &= 180^\circ - 2(\angle TAB) = 180^\circ - 2(\angle PAE + \angle EAB) = 180^\circ - 2(\angle ECP + \angle ACB) \\ &= 180^\circ - 2(90^\circ - \angle PZB) = 2(\angle PZB) = \angle PZB + \angle BPZ = \angle PBA.\end{aligned}$$

Thus T, P, B, Z are concyclic, and since P, E, B, Z are also concyclic then T, E, B, Z are concyclic as required.

$\square$

Problem 3. Alice and Bob play the following game: Alice picks a set $A = \{1, 2, \ldots, n\}$ for some natural number $n \geqslant 2$. Then starting with Bob, they alternatively choose one number from the set A, according to the following conditions: initially Bob chooses any number he wants, afterwards the number chosen at each step should be distinct from all the already chosen numbers, and should differ by 1 from an already chosen number. The game ends when all numbers from the set A are chosen. Alice wins if the sum of all of the numbers that she has chosen is composite. Otherwise Bob wins. Decide which player has a winning strategy.

Solution. To say that Alice has a winning strategy means that she can find a number n to form the set A, so that she can respond appropriately to all choices of Bob and always get at the end a composite number for the sum of her choices. If such n does not exist, this would mean that Bob has a winning strategy instead.

Alice can try first to check the small values of n. Indeed, this gives the following winning strategy for her: she initially picks $n = 8$ and responds to all possible choices made by Bob as in the list below (in each row the choices of Bob and Alice are given alternatively, starting with Bob):

1 2 3 4 5 6 7 8
2 3 1 4 5 6 7 8
2 3 4 1 5 6 7 8
3 2 1 4 5 6 7 8
3 2 4 5 1 6 7 8
3 2 4 5 6 1 7 8
4 5 3 6 2 1 7 8
4 5 3 6 7 8 2 1
4 5 6 7 3 2 1 8
4 5 6 7 3 2 8 1
4 5 6 7 8 3 2 1
5 4 3 2 1 6 7 8
5 4 3 2 6 7 1 8
5 4 3 2 6 7 8 1
5 4 6 3 2 1 7 8
5 4 6 3 7 8 2 1
6 7 5 4 3 8 2 1
6 7 5 4 8 3 2 1
6 7 8 5 4 3 2 1
7 6 8 5 4 3 2 1
7 6 5 8 4 3 2 1
8 7 6 5 4 3 2 1

In all cases, Alice's sum is either an even number greater than 2, or else 15 or 21, thus Alice always wins.

Problem 4. Find all pairs (p, q) of prime numbers such that

$$1 + \frac{p^q - q^p}{p + q}$$

is a prime number.

Solution. It is clear that $p \neq q$. We set

$$1 + \frac{p^q - q^p}{p + q} = r$$

and we have that

$$p^q - q^p = (r - 1)(p + q). \tag{3}$$

From Fermat's Little Theorem we have

$$p^q - q^p \equiv -q \pmod{p}.$$

Since we also have that

$$(r - 1)(p + q) \equiv -rq - q \pmod{p},$$

from (3) we get that

$$rq \equiv 0 \pmod{p} \Rightarrow p \mid qr,$$

hence $p \mid r$, which means that $p = r$. Therefore, (3) takes the form

$$p^q - q^p = (p - 1)(p + q). \tag{4}$$

We will prove that $p = 2$. Indeed, if p is odd, then from Fermat's Little Theorem we have

$$p^q - q^p \equiv p \pmod{q}$$

and since

$$(p - 1)(p + q) \equiv p(p - 1) \pmod{q},$$

we have

$$p(p - 2) \equiv 0 \pmod{q} \Rightarrow q \mid p(p - 2) \Rightarrow q \mid p - 2 \Rightarrow q \leq p - 2 < p.$$

Now, from (4) we have

$$p^q - q^p \equiv 0 \pmod{p - 1} \Rightarrow 1 - q^p \equiv 0 \pmod{p - 1} \Rightarrow q^p \equiv 1 \pmod{p - 1}.$$

Clearly $\gcd(q, p - 1) = 1$ and if we set $k = ord_{p-1}(q)$, it is well-known that $k \mid p$ and $k < p$, therefore $k = 1$. It follows that

$$q \equiv 1 \pmod{p - 1} \Rightarrow p - 1 | q - 1 \Rightarrow p - 1 \leq q - 1 \Rightarrow p \leq q$$

a contradiction.
Therefore, $p = 2$ and (4) transforms to

$$2^q = q^2 + q + 2.$$

We can easily check by induction that for every positive integer $n \geq 6$ we have $2^n > n^2 + n + 2$. This means that $q \leq 5$ and the only solution is for $q = 5$. Hence the only pair which satisfy the condition is $(p, q) = (2, 5)$.

$\square$

Comment by the PSC. From the problem condition, we get that p^q should be bigger than q^p, which gives

$$q \ln p > p \ln q \iff \frac{\ln p}{p} > \frac{\ln q}{q}.$$

The function $\dfrac{\ln x}{x}$ is decreasing for $x > e$, thus if p and q are odd primes, we obtain $q > p$.