9TH EUROPEAN MATHEMATICAL CUP

12th December 2020 - 20th December 2020

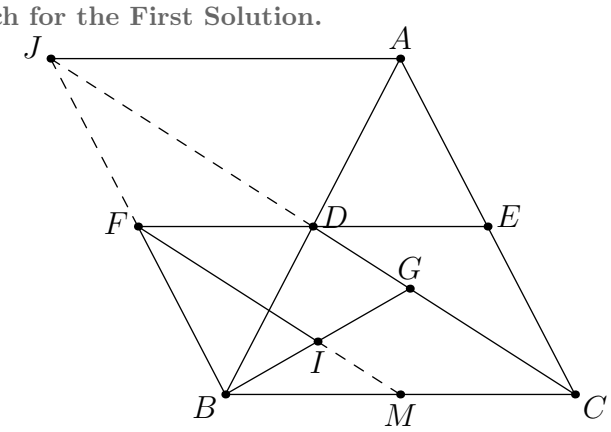
Junior Category

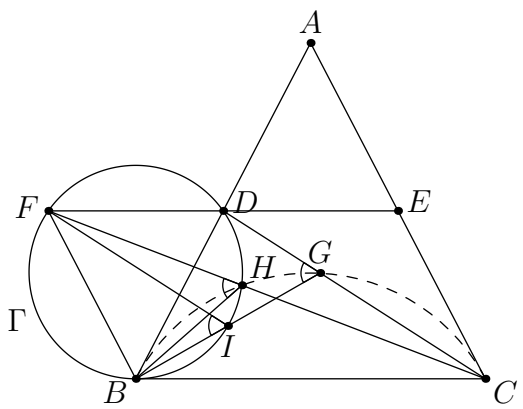
Problems and Solutions

Problem 1. Let ABC be an acute-angled triangle. Let D and E be the midpoints of sides $\overline{AB}$ and $\overline{AC}$ respectively. Let F be the point such that D is the midpoint of $\overline{EF}$. Let Γ be the circumcircle of triangle FDB. Let G be a point on the segment $\overline{CD}$ such that the midpoint of $\overline{BG}$ lies on Γ. Let H be the second intersection of Γ and FC. Show that the quadrilateral $BHGC$ is cyclic.

(Art Waeterschoot, Belgium)

Sketch for the First Solution.





First Solution. Since D and E are midpoints, the diagonals $\overline{AB}$ and $\overline{EF}$ of the quadrilateral $AFBE$ bisect each other, so $AFBE$ is a parallelogram. Hence $BF \parallel AE$.

2 points.

Lemma. If I is the second intersection of Γ and $\overline{BG}$, then $FI \parallel CD$. (We will present two different proofs.)

First proof. Let J be the point such that $BCAJ$ is a parallelogram. Since $BF \parallel AE$, we have that B, F, J are colinear.

2 points.

Since D is the midpoint of $\overline{AB}$, C, D, J are collinear.

1 point.

As F and I are midpoints of $\overline{BJ}$ and $\overline{BG}$, then $FI \parallel CD$. $\square$

2 points.

Second proof. Let M be the midpoint of $\overline{BC}$. As

$$|MC| = \frac{|BC|}{2} = |DE| = |DF|$$

and $FD \parallel MC$, then $MCDF$ is a parallelog., so $MF \parallel CD$.

2 points.

As M and I are midpoints of $\overline{BC}$ and $\overline{BG}$, then $MI \parallel CD$.

2 points.

Hence M, I and F are collinear and $FI \parallel CD$. $\square$

1 point.

Now as we know that $FI \parallel CD$, we have $\angle BIF = \angle BGD$.

1 point.

As $BIHF$ is a cyclic quadrilateral, we have $\angle BIF = \angle BHF$.

1 point.

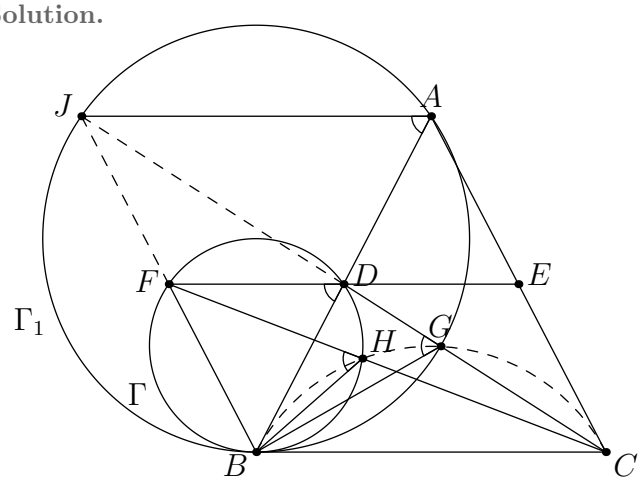
Hence

$$\angle CHB = 180^\circ - \angle BHF = 180^\circ - \angle BGD = \angle CGB,$$

so $BHGC$ is cyclic as desired.

1 point.

Sketch for the Second Solution.



Second Solution. Since D and E are midpoints, the diagonals $\overline{AB}$ and $\overline{EF}$ of the quadrilateral $AFBE$ bisect each other, so $AFBE$ is a parallelogram. Hence $BF \parallel AE$.

2 points.

Let J be the point such that $BCAJ$ is a parallelogram. Since $BF \parallel AE$, we have that B, F, J are collinear.

2 points.

Since D is the midpoint of $\overline{AB}$, C, D, J are collinear.

1 point.

Now let Γ_1 be the circumcircle of triangle JAB. As F and D are midpoints of $\overline{BJ}$ and $\overline{BA}$, and the midpoint of $\overline{BG}$ lies on Γ, we can redefine G as the second intersection of Γ_1 and CJ.

2 points.

As $AJBG$ is a cyclic quadrilateral, we have $\angle BGJ = \angle BAJ$.

1 point.

As FD is parallel to JA, we have $\angle BAJ = \angle BDF$.

0 points.

As $BHDF$ is a cyclic quadrilateral, we have $\angle BDF = \angle BHF$.

1 point.

Hence

$$\angle CHB = 180^\circ - \angle BHF = 180^\circ - \angle BGD = \angle CGB,$$

so $BHGC$ is cyclic as desired.

1 point.

Notes on marking:

- If a student has a partial solution with analytic methods, only points for proving facts that can be expressed in geometric ways and lead to a complete solution can be awarded.

Problem 2. A positive integer $k \geqslant 3$ is called *fibby* if there exists a positive integer n and positive integers $d_1 < d_2 < \ldots < d_k$ with the following properties:

- $d_{j+2} = d_{j+1} + d_j$ for every j satisfying $1 \leqslant j \leqslant k-2$,
- $d_1, d_2, \ldots, d_k$ are divisors of n,
- any other divisor of n is either less than d_1 or greater than d_k.

Find all fibby numbers.

(Ivan Novak)

Solution. Note that $(1, 2, 3, 5)$ is a sequence of length 4 such that all its elements are divisors of 30 and every other divisor of 30 is either less than 1 or greater than 5. Also $3 = 1 + 2$ and $5 = 2 + 3$, which means 4 is fibby. Consequently, 3 is also fibby.

1 point.

Suppose there exist positive integers n, $d_1 < d_2 < \ldots < d_k$ satisfying the problem's conditions, with $k \geqslant 5$. Suppose for the sake of contradiction that d_j is even for some $j \geqslant 3$. Then $\frac{d_j}{2}$ is also a divisor of n.

1 point.

However,

$$d_1 \leqslant d_{j-2} < \frac{d_{j-1} + d_{j-2}}{2} = \frac{d_j}{2} < d_{j-1} < d_k.$$

This implies $\frac{d_j}{2}$ is a divisor of n which is neither less than d_1 nor greater than d_k and is distinct from the numbers $d_1, d_2, \ldots, d_k$, which is a contradiction.

6 points.

This implies that d_3 and d_4 are odd. However, this means that $d_5 = d_3 + d_4$ is even, which is a contradiction. Therefore, any number greater than 4 is not fibby.

2 points.

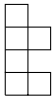
Notes on marking:

- The part of the proof where we prove all $k \geq 5$ are not fibby is worth **9 points**. It may happen that a contestant proves a weaker statement in that direction.
  - If a contestant proves that there exists C such that no $k \geq C$ is fibby, they should get **1 point**.
  - If the C above is explicit, they should get an additional **1 point**.
  - If in addition $C = 6$, they should get **1 point** more.

  The points above (at most **3 points**) are not additive with the points for proving $C = 5$ in the official solution. Thus, without using ideas that can solve the $C = 5$ case, the contestant should not get more than **1 point** for the construction, plus the points above if applicable.
- Many solutions proceed by cases on the parity of d_1 and d_2. However, in all solutions that the Problem Selection Committee were aware of, the only parity that matters is the parity of some $d_j, j \geq 3$.

  Thus, stating and proving that some of d_3, d_4 and d_5 is even is worth **2 points**, as in the official solution, and no other points are awarded for parity concerns.

Problem 3. Two types of tiles, depicted on the figure below, are given.

Tile F:  Tile Z: 

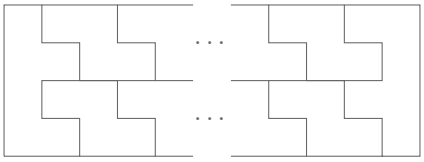
Find all positive integers n such that an $n \times n$ board consisting of n^2 unit squares can be covered without gaps with these two types of tiles (rotations and reflections are allowed) so that no two tiles overlap and no part of any tile covers an area outside the $n \times n$ board.

(Art Waeterschoot)

Solution. We claim such a tiling exists whenever n is divisible by 4 and greater than 4.

0 points.

We now prove the existence of a tiling in the case where n is divisible by 4 and greater than 4. The figure below shows that if $k \geq 1$, we can tile a $(2k+1) \times 4$-rectangle.



1 point.

By gluing a 3×4 rectangle to the above tiling, we get a tiling of any $(4k+4) \times 4$ rectangle, where $k \geqslant 1$. We can now stack $k+1$ such rectangles next to each other to obtain a $(4k+4) \times (4k+4)$ square, which proves the claim.

1 point.

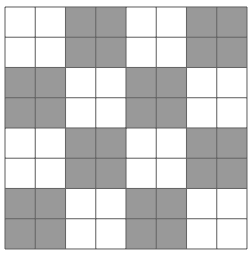
Suppose we can tile a $n \times n$ square with the given tiles. Let a and b be the number of F-tiles and Z-tiles used in the tiling, respectively. Then $6a + 4b = n^2$, which implies n is even. This implies that a is also even. Let $n = 2k$, where k is a positive integer.

0 points.

Consider the following colouring of the square: divide up the square into k^2 smaller squares of size 2×2 and colour these squares with a chessboard colouring (see the figure below). Every F-tile covers exactly 3 black unit squares and every Z-tile covers an odd number of black unit squares.

1 point.

Because there are an even number of black squares, we obtain that a and b have equal parity. Since a is even, this implies that b is even.

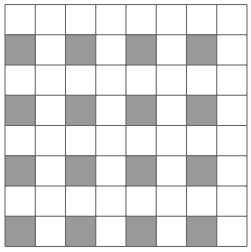


3 points.

Now colour all unit squares in an even row and odd column black (see the figure below). Now every F-tile covers an even number of black unit squares and every Z-tile covers exactly one black unit square.

1 point.

Since the number of black squares is k^2, we obtain that b and k^2 have equal parity. Since b is even, this implies k is even.



3 points.

Therefore, n is a multiple of 4.

0 points.

Furthermore, it is easily seen a 4×4-square cannot be tiled, as there are no positive integers (a, b) such that b is even and $6a + 4b = 16$.

0 points.

Notes on marking:

- Colouring a square in a certain way without drawing any relevant conclusions from the colouring is worth **0 points**.
- Another possible solution is to consider a colouring with 4 colours by dividing up into small 2×2-squares. In fact this is equivalent to our solution, because is the same as considering both colourings above at once. Considering such a colouring and drawing the same conclusions is worth the same amount of points as considering the colourings one by one.
- If a student doesn't check the case when $n = 4$, they can score at most **9 points** on the problem.
- The standard chessboard colouring gives only that a is even, which is considered trivial by the Jury, thus it is worth **0 points**.
- If a student has another colouring which proves that $2|b$, this is worth **4 points**, as in the official solution.
- If a student has another colouring which proves that $4|a$, this is worth **4 points**, as in the official solution.

Problem 4. Let a, b, c be positive real numbers such that $ab+bc+ac = a+b+c$. Prove the following inequality:

$$\sqrt{a+\frac{b}{c}}+\sqrt{b+\frac{c}{a}}+\sqrt{c+\frac{a}{b}} \leqslant \sqrt{2}\cdot\min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}.$$

(Dorlir Ahmeti)

First Solution. We can rewrite the inequality as

$$\sum_{cyc} 2\sqrt{2\left(a+\frac{b}{c}\right)} \leqslant 4\cdot\min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}$$

and distinguish two cases based on what the right hand side is.

Case 1. $\min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\} = \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$.
Using *AM-GM* inequality, we have

$$\sum_{cyc} 2\sqrt{2\left(a+\frac{b}{c}\right)} \leqslant \sum_{cyc}\left(2+a+\frac{b}{c}\right) = 6+a+b+c+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}.$$

2 points.

Hence, it is enough to prove

$$6+a+b+c+\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \leqslant 4\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$$
$$\iff 6+a+b+c \leqslant 3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right). \tag{1}$$

Applying *AM-GM* inequality we obtain

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geqslant 2\cdot 3\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}} = 6 \tag{2}$$

and using *Cauchy-Schwarz* inequality together with the condition allows us to conclude:

$$(ab+bc+ac)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geqslant (a+b+c)^2 = (a+b+c)(ab+bc+ac)$$
$$\implies \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geqslant a+b+c. \tag{3}$$

2 points.

Combining results (2) and (3) yields (1).

Case 2. $\min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\} = \frac{b}{a}+\frac{c}{b}+\frac{a}{c}$.
Using *AM-GM* inequality, we have

$$\sum_{cyc} 2\sqrt{2\left(a+\frac{b}{c}\right)} = \sum_{cyc} 2\sqrt{\frac{2a}{c}\left(c+\frac{b}{a}\right)} \leqslant \sum_{cyc}\left(\frac{2a}{c}+c+\frac{b}{a}\right) = a+b+c+3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

4 points.

Hence, it is enough to prove

$$a+b+c+3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) \leqslant 4\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$$
$$\iff a+b+c \leqslant \frac{b}{a}+\frac{c}{b}+\frac{a}{c}.$$

Using *Cauchy-Schwarz* inequality together with the condition allows us to conclude

$$(ab+bc+ac)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) \geqslant (a+b+c)^2 = (a+b+c)(ab+bc+ac)$$
$$\implies \frac{b}{a}+\frac{c}{b}+\frac{a}{c} \geqslant a+b+c$$

2 points.

which is exactly what we wanted to prove.

Second Solution. Using the substitution $m = \min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}$, we can rewrite the inequality as

$$\frac{1}{3}\left(\sqrt{a+\frac{b}{c}}+\sqrt{b+\frac{c}{a}}+\sqrt{c+\frac{a}{b}}\right) \leqslant \frac{m\sqrt{2}}{3}.$$

Recognizing the left hand side as an arithmetic mean, we may apply the *QM-AM* inequality to obtain

$$\frac{1}{3}\left(\sqrt{a+\frac{b}{c}}+\sqrt{b+\frac{c}{a}}+\sqrt{c+\frac{a}{b}}\right) \leqslant \sqrt{\frac{a+\frac{b}{c}+b+\frac{c}{a}+c+\frac{a}{b}}{3}}.$$

We're now left with proving

$$\frac{a+\frac{b}{c}+b+\frac{c}{a}+c+\frac{a}{b}}{3} \leqslant \left(\frac{m\sqrt{2}}{3}\right)^2$$

which can be written as:

$$\frac{3}{2}(a+b+c)+\frac{3}{2}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \leqslant m^2. \tag{1}$$

1 point.

We distinguish two cases based on the value of m:

Case 1. $m = \frac{b}{a}+\frac{c}{b}+\frac{a}{c}$.

Expanding the right hand side of (1) and cancelling out $\frac{3}{2}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$ turns the inequality into

$$\frac{3}{2}(a+b+c) \leqslant \frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}+\frac{1}{2}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

Multiplying both sides by $2(ab+bc+ac)$ and making use of the given condition on the left hand side gives us:

$$3(a+b+c)^2 \leqslant 2\left(\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}\right)(ab+bc+ac)+\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)(ab+bc+ac).$$

We may now apply *Cauchy-Schwarz* inequality to obtain $\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)(ab+bc+ac) \geqslant (a+b+c)^2$

2 points.

and this leaves us with proving the following:

$$(a+b+c)^2 \leqslant \left(\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}\right)(ab+bc+ac). \tag{2}$$

We now make use of a well known lemma:

Lemma 1. *For positive real numbers x, y, z one has* $\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geqslant \frac{x+y+z}{\sqrt[3]{xyz}}$.

Proof. Applying *AM-GM* inequality we obtain:

$$\frac{x}{y}+\frac{x}{y}+\frac{y}{z} \geqslant 3\sqrt[3]{\frac{x^2y}{y^2z}} = \frac{3x}{\sqrt[3]{xyz}},$$
$$\frac{y}{z}+\frac{y}{z}+\frac{z}{x} \geqslant 3\sqrt[3]{\frac{y^2z}{z^2x}} = \frac{3y}{\sqrt[3]{xyz}},$$
$$\frac{z}{x}+\frac{z}{x}+\frac{x}{y} \geqslant 3\sqrt[3]{\frac{z^2x}{x^2y}} = \frac{3z}{\sqrt[3]{xyz}}.$$

Summing up the above three inequalities finishes the proof of the lemma. □

3 points.

Applying the lemma we obtain $\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2} \geqslant \frac{a^2+b^2+c^2}{\sqrt[3]{a^2b^2c^2}}$ and applying *AM-GM* we obtain $ab+bc+ac \geqslant 3\sqrt[3]{a^2b^2c^2}$, which together used in (2) mean that we only need to prove

$$(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$$

and this is equivalent to $(a-b)^2+(b-c)^2+(a-c)^2 \geqslant 0$.

1 point.

Case 2. $m = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$.
Expanding the right hand side of (1) turns the inequality into

$$\frac{3}{2}(a+b+c) + \frac{3}{2}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \leqslant \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + 2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right). \tag{3}$$

Since $m = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$, we have that $\frac{3}{2}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \leqslant \frac{3}{2}\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)$ and using this in (3), we're left with proving:

$$\frac{3}{2}(a+b+c) \leqslant \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + \frac{1}{2}\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right).$$

1 point.

The rest of the proof is now analogous to the steps we used to solve the first case, namely multiplying both sides by $2(ab+bc+ac)$ and making use of the given condition, applying *Cauchy-Schwarz* inequality to prove $\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)(ab+bc+ac) \geqslant (a+b+c)^2$, making use of the lemma to prove $\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geqslant \frac{a^2+b^2+c^2}{\sqrt[3]{a^2b^2c^2}}$, making use of *AM-GM* inequality to obtain $ab+bc+ac \geqslant 3\sqrt[3]{a^2b^2c^2}$ and finally proving $(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$.

2 points.

Third Solution. Let $m = \min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}$ and $n = \max\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}$.

0 points.

Using *Cauchy-Schwarz* inequality, we obtain the following:

$$\begin{aligned}\sqrt{a+\frac{b}{c}}+\sqrt{b+\frac{c}{a}}+\sqrt{c+\frac{a}{b}} &= \sqrt{\left(\sqrt{\frac{ac+b}{c}}+\sqrt{\frac{ab+c}{a}}+\sqrt{\frac{bc+a}{b}}\right)^2} \\ &\leqslant \sqrt{(ac+b+ab+c+bc+a)\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)}\end{aligned}$$

2 points.

Now by using $ab+bc+ac = a+b+c$, we get:

$$\sqrt{(ac+b+ab+c+bc+a)\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)} = \sqrt{2\,(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}.$$

Therefore, we want to show

$$\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \leqslant m. \tag{1}$$

0 points.

We proceed by proving

$$m^2 \geqslant 3+2n. \tag{2}$$

Proof. Using *AM-GM* inequality, we get the following:

$$\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2} \geqslant 3.$$

Applying this result, we see that

$$\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)^2 = \frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geqslant 3+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

Analogously, we also get that $\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2 \geqslant 3+2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$, which proves (2). □

2 points.

Now $m \leqslant n$ along with (2) yields

$$\begin{aligned}\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} &= \sqrt{3+\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)} \\ &= \sqrt{3+m+n} \\ &\leqslant \sqrt{3+2n} \\ &\leqslant \sqrt{m^2} = m\end{aligned}$$

which is exactly (1).

6 points.

Notes on marking:

- In the third solution, considering only one case for $m \neq n$ and completing the proof is worth **8 points**. Full points are awarded if the analogy to the other case is mentioned.
- Proving $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geqslant 3$ should not be awarded any points as this claim is considered trivial.
- In the first solution, proving $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geqslant a+b+c$ (or the analogous version) and not applying this inequality in both cases such that the application leads to the solution should only be awarded **2 points**.

9TH EUROPEAN MATHEMATICAL CUP

12th December 2020 - 20th December 2020

Senior Category

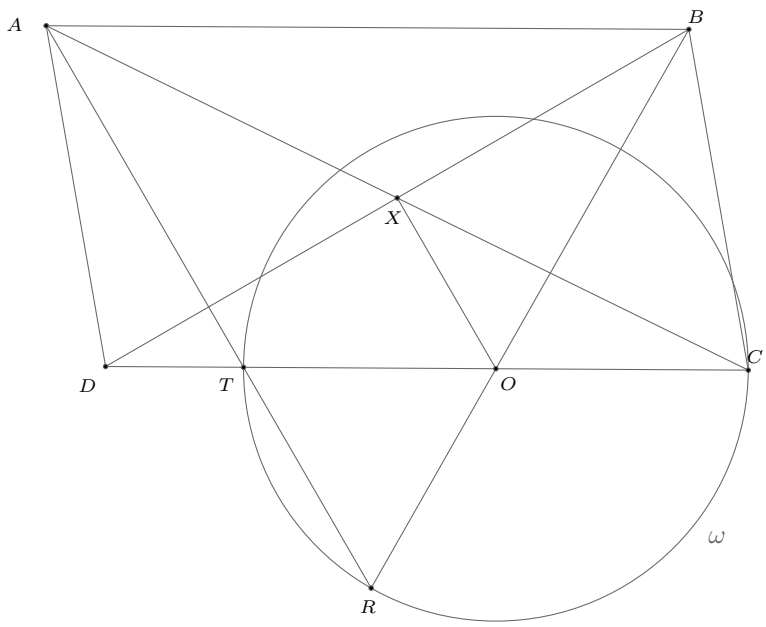
Problems and Solutions

Problem 1. Let $ABCD$ be a parallelogram in which $|AB| > |BC|$. Let O be a point on the line CD such that $|OB| = |OD|$. Let ω be a circle with center O and radius $|OC|$. If T is the second intersection of ω and CD, prove that AT, BO and ω are concurrent.

First Solution. Let R denote the intersection od ω and line BO such that O is located between B and R. We will prove that A, T and R are collinear.

0 points.

Let X be the intersection of the diagonals of $ABCD$.



We know that X is the midpoint of $\overline{AC}$ and O is the midpoint of $\overline{TC}$ so we conclude that $XO||AT$.

2 points.

X is also the midpoint of $\overline{BD}$ so, since triangle OBD is isosceles, $OX \perp BD$.

2 points.

This means that $AT \perp BD$.

1 point.

Now because of $|DO| = |BO|$ we have

$$\angle DOR = 2\angle ODB$$

and because $|OT| = |OR|$ we have

$$\angle OTR = 90^o - \angle ODB$$

2 points.

Finally we have

$$\angle ATD = 90^o - \angle BDC = \angle OTR$$

and so A, T and R are collinear as desired.

3 points.

Second Solution. Define R as the intersection of the ray BO with ω such that O is between B and R. We will prove that A, T and R are collinear.

0 points.

Since $|BO| = |DO|$ and $|OR| = |OC|$, we have:

$$|BR| = |BO| + |OR| = |DO| + |OC| = |CD| = |BA|.$$

Therefore, triangle BRA is isosceles.

5 points.

Now, due to the triangles TOR and BRA being isosceles, we have:

$$|\angle BRA| = \frac{180^\circ - |\angle RBA|}{2}$$

and

$$\frac{180^\circ - |\angle ROT|}{2} = |\angle ORT| = |\angle BRT|$$

.

2 points.

Finally, since $|\angle RBA| = |\angle TOR|$, we have

$$|\angle BRA| = |\angle BRT|$$

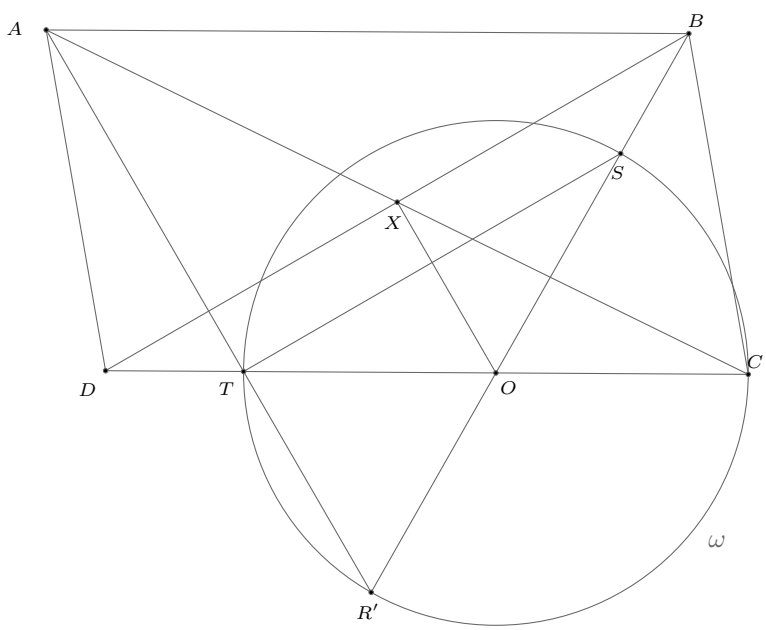
, so R, T and A are collinear, which proves the claim.

3 points.

Third Solution. Let R' be the intersection of line AT and ω different from T. We will prove that points B, O and R' are collinear.

0 points.

Let X be the intersection of the diagonals of the parallelogram $ABCD$.



Now as in the first solution we conclude that $XO||AT$ and $OX \perp BD$, which leads to AT being perpendicular to BD.

5 points.

Let S be the intersection of ω and ray OB. Since triangles ODB and OTS are isosceles with $\angle DOB = \angle TOS$, these triangles are similar, which means that $TS||BD$.

2 points.

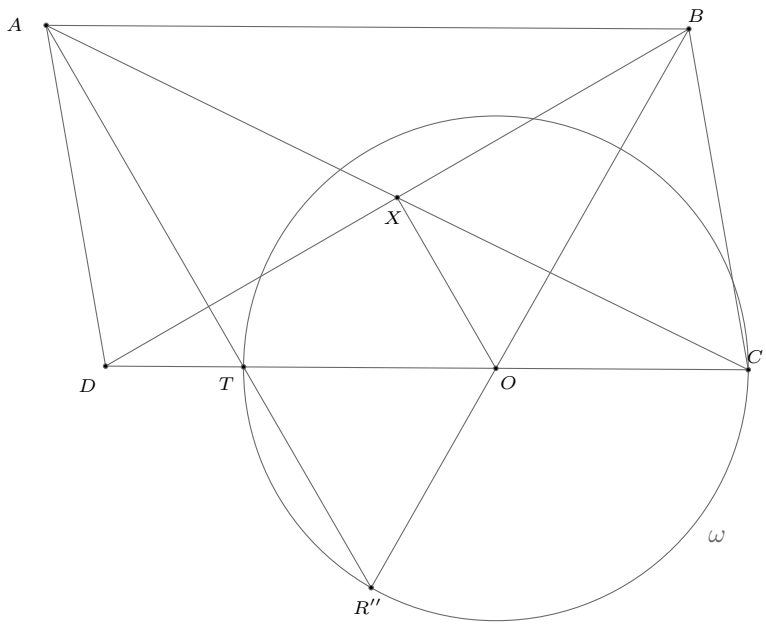
From this it follows that $ST \perp AT$, i.e. $\angle STR' = 90^\circ$. This means that $\overline{SR'}$ is the diameter of ω, and as we know that B, S and O are collinear, we conclude that B, O and R' are collinear.

3 points.

Fourth Solution. Let R'' be the intersection of lines AT and BO. We will show that R'' lies on the circle ω.

0 points.

Let X be the intersection of the diagonals of the parallelogram $ABCD$.



As in the first solution we conclude that $XO||AT$, $OX \perp BD$ and $AT \perp BD$.

5 points.

Denote $\angle TR''O = \alpha$. Since $AT \perp BD$, we have

$$\angle OBD = 90^\circ - \alpha.$$

Now, due to the triangle ODB being isosceles, we have

$$\angle ODB = 90^\circ - \alpha.$$

2 points.

Using again the fact that $AT \perp BD$, it follows that

$$\angle R''TO = \angle ATD = \alpha.$$

We can now conclude that $|OT| = |OR''|$, which proves the claim.

3 points.

Notes on marking:

- Points from different solutions are not additive. Student's score should be the maximum of points scored over all solutions.
- If a student has a partial solution with analytic methods, only points for proving facts that can be expressed in geometric ways and lead to a complete solution can be awarded.

Problem 2. Let n and k be positive integers. An n-tuple $(a_1, a_2, \ldots, a_n)$ is called a *permutation* if every number from the set $\{1, 2, \ldots, n\}$ occurs in it exactly once. For a permutation $(p_1, p_2, \ldots, p_n)$, we define its *k-mutation* to be the n-tuple

$$(p_1 + p_{1+k}, p_2 + p_{2+k}, \ldots, p_n + p_{n+k}),$$

where indices are taken modulo n. Find all pairs (n, k) such that every two distinct permutations have distinct k-mutations.

Remark: For example, when $(n, k) = (4, 2)$, the 2-mutation of $(1, 2, 4, 3)$ is $(1+4, 2+3, 4+1, 3+2) = (5, 5, 5, 5)$.

(Borna Šimić)

First Solution. Let f denote the function that, when given a permutation, returns its k-mutation.
Let $M(a, b)$ denote the greatest common divisor of a and b.
The answer is all (n, k) such that $n/M(n, k)$ is odd.

Suppose that $n/M(n, k)$ is odd.
Consider permutations p, q such that $f(p) = f(q)$. Suppose for the sake of contradiction that there exists some $t \leqslant n$ such that $p_t > q_t$. We have:

$$p_t + p_{t+k} = q_t + q_{t+k}$$

so we must have $p_{t+k} < q_{t+k}$, and $p_{t+2k} > q_{t+2k}$. Inductively, we obtain $p_{t+dk} < q_{t+dk}$ for all odd d (where the indices are taken modulo n).

2 points.

However, $n/M(n, k)$ is odd and we have $p_{t+nk/M(n,k)} = p_t$ and $q_{t+nk/M(n,k)} = q_t$. However, then $p_t < q_t$, which is a contradiction. Therefore, $p = q$, which proves that all (n, k) for which $n/M(n, k)$ is odd are solutions.

3 points.

We will now show that when $n/M(n, k)$ is even, there exist distinct permutations p, q such that $f(p) = f(q)$.
Firstly, fix n, and for $(n, k) = (2m, 1)$ for some $m \in \mathbb{N}$ take:

$$p^1 = (1, 2, 3, \ldots 2m-1, 2m)$$

$$q^1 = (2, 1, 4, \ldots 2m, 2m-1)$$

It's easy to see that $f(p^1) = f(q^1)$.

1 point.

Now, if $k = 2u - 1$ for some $u \geqslant 1$ such that $M(n, 2u-1) = 1$, define permutations p^u, q^u by taking

$$p^u_{(m-1)k+1} = p^1_m \text{ and } q^u_{(i-1)k+1} = q^1_i,$$

where indices are taken modulo n. (For example, for p^1 and p^u, $p^u_1 = p^1_1 = 1, p^u_{k+1} = p^1_2 = 2$ and so on).
As k and n are relatively prime, p^u and q^u are well defined, because the map $x \mapsto (x-1)k + 1$ is a bijection on the set of residues modulo n. Furthermore, it's easy to see that $f(p^u) = f(q^u)$ holds, because $f(p^1) = f(q^1)$ holds.

2 points.

Finally, a construction for $(n, 2u-1)$ can be expanded to a construction for $(ln, l(2u-1))$, by defining $p(lj) = p^u(j)$ and $q(lj) = q^u(j)$ for every j, and setting $p(x) = q(x)$ for x which are not divisible by l (it is not important how p and q are defined on the set of numbers not divisible by l, it's only important that they are equal on this set). Since $f(p^u) = f(q^u)$, we conclude that $f(p) = f(q)$ also holds.

Since any pair of positive integers (n, k) for which $n/M(n, k)$ is even can be written in this form, we've proved the claim.

2 points.

Second Solution. Let the notation be the same as in the first solution. Let d be an odd positive integer. Consider some permutations p, q such that $f(p) = f(q)$. This gives us the following sequence of equations for $i = 1, 2, \ldots, n$:

$$p_i + p_{i+k} = q_i + q_{i+k} \tag{1}$$

$$p_{i+k} + p_{i+2k} = q_{i+k} + q_{i+2k} \tag{2}$$

$$\vdots \quad = \quad \vdots$$

$$p_{i+(d-2)k} + p_{i+(d-1)k} = q_{i+(d-2)k} + q_{i+(d-1)k} \tag{$d-1$}$$

$$p_{i+(d-1)k} + p_{i+dk} = q_{i+(d-1)k} + q_{i+dk} \tag{d}$$

We telescope the equations: $(d-1)-(d-2)+(d-3)-\ldots+\ldots-(1)$. Since d is odd, we obtain:

$$p_{i+(d-1)k} - p_i = q_{i+(d-1)k} - q_i$$

We subtract that from (d) and obtain: $p_{i+dk} + p_i = q_{i+dk} + q_i$.

2 points.

Since this equality holds for every odd d, it also holds for $n/M(n,k)$. Since $p_{i+nk/M(n,k)} = p_i$ and $q_{i+nk/M(n,k)} = q_i$, we conclude that $2p_i = 2q_i$ for all i. Therefore, $p = q$.

3 points.

The case where $n/M(n,k)$ is even is the same as in the first solution.

5 points.

Notes on marking:

- Note that the set of solutions can also be characterized as the set of all pairs (n,k) such that $\nu_2(n) \leqslant \nu_2(k)$, where $\nu_2(x)$ denotes the largest nonnegative integer y such that $2^y \mid x$. Of course, this characterization or any other trivially equivalent characterization of the set of solutions is valid.

Problem 3. Let p be a prime number. Troy and Abed are playing a game. Troy writes a positive integer X on the board, and gives a sequence $(a_n)_{n\in\mathbb{N}}$ of positive integers to Abed. Abed now makes a sequence of moves. The n-th move is the following:

Replace Y currently written on the board with either $Y + a_n$ or $Y \cdot a_n$.

Abed wins if at some point the number on the board is a multiple of p. Determine whether Abed can win, regardless of Troy's choices, if

a) $p = 10^9 + 7$;

b) $p = 10^9 + 9$.

Remark: Both $10^9 + 7$ and $10^9 + 9$ are prime.

(Ivan Novak)

Solution. We will prove that Abed cannot win in either case.

0 points.

We now explain Troy's strategies. Throughout the solution, we will use fractions modulo p.

a) Suppose $p = 10^9 + 7$. Note that $p \equiv 2 \pmod 3$. Let $X = 2$. We will define the sequence $(a_n)_{n\in\mathbb{N}}$ recursively. Note that neither 2 nor $2 - 1$ is divisible by p.

Suppose we've defined $a_1, \ldots, a_{n-1}$, where $n \in \mathbb{N}$, and suppose that whatever Abed's first $n - 1$ moves are, the number on the board after these $n - 1$ moves is congruent to Y modulo p, and neither Y nor $Y - 1$ are divisible by p.
We now prove that there exists a positive integer k such that $Y + k \equiv Yk \pmod p$, and neither Yk nor $Yk - 1$ are not divisible by p.

Indeed, let $k \equiv \frac{Y}{Y-1} \pmod p$. Note that this is well defined since $Y - 1$ is not divisible by p. Then $Y + k \equiv Yk \equiv \frac{Y^2}{Y-1} \pmod p$. Note that $\frac{Y^2}{Y-1} \not\equiv 0 \pmod p$ since $Y \not\equiv 0 \pmod p$.

1 point.

Suppose for the sake of contradiction that $\frac{Y^2}{Y-1} \equiv 1 \pmod p$. This implies that $p \mid Y^2 - Y + 1$. However, this would imply $p \mid (-Y)^3 - 1$.
This means that $\operatorname{ord}_p(-Y) \mid 3$. Since $p \equiv 2 \pmod 3$ and $\operatorname{ord}_p(-Y) \mid p - 1$, it follows that $\operatorname{ord}_p(-Y) \neq 3$. This forces $\operatorname{ord}_p(-Y) = 1$. However, then $Y \equiv -1 \pmod p$, which implies $Y^2 - Y + 1 \equiv 3 \not\equiv 0 \pmod p$. Therefore, $\frac{Y^2}{Y-1} \not\equiv 1 \pmod p$.

2 points.

We define $a_n := k$. No matter what Abed's first n moves are, the number on the board after n moves is congruent to $\frac{Y^2}{Y-1}$ modulo p, which is not congruent to 0 or 1 modulo p. Therefore, Abed cannot win after n steps. Since this claim is true for any positive integer n, we conclude that Abed cannot win.

1 point.

b) Suppose $p = 10^9 + 9$. Note that $p \equiv 1 \pmod 4$, which means that there exists a positive integer z such that $z^2 \equiv -1 \pmod p$. Then there also exists a positive integer t such that $(2t - 1)^2 \equiv -1 \pmod p$.
Let $X = t$. Note that neither X nor $X - 1$ are divisible by p, and note that $4X^2 - 4X + 2 \equiv 0 \pmod p$.

1 point.

Let $a_1 \equiv \frac{X}{X-1} \pmod p$. Then $a_1 + X \equiv a_1 X \equiv \frac{X^2}{X-1} \pmod p$. Therefore, whatever Abed's first move is, the number on the board after the first move will be congruent to $\frac{X^2}{X-1}$ modulo p. Furthermore, $\frac{X^2}{X-1}$ is not divisible by p since X isn't.
Suppose for the sake of contradiction that $\frac{X^2}{X-1} \equiv 1 \pmod p$. Then $4X^2 - 4X + 4 \equiv 0 \pmod p$, but, by definition of X, $4X^2 - 4X + 2 \equiv 0 \pmod p$, which implies $2 \equiv 0 \pmod p$, which is a contradiction. Therefore, $\frac{X^2}{X-1} \not\equiv 1 \pmod p$.

1 point.

Let $a_2 \equiv \frac{X^2}{X^2-X+1} \pmod p$. Note that this is well defined since $X^2 - X + 1 \not\equiv 0 \pmod p$. Whatever Abed's second move is, the number on the board will be congruent to $\frac{X^2}{X-1} + \frac{X^2}{X^2-X+1} \equiv \frac{X^4}{(X^2-X+1)(X-1)} \pmod p$. Now note that

$$\frac{X^4}{(X^2 - X + 1)(X - 1)} \equiv X \pmod p \iff X^3 \equiv (X^2 - X + 1)(X - 1) \pmod p \iff 2X^2 - 2X + 1 \equiv 0 \pmod p,$$

which is true by definition of X. Therefore, whatever Abed's first two moves are, the number written on the board after the first two moves will be congruent to X modulo p.

4 points.

Thus, if we define $a_{2j-1} := a_1$ and $a_{2j} := a_2$ for $j \geqslant 2$, no matter what moves Abed makes, the number on the board will never be divisible by p.

0 points.

Notes on marking:

- Part a) is worth **4 points**, and part b) is worth **6 points**.
- The idea of making it impossible for Abed to affect the numbers on the board modulo p, although used in both parts, is worth **0 points** on its own.
- In part a), if a student doesn't prove that $x^2 - x + 1$ doesn't have prime divisors of the form $3k + 2$, but instead states that this fact is well known and checks that $10^9 + 7$ is of the form $3k + 2$, they should be awarded all the points intended for this part.
- In part b), the idea of 2-periodicity of the game state is worth **0 points** on its own.
- Due to overlapping arguments, if a student solves b), but does not solve a), then they get **0 points** for the very first point in part a). This point is then merged with the second block of **2 points** in part a).

Problem 4. Let $\mathbb{R}^+$ denote the set of all positive real numbers. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$xf(x+y) + f(xf(y)+1) = f(xf(x))$$

for all $x, y \in \mathbb{R}^+$.

(Amadej Kristjan Kocbek, Jakob Jurij Snoj)

First Solution. Let f be a function satisfying the equation. We split the solution into a series of claims.

Claim 1. $f(x) < f(f(1))$ for all $x > 1$.

Proof. Substituting $x = 1$ gives

$$f(y+1) + f(f(y)+1) = f(f(1)). \tag{3}$$

Since the function only attains positive values, we have $f(y+1) < f(f(1))$ for all y, and the conclusion follows. □

Claim 2. The function f is injective.

Proof. Assume the contrary and choose $a < b$ such that $f(a) = f(b)$. Substituting $y = a$ and, afterwards, $y = b$ into the original equations and comparing the equations gives

$$f(x+a) = f(x+b) \quad \text{for all } x \in \mathbb{R}^+.$$

Hence, f is periodic for all $x \geqslant P$ for some constant $P \in \mathbb{R}^+$ with period $p = b - a$. Fix some $x_1, y_1 \in \mathbb{R}^+$ with $x_1 > P$ and pick a positive integer n such that $(x_1 + np)f(x_1 + y_1) \geqslant f(f(1))$ and $(x_1 + np)f(x_1) > 1$. Substituting $x = x_1 + np, y = y_1$ gives

$$(x_1+np)f((x_1+np)+y_1) + f((x_1+np)f(y_1)+1) = f((x_1+np)f(x_1+np))$$
$$(x_1+np)f(x_1+y_1) + f((x_1+np)f(y_1)+1) = f((x_1+np)f(x_1))$$

after using periodicity to simplify the equation. Due to our choice of n and the function only attaining positive values, we have

$$f((x_1+np)f(x_1)) > (x_1+np)f(x_1+y_1) > f(f(1)).$$

However, since we have $(x_1 + np)f(x_1) > 1$, Claim 1 implies $f((x_1 + np)f(x_1)) < f(f(1))$, leading to a contradiction. Therefore, such a and b do not exist and f is injective. □

2 points.

Claim 3. $f(f(x)) = x$ for all $x \in \mathbb{R}^+$.

Proof. We substitute $y = f(y)$ into (1). Comparing the resulting equation with (1) gives:

$$f(f(y)+1) + f(f(f(y))+1) = f(y+1) + f(f(y)+1)$$
$$f(f(f(y))+1) = f(y+1)$$

Using injectivity, we get $f(f(y)) = y$ for all $y \in \mathbb{R}^+$. □

1 point.

Claim 4. For all $x \in \mathbb{R}^+$, $xf(x) \leqslant 1$. In particular, $f(a) \leqslant \frac{1}{x}$ for all $a \geqslant x$.

Proof. Assume the contrary - there exists some $c \in \mathbb{R}^+$ such that $cf(c) > 1$. Substituting $y = f(y)$ and using Claim 3, we transform the original equation:

$$xf(x+f(y)) + f(xy+1) = f(xf(x)).$$

Substituting $x = c, y = \frac{cf(c)-1}{c}$ into the above equation gives $cf(c + f((cf(c)-1)/c)) = 0$ after cancellation of the terms, a clear contradiction. The second part of the claim follows immediately. □

1 point.

Claim 5. For all $x \in \mathbb{R}^+$, we have $f(xf(x)) \leqslant 1$.

Proof. We notice

$$f(xf(x)) = xf(x+y) + f(xf(y)+1) < xf(x+y) + 1 \leqslant \frac{x}{x+y} + 1,$$

where the inequalities hold due to Claim 1 and Claim 4, respectively, as well as the identity $f(f(1)) = 1$. Assume there exists a c such that $f(cf(c)) > 1$: therefore, it should hold that

$$f(cf(c)) < \frac{c}{c+y} + 1.$$

However, the left hand side of the above inequality is independent of y. Thus, for y sufficiently large, the opposite direction of the inequality will hold since $c/(c+y)$ can get arbitrarily small, which leads to a contradiction. □

1 point.

Claim 6. For all $x \in \mathbb{R}^+$, $f(xf(x)) \geqslant 1$.

Proof. Assume the contrary. Therefore, there exists some a such that $f(af(a)) < 1$, let $f(af(a)) = 1 - e$. By Claim 4, there exists a $Y \in \mathbb{R}^+$ such that $f(y+1) < e$ for all $y > Y$. Let $d > Y$. Observing (1) after substituting $y = d$, we notice

$$f(f(d)+1) = 1 - f(d+1) > 1 - e.$$

Substituting $x = a, y = f\left(\frac{f(d)}{a}\right)$ into the original equation gives

$$1 - e = f(af(a)) = af\left(a + f\left(\frac{f(d)}{a}\right)\right) + f(f(d)+1) > 1 - e,$$

a contradiction. □

4 points.

Finally, observe Claims 5 and 6 together yield $f(xf(x)) = 1$ for all $x \in \mathbb{R}^+$. By injectivity, $xf(x)$ is constant, hence $f(x) = \frac{c}{x}$ for some constant $c \in \mathbb{R}^+$. By checking, we see $c = 1$ yields the only valid solution, $f(x) = \frac{1}{x}$.

1 point.

Second Solution. We present an alternative way of proving $f(xf(x))$ is constant after obtaining the first four claims of the first solution.

Assume there exist a and b such that $f(af(a)) - f(bf(b)) \neq 0$. Without loss of generality, we can assume $f(af(a)) - f(bf(b)) > 0$. We now substitute (x, y) with $\left(a, f\left(\frac{x}{a}\right)\right)$ and $\left(b, f\left(\frac{x}{b}\right)\right)$ and subtract the resulting equations to obtain

$$\begin{aligned}
f(af(a)) - f(bf(b)) &= af\left(a + f\left(\frac{x}{a}\right)\right) - bf\left(b + f\left(\frac{x}{b}\right)\right) + f\left(af\left(f\left(\frac{x}{a}\right)\right) + 1\right) - f\left(bf\left(f\left(\frac{x}{b}\right)\right) + 1\right) \\
&= af\left(a + f\left(\frac{x}{a}\right)\right) - bf\left(b + f\left(\frac{x}{b}\right)\right) + f\left(a \cdot \frac{x}{a} + 1\right) - f\left(b \cdot \frac{x}{b} + 1\right) \\
&= af\left(a + f\left(\frac{x}{a}\right)\right) - bf\left(b + f\left(\frac{x}{b}\right)\right).
\end{aligned}$$

1 point.

This shows that, as x varies, the expression $af\left(a + f\left(\frac{x}{a}\right)\right) - bf\left(b + f\left(\frac{x}{b}\right)\right)$ is constant. As f is an involution and thus surjective, we can choose a number $x_1 \in \mathbb{R}^+$ such that $a + f\left(\frac{x_1}{a}\right) > \frac{a}{f(af(a)) - f(bf(b))}$. Substituting x with x_1 in the above equation and using Claim 4, we obtain

$$\begin{aligned}
f(af(a)) - f(bf(b)) &= af\left(a + f\left(\frac{x_1}{a}\right)\right) - bf\left(b + f\left(\frac{x_1}{b}\right)\right) \\
&< af\left(a + f\left(\frac{x_1}{a}\right)\right) \\
&\leqslant \frac{a}{a + f\left(\frac{x_1}{a}\right)} \\
&< f(af(a)) - f(bf(b)),
\end{aligned}$$

which leads to a contradiction. Therefore, $f(xf(x))$ is constant.

4 points.

As in the first solution, this now implies $xf(x)$ is constant, therefore, f is of the form $f(x) = \frac{c}{x}$ for some constant c. We can easily check $f(x) = \frac{1}{x}$ is the only valid solution.

1 point.

Notes on marking:

- If a student doesn't check that $f(x) = \frac{1}{x}$ is indeed a solution or at least mention that it can be easily checked, they should lose **1 point**.
- Points from two marking schemes are not additive.