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# COMMON FIXED POINTS FOR TWO $T_f$ KANNAN TYPE CONTRACTIONS IN A COMPLETE METRIC SPACE

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**Abstract.** The focus in this paper are statements about common fixed points for two $T_f$ Kannan type contractions in a complete metric space $(X,d)$. In doing so we defined $T$ as continuous, injection and subsequentially convergent mapping, and $f$ as a function of the class $\Theta$ continuous monotony non-decreasing functions $f:[0,+\infty)\to[0,+\infty)$ such that $f^{-1}(0)=\{0\}$, and furthermore $f$ is sub-additive i.e. $f(x+y)\le f(x)+f(y)$, for all $x,y\in[0,+\infty)$.

## 1. INTRODUCTION

The Banach fixed-point theorem, as well as its generalizations presented by R. Kannan ([4]), S. K. Chatterjea ([7]) and P. V. Koparde, B. B. Waghmode ([3]), are well known. S. Moradi and D. Alimohammadi [9] generalized R. Kannan results using the sequentially convergent mappings. Using the sequentially convergent mappings, some generalizations of R. Kannan, S. K. Chatterjea and P. V. Koparde, B. B. Waghmode are proved [1], and also proved results about sharing fixed point for two R. Kannan, S. K. Chatterjea и P. V. Koparde, B. B. Waghmode types of mapping [5], 2006.

S. Moradi and A. Beiranvand introduced the concept of $T_f$ contractive mapping, [8], 2010, applying the $\Theta$ class of continuous monotony non-decreasing functions $f:[0,+\infty)\to[0,+\infty)$ such that $f^{-1}(0)=\{0\}$. For $f\in\Theta$, $f^{-1}(0)=\{0\}$ implies that $f(t)>0$, for all $t>0$. S. Moradi and A.Beiranvand proved that if $S$ is $T_f$ contractive mapping, then $S$ has a unique fixed point. M. Kir and H. Kiziltunc, 2014 generalized the S. Moradi and A. Beiranvand result about R. Kannan and S. K. Chatterjea types of mapping.

In our further observations we will present several results about sharing fixed points of two $T_f$ Kannan type contractions in a complete metric space, such that the function $f$, $f$ is a function of $\Theta$ class, and additionally we will suppose that it is subadditive, i.e. $f(a+b)\le f(a)+f(b)$, for all $a,b\in[0,+\infty)$.

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## 2. MAIN RESULTS

**Definition 1 ([8]).** Let $(X,d)$ be a metric space. A mapping $T:X\to X$ is said sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ also is convergence. A mapping $T$ is said subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ has a convergent subsequence.

**Definition 2 ([8]).** Let $(X,d)$ be a metric space, $S,T:X\to X$ and $f\in\Theta$. A mapping $S$ is said $T_f-$contraction if there exist $\alpha\in(0,1)$ such that

$$f(d(TSx,TSy))\le\alpha f(d(Tx,Ty)),$$

for all $x,y\in X$.

**Theorem 1.** Let $(X,d)$ be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ is such that $f(a+b)\le f(a)+f(b)$, for all $a,b\in[0,+\infty)$ and the mapping $T:X\to X$ be continuous, injection and subsequentially convergent. If there exist $\alpha>0,\beta\ge 0$ such that $2\alpha+\beta\in(0,1)$ and

$$f(d(TS_1x,TS_2y))\le\alpha(f(d(Tx,TS_1x))+f(d(Ty,TS_2y)))+\beta f(d(Tx,Ty)) \quad (1)$$

for all $x,y\in X$, then $S_1$ and $S_2$ have a unique sharing fixed point.

**Proof.** Let $x_0$ be any point of $X$ and let the sequence $\{x_n\}$ be defined as

$$x_{2n+1}=S_1x_{2n},\ x_{2n+2}=S_2x_{2n+1},\ n=0,1,2,3,\dots.$$

If there exists $n\ge 0$, such that $x_n=x_{n+1}=x_{n+2}$, then it is easy to be proven that $u=x_n$ is a sharing fixed point for $S_1$ and $S_2$. Therefore, let us assume that there no exist three consecutive equal terms of the sequence $\{x_n\}$. Then by applying the inequality (1), it is easy to prove the following inequalities:

$$f(d(Tx_{2n+1},Tx_{2n}))\le\alpha(f(d(Tx_{2n+1},Tx_{2n}))+f(d(Tx_{2n-1},Tx_{2n})))+\\+\beta f(d(Tx_{2n},Tx_{2n-1}))$$

and

$$f(d(Tx_{2n-1},Tx_{2n}))\le\alpha(f(d(Tx_{2n-2},Tx_{2n-1}))+f(d(Tx_{2n-1},Tx_{2n})))+\\+\beta f(d(Tx_{2n-2},Tx_{2n-1}))$$

.

The above stated implies that for each $n=0,1,2,\dots$ and $\lambda=\frac{\alpha+\beta}{1-\alpha}\in(0,1)$ holds:

$$f(d(Tx_{n+1},Tx_n))\le\lambda f(d(Tx_n,Tx_{n-1})). \quad (2)$$

Moreover, the inequality (2) implies

$$f(d(Tx_{n+1},Tx_n))\le\lambda^n f(d(Tx_1,Tx_0)), \quad (3)$$

for each $n=0,1,2,\dots$.So, the properties of metrics, the properties of the function $f$ and the inequality (3) imply

$$f(d(Tx_n,Tx_m)) \le \frac{\lambda^m}{1-\lambda} f(d(Tx_1,Tx_0)),$$

for all $m,n \in \mathbb{N}$, $n > m$. According to this, the sequence $\{Tx_n\}$ is Caushy sequence, and since $(X,d)$ is a complete metric space, it is convergent sequence. The mapping $T: X \to X$ is subsequentially convergent, therefore the sequence $\{x_n\}$ consists a convergent sub-sequence $\{x_{n(k)}\}$, i.e. it exists $u \in X$ such that $\lim_{k\to\infty} x_{n(k)} = u$. The continuity of $T$ implies that $\lim_{k\to\infty} Tx_{n(k)} = Tu$. But, $\{Tx_{n(k)}\}$ is a subsequence of the convergent sequence $\{Tx_n\}$. So,

$$\lim_{n\to\infty} Tx_n = \lim_{k\to\infty} Tx_{n(k)} = Tu.$$

Next, we will prove that $u \in X$ is a fixed point for the mapping $S_1$.

$$\begin{aligned} f(d(Tu,TS_1u)) &\le f(d(Tu,Tx_{2n+2})) + f(d(Tx_{2n+2},TS_1u)) \\ &= f(d(Tu,Tx_{2n+2})) + f(d(TS_2x_{2n+1},TS_1u)) \\ &\le f(d(Tu,Tx_{2n+2})) + \alpha(f(Tu,TS_1u) + f(d(Tx_{2n+1},TS_2x_{2n+1}))) + \\ &\quad + \beta f(d(Tu,Tx_{2n+1})) \\ &= f(d(Tu,Tx_{2n+2})) + \alpha(f(Tu,TS_1u) + f(d(Tx_{2n+1},Tx_{2n+2}))) + \\ &\quad + \beta f(d(Tu,Tx_{2n+1})) \end{aligned}$$

The mappings $f$ and $T$ are continuous, and applying the properties of metric for $n \to \infty$ in the last inequality, we get that

$$f(d(Tu,TS_1u)) \le \alpha f(Tu,TS_1u) + (1+\alpha+\beta) f(0)$$

But, $1-\alpha > 0$ and $f^{-1}(0) = \{0\}$, therefo.re the last inequality implies $d(Tu,TS_1u) = 0$, that is $TS_1u = Tu$. Finally, since $T$ is injection it holds that $S_1u = u$, that $u$ is a fixed point for the mapping $S_1$. Analogously, $u$ is fixed point for $S_2$, i.e. $u$ is a sharing fixed point for the both $S_1$ and $S_2$ mapping.

Further, we will prove that $S_1$ and $S_2$ have the unique sharing fixed point. Let $v \in X$ be fixed point for $S_2$, i.e. $S_2v = v$. Then

$$\begin{aligned} f(d(Tu,Tv)) &= f(d(TS_1u,TS_2v)|) \\ &\le \alpha(f(d(Tu,TS_1u)) + f(d(Tv,TS_2v))) + \beta f(d(Tu,Tv) \\ &= \alpha(f(d(Tu,Tu)) + f(d(Tv,Tv))) + \beta f(d(Tu,Tv) \\ &= 2\alpha f(0) + \beta f(d(Tu,Tv). \end{aligned}$$

And since $1-\beta > 0$ and $f^{-1}(0) = \{0\}$, applying the last inequality we get that $d(Tu,Tv) = 0$, i.e. it holds that $Tu = Tv$.

But since $T$ is injection, we get that $u = v$, that is $S_1$ and $S_2$ have a unique sharing fixed point. ■

**Corollary 1.** Let $(X,d)$ be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ is such that $f(a+b)\le f(a)+f(b)$, for all $a,b\in[0,+\infty)$ and the mapping $T:X\to X$ be continuous, injection and subsequentially convergent. If it exists $\lambda\in(0,1)$ such that

$$f(d(TS_1x,TS_2y))\le\lambda\sqrt[3]{f(d(Tx,TS_1x))\cdot f(d(Ty,TS_2y))\cdot f(d(Tx,Ty)}$$

for all $x,y\in X$, then $S_1$ and $S_2$ have an unique sharing fixed point.

**Proof.** The arithmetic-geometric mean inequality implies that

$$f(d(TS_1x,TS_2y))\le\tfrac{\lambda}{3}(f(d(Tx,TS_1x))+f(d(Ty,TS_2y))+f(d(Tx,Ty))),$$

for all $x,y\in X$. Applying the Theorem 1 for $\alpha=\beta=\frac{\lambda}{3}$ we get the above corollary. ■

**Corollary 2.** Let $(X,d)$ be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ be such that $f(a+b)\le f(a)+f(b)$,for all $a,b\in[0,+\infty)$ and the mapping $T:X\to X$ be continuous, injection and subsequentially convergent. If there exist $\alpha>0,\beta\ge0$ such that $2\alpha+\beta\in(0,1)$ and

$$f(d(TS_1x,TS_2y))\le\alpha\frac{f^2(d(Tx,TS_1x))+f^2(d(Ty,TS_2y))}{f(d(Tx,TS_1x))+f(d(Ty,TS_2y))}+\beta f(d(Tx,Ty),$$

for all $x,y\in X$, then $S_1$ and $S_2$ have a unique sharing fixed point.

**Proof.** The inequality (1) is a direct implication of the given inequality. ■

**Corollary 3.** Let $(X,d)$ be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ be such that $f(a+b)\le f(a)+f(b)$,for all $a,b\in[0,+\infty)$ and the mapping $T:X\to X$ be continuous, injection and subsequentially convergent. If it exists $\alpha\in(0,\frac{1}{2})$ such that

$$f(d(TS_1x,TS_2y))\le\alpha(f(d(Tx,TS_1x))+f(d(Ty,TS_2y))),$$

for all $x,y\in X$, then $S_1$ and $S_2$ have a unique sharing fixed point.

**Proof.** The proof is directly implied by Theorem 1, for $\beta=0$. ■

**Corollary 4.** Let $(X,d)$ be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ be such that $f(a+b)\le f(a)+f(b)$,for all $a,b\in[0,+\infty)$. If there exist $\alpha>0,\beta\ge0$ so that $2\alpha+\beta\in(0,1)$ and

$$f(d(S_1x,S_2y))\le\alpha(f(d(x,S_1x))+f(d(y,S_2y)))+\beta f(d(x,y))$$

for all $x,y\in X$, then $S_1$ and $S_2$ have a unique sharing fixed point.

**Proof.** The mapping $T:X\to X$ determined as $Tx=x$ е непрекинато, инјекција и subsequentially convergent.

Therefore, the proof is directly implied by Theorem 1, for $Tx=x$. ■

**Corollary 5.** Let $(X,d)$ be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ be such that $f(a+b)\le f(a)+f(b)$ ,for all $a,b\in[0,+\infty)$. If it exists $\alpha\in(0,\frac{1}{2})$ so that

$f(d(S_1x,S_2y))\le\alpha(f(d(x,S_1x))+f(d(y,S_2y)))$, for all $x,y\in X$ it holds , then $S_1$ and $S_2$ have a unique sharing fixed point.

**Proof.** Direct implication from the Corollary 3 for $Tx=x$ or the Corollary 4 for $\beta=0$. ■

**Corollary 6.** Let $(X,d)$ be a complete metric space $S_1,S_2:X\to X$, $f\in\Theta$ be such that $f(a+b)\le f(a)+f(b)$ ,for all $a,b\in[0,+\infty)$ and the mapping $T:X\to X$ be continuous, injection and subsequentially convergent. If there exist $p,q\in\mathbb{N}$ and $\alpha>0,\beta\ge 0$ such that $2\alpha+\beta\in(0,1)$ and

$$f(d(TS_1^px,TS_2^qy))\le\alpha(f(d(Tx,TS_1^px))+f(d(Ty,TS_2^qy)))+\beta f(d(Tx,Ty))$$

for all $x,y\in X$, then $S_1$ and $S_2$ have a unique sharing fixed point.

**Proof.** The Theorem1 implies that the mappings $S_1^p$ and $S_2^q$ have a unique common fixed point $u\in X$. So, $S_1^pu=u$ and therefore

$$S_1u=S_1(S_1^pu)=S_1^p(S_1u),$$

that is $S_1u$ is a fixed point for $S_1^p$. Analogously, $S_2^qu=u$ implies that

$$S_2u=S_2(S_2^qu)=S_2^q(S_2u),$$

that is $S_2u$ a fixed point for $S_2^q$. But, the proof of the Theorem1 implies that $S_1^p$ and $S_2^q$ *has* unique fixed points. Therefore $S_1u=u$ and $S_2u=u$. According to this, $u\in X$ is a common fixed point for $S_1$ and $S_2$.

For $v\in X$ is an arbitrary fixed point for $S_1$ and $S_2$ , we get that it is also a common fixed point for $S_1^p$ and $S_2^q$. But the mappings $S_1^p$ and $S_2^q$ have a unique common fixed point, and therefore $v=u$. ■

**Remark.** The function $f:[0,+\infty)\to[0,+\infty)$ defined as $f(t)=t$, for each $t\in[0,1)$, it is a function of $\Theta$ class and it is a subadditive. Moreover, each sequentially convergent mapping $T:X\to X$ is sub-sequentially convergent mapping. Therefore the Theorem1 and the Corollaries 1-6 [5], are directly implied by the above proved the Theorem 1 and the Corollaries 1-6, respectively.

## CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

## AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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# Some fixed point theorems in S-complete spaces

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**Abstract**

In this article we prove the existence and the uniqueness of a fixed point for a self-map $f$ on a $S$-complete space $(X, S)$ such that for all $x, y, z \in X$, $S(fx, fy, fz) \leq C \sum_{cyc}(S(x, y, y)+S(y, z, y)+S(x, z, x))$ for $0 \leq C < 1/6$, or $S(fx, fy, fz) \leq C \max\{S(a, b, c)|a, b, c \in \{x, y, z\}\}$ for $0 \leq C < 1$.

**Mathematics Subject Classification**. 45H10, 54H25
**Keywords**. $S$-metric space, $S$-complete spaces, Fixed point

## 1 Introduction

The Banach fixed point theorem [1] is a very simple and powerful theorem with a wide range of applications. It has been used by many authors for solving linear, nonlinear differential and integral equations and more recent in partial equations, fractional equations, dynamic systems, Cauchy boundary problems. Through the years this theorem has been generalized and extended by many authors in various ways and directions.
In 1963 Gahler [2] gave the concept of 2-metric space. We refer to [4] for fixed points in 2-metric space. Further, in 1992 Dhage [3] introduced the concept of $D$-metric spaces, as a modification of the concept of 2-metric space. In 2003 Mustafa and Sims in their paper [5] demonstrated that most of the claims concerning the fundamental topological properties of $D$-metric spaces are incorrect. They made an attempt to fix these problems in 2005 in [6] and these attempts resulted with the introduction of the concept of $G$-metric space. They proved the existence of fixed points of various contraction type mappings. Other authors also did some research in the area of fixed point concerning $G$-metric spaces, c.f. [7]. In 2007 Sedghi in [8] modified the concept of $D$-metric space and introduced the concept of $D^*$-metric space. He also proved a fixed point theorem in $D^*$-metric space. Later in 2012 Sedghi, Shobe and Aliouche [9] introduced the concept of $S$-metric space which differs from the previous type spaces and proved some fixed point theorems in $S$-metric spaces.

As a motivation for this research we present a simple example of the use of fixed point technique in solving Cauchy initial value problem. We start with countinious real valued function on $[a, b] \times [c, d]$. The Cauchy initial value problem consists of finding a continuously differentiable function $y$ on $[a, b]$ satisfying

the equations

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0. \tag{1}$$

We consider the Banach space $C[a, b]$ of countinuous real valued functions equiped with the norm $\|y\| = \sup\{y(x)|x \in [a, b]\}$. Integrating the equation (1) one obtais the following

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt. \tag{2}$$

The problem (2) is equivalent with the problem (1). We define the operator $T : C[a, b] \to C[c, d]$ with

$$Ty = y_0 + \int_{x_0}^{x} f(t, y(t))dt. \tag{3}$$

Hence solving (3) reduces to finding fixed point of the operator $T$.

The main problem of finding the fixed points of similarly defined operators is to find conditions concerning the operator $T$ such that it becomes some kind of contraction.

## 2 Preliminaries

The notation, definitions and elementary results given in this section are from [9]. We give the basic definitions concerning $S$-metric space, i.e. $S$-convergent sequence, $S$-Cauchy sequence and $S$-complete space, then some known examples and basic properties of $S$-metric spaces.

**Definition 2.1.** *Let $X$ be a nonempty set. A function $S : X^3 \to \mathbb{R}_0^+$ is called an $S$-metric on $X$, if for each $x, y, z, a \in X$ the following conditions are satisfied*

$(S1)$ $S(x, y, z) = 0$ *if and only if* $x = y = z$; *and*

$(S2)$ $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

*The pair $(X, S)$ is called an $S$-metric space.*

**Lemma 2.2.** *([9]) Let $(X, S)$ be an $S$-metric space. Then $S(x, x, y) = S(y, y, x)$, for all $x, y \in X$.*

**Definition 2.3.** *A sequence $(x_n)_{n=1}^{\infty}$ in a $S$-metric space $(X, S)$ is called $S$-convergent, if there is $x \in X$ such that $S(x_n, x_n, x) \to 0$, as $n \to \infty$.*

**Definition 2.4.** *A sequence $(x_n)_{n=1}^{\infty}$ in a $S$-metric space $(X, S)$ is called $S$-Cauchy, if $S(x_n, x_n, x_m) \to 0$, as $n, m \to \infty$.*

**Definition 2.5.** *An $S$-metric space $(X, S)$ is called $S$-complete, if every $S$-Cauchy sequence in $X$ converges in $X$.*

Concerning the topology one has the following.

**Definition 2.6.** *([9]) Let $(X, S)$ be an S-metric space. For $x \in X$ and $r > 0$, the open ball $B_S(x, r)$ is the set*

$$B_S(x, r) = \{y \in X \, : \, S(y, y, x) < r\}.$$

Let $\tau$ be the set of all $A \subset X$ such that for all $x \in A$ there exist $r > 0$ and $B_S(x, r) \subset A$. Then $\tau$ is a topology on $X$ induced by the $S$-metric $S$.

**Lemma 2.7.** *([9]) Let $(X, S)$ be an S-metric space. For $x \in X, r > 0$ the ball $B_S(x, r)$ is an open subset of $X$.*

Trivially but worth mentioning is the fact that the sequence convergence in $(X, S)$ is equivalent with $\tau$ convergence, where $\tau$ is induced by the $S$-metric $S$.

**Example 2.8.** *Let $X = \mathbb{R}^n$ and $\| \cdot \|$ be a norm on $X$. Then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on $X$.*

**Example 2.9.** *Let $X$ be a nonempty set and $d$ be an ordinary metric on $X$. Then $S(x, y, z) = d(x, z) + d(y, z)$ is an S-metric on $X$.*

**Example 2.10.** *Let $X = [0, 1]$. We define $S : X^3 \to \mathbb{R}_0^+$ by:*

$$S(x, y, z) = \begin{cases} 0 & , x = y = z \\ \max\{x, y, z\} & , otherwise \end{cases}.$$

*Then $S$ is an S-metric on $X$.*

**Example 2.11.** *Let $X$ be a nonempty set and $d_1, d_2$ are two metrics on $X$. Then $S(x, y, z) = d_1(x, z) + d_2(y, z)$ is an S-metric on $X$.*

**Example 2.12.** *If $X$ is a vector space over $\mathbb{R}$ and $\| \cdot \|$ is a norm on $X$, then $S(x, y, z) = \|\alpha y + \beta z - \gamma x\| + \|y - z\|$, where $\alpha + \beta = \gamma$ for every $\alpha, \beta \geq 1$, is an S-metric on $X$.*

# 3 Main results

The main idea of this article is to define certain self-mapings on $(X, S)$ and to prove the existence and the uniqueness of their fixed points.

**Theorem 3.1.** *Let $f$ be a self-map on an S-complete space $(X, S)$ such that $S(fx, fy, fz) \leq C \sum_{cyc}(S(x, y, y) + S(y, z, y) + S(x, z, x))$ for all $x, y, z \in X$, where $0 \leq C < 1/6$. Then $f$ has a unique fixed point. The sum is over all cyclic permutations of the triple $(x, y, z)$.*

*Proof.* Plugging $y = x$ in to the condition for the mapping $f$ and one obtains

$$S(fx, fx, fz) \leq 2C(S(x, z, x) + S(z, x, z)) + C(S(x, z, z) + S(z, x, x)). \quad (4)$$

Directly from $(S2)$ in the definition (2.1) it follows that $S(x,y,x) \leq S(y,y,x)$ and $S(x,y,y) \leq S(x,x,y)$ for all $x,y \in X$. Applying these inequalities in (4) we obtain,

$$S(fx,fx,fz) \leq 2C(S(z,z,x)+S(x,x,z)) + C(S(x,x,z)+S(z,z,x)).$$

Lemma 2.2 implies

$$S(fx,fx,fz) \leq 6CS(x,x,z) = MS(x,x,z). \tag{5}$$

where $0 \leq 6C = M < 1$.

Let $x_0 \in X$ be an arbitrary point. We take the orbit of $x_0$, i.e. the sequence $(x_n)_{n=1}^{\infty}$ defined with the conditions: $x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), \ldots, x_n = f^n(x_0), \cdots$. For simplicity, we use the notation $fx_n$ for $f(x_n)$. Then we have

$$\begin{aligned} S(x_n,x_n,x_{n+1}) &\leq MS(x_{n-1},x_{n-1},x_n) \leq M^2S(x_{n-2},x_{n-2},x_{n-1}) \\ &\leq M^nS(x_0,x_0,x_1). \end{aligned}$$

Thus, since $0 \leq M < 1$,

$$\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = 0. \tag{6}$$

Next, we will show that the sequence $(x_n)_{n=1}^{\infty}$ is $S$-Cauchy. From the definition of $(x_n)_{n=1}^{\infty}$ and $(S2)$ we obtain

$$\begin{aligned} S(x_n,x_n,x_m) &\leq S(x_n,x_n,fx_n) + S(x_n,x_n,fx_n) + S(x_m,x_m,fx_n) \\ &= 2S(x_n,x_n,fx_n) + S(x_m,x_m,fx_n) \\ &\leq 2S(x_n,x_n,fx_n) + S(x_m,x_m,fx_m) + S(x_m,x_m,fx_m) + S(fx_n,fx_n,fx_m) \\ &= 2[S(x_n,x_n,fx_n) + S(x_m,x_m,fx_m)] + S(fx_n,fx_n,fx_m). \end{aligned}$$

Choosing $x = x_n$ and $z = x_m$ in inequality (5), one obtains $S(fx_n,fx_n,fx_m) \leq MS(x_n,x_n,x_m)$, hence

$$S(x_n,x_n,x_m) \leq 2[S(x_n,x_n,fx_n) + S(x_m,x_m,fx_m)] + MS(x_n,x_n,x_m).$$

$$(1-M)S(x_n,x_n,x_m) \leq 2[S(x_n,x_n,fx_n) + S(x_m,x_m,fx_m)],$$

i.e.

$$S(x_n,x_n,x_m) \leq 2/(1-M)[S(x_n,x_n,fx_n) + S(x_m,x_m,fx_m)].$$

From (6) it follows that $S(x_n,x_n,x_m) \to 0$, as $n,m \to \infty$, i.e. the sequence $(x_n)_{n=1}^{\infty}$ is S-Cauchy. Since $(X,S)$ is $S$-complete, there exists $p \in X$ such that

$$\lim_{n\to\infty} S(x_n,x_n,p) = 0. \tag{7}$$

Next we will prove that $p \in X$ is a fixed point for $f$. Using $(S2)$ and Lemma 2.2 we obtain

$$\begin{aligned} S(fp,fp,p) &\leq S(fp,fp,fx_n) + S(fp,fp,fx_n) + S(p,p,fx_n) \\ &= 2S(fp,fp,fx_n) + S(p,p,fx_n) \\ &= 2S(fx_n,fx_n,fp) + S(fx_n,fx_n,p). \end{aligned}$$

We choose $x = x_n$ and $z = p$ in (5) and obtain $S(fx_n, fx_n, fp) \leq MS(x_n, x_n, p)$. Hence, $S(fp, fp, p) \leq 2MS(x_n, x_n, p) + S(fx_n, fx_n, p)$. On the other hand, $S(fx_n, fx_n, p) \leq 2S(fx_n, fx_n, x_n) + S(p, p, x_n) = 2S(x_n, x_n, fx_n) + S(x_n, x_n, p)$. Thus,

$$S(fp, fp, p) \leq (2M + 1)S(x_n, x_n, p) + 2S(x_n, x_n, x_{n+1}). \tag{8}$$

As $n \to \infty$, from (6) and (7) it follows that $S(fp, fp, p) = 0$, i.e. $fp = p$. Next we will show the uniqueness of the fixed point $p$. Let $q \in X$ be another fixed point for $f$. Then we choose $x = p$ and $z = q$ in (5) and obtain $S(p, p, q) = S(fp, fp, fq) \leq MS(p, p, q)$, i.e. $(1 - M)S(p, p, q) \leq 0$. From $0 \leq M < 1$ it follows that $S(p, p, q) = 0$. Thus, $p = q$. □

**Theorem 3.2.** *Let $f$ be a self-map on an $S$-complete space $(X, S)$ such that $S(fx, fy, fz) \leq C \max\{S(a, b, c)|a, b, c \in \{x, y, z\}\}$ for all $x, y, z \in X$, where $0 \leq C < 1$. Then $f$ has a unique fixed point.*

*Proof.* If we set $y = x$ in the inequality defining $f$ we obtain

$$\begin{aligned} S(fx, fx, fz) &\leq C \max\{S(a, b, c)|a, b, c \in \{x, z\}\} \\ &= C \max\{S(x, x, x), S(x, z, z), S(z, x, z), S(z, z, x), \\ &\quad S(x, x, z), S(x, z, x), S(z, x, x), S(z, z, z)\} \end{aligned}$$

From Lemma 2.2 and $(S2)$ we obtain $S(x, z, z) \leq S(x, x, z), S(z, x, z) \leq S(x, x, z)$, $S(x, z, x) \leq S(z, z, x) = S(x, x, z)$ and $S(z, x, x) \leq S(z, z, x) = S(x, x, z)$. Hence, $S(fx, fx, fz) \leq CS(x, x, z)$. The rest of the proof is analogous to the previous theorem.

□

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# Moore-Penrose Hermitian elements in rings with involution

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**Abstract**

A Moore-Penrose invertible element in rings with involution is Moore-Penrose Hermitian if Moore-Penrose inverse is equal to the element itself. In this paper, we present a number of new characterizations of Moore-Penrose Hermitian elements in rings with involution in purely algebraic terms.

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## 1 Introduction

Let $\mathcal{R}$ be an associative ring with unity 1, and let $a \in \mathcal{R}$. Then $a$ is group invertible if there is $a^\sharp \in \mathcal{R}$ such that

$$(1)\ aa^\sharp a = a, \quad (2)\ a^\sharp a a^\sharp = a^\sharp, \quad (3)\ aa^\sharp = a^\sharp a.$$

Recall that $a^\sharp$ is uniquely determined by previous equations. We use $\mathcal{R}^\sharp$ to denote the set of all group invertible elements of $\mathcal{R}$.

An involution $a \to a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(1)\ (a^*)^* = a, \quad (2)\ (a+b)^* = a^* + b^*, \quad (3)\ (ab)^* = b^* a^*.$$

In the rest of the paper, we assume that $\mathcal{R}$ is a ring with involution. An element $a \in \mathcal{R}$ satisfying $aa^* = a^* a$ is called normal.

The Moore-Penrose inverse of $a$, is the unique element $a^\dagger$ satisfying the equations

$$(1)\ aa^\dagger a = a, \quad (2)\ a^\dagger aa^\dagger = a^\dagger, \quad (3)\ aa^\dagger = (aa^\dagger)^*, \quad (4)\ a^\dagger a = (a^\dagger a)^*.$$

The subset of $\mathcal{R}$ consisting of elements of $\mathcal{R}$ that have a Moore-Penrose inverse will be denoted by $\mathcal{R}^\dagger$.

Several characterizations of elements $a \in \mathcal{R}^\dagger$ such that $aa^\dagger = a^\dagger a$ can be found in the literature (see [1], [6], [7], [8] and [10]). These elements are called EP. For a ring with involution $\mathcal{R}$, we will denote $\mathcal{R}^{EP} = \{a \in \mathcal{R}^\dagger : aa^\dagger = a^\dagger a\}$.

An element $a \in \mathcal{R}^\dagger$ satisfying $a^*a^\dagger = a^\dagger a^*$ is called star-dagger.

A characterization of nonnegative matrices which are equal to their Moore-Penrose inverse is derived by Berman in [2]. Many years ago, the concept of Moore-Penrose Hermitian elements in $C^*-$algebras was introduced by E. Boasso [3]: Let $\mathcal{A}$ be a $C^*$-algebra. A regular element $a \in \mathcal{A}$ is Moore-Penrose Hermitian if $a^\dagger = a$. The definition of Moore-Penrose Hermitian elements can be generalized to elements in rings with involution.

**Definition 1.1** *If $\mathcal{R}$ is a ring with involution, and $a^\dagger$ is the Moore-Penrose inverse of $a \in \mathcal{R}^\dagger$, then the element $a$ is called Moore-Penrose Hermitian if $a^\dagger = a$.*

The following result is well-know and frequently used in the rest of the paper.

**Theorem 1.1** *[4, 9] For any $a \in \mathcal{R}^\dagger$, the following is satisfied:*

(i) $(a^\dagger)^\dagger = a$,

(ii) $a^* = a^\dagger aa^* = a^*aa^\dagger$.

In [2] and [5], authors used the representation of complex matrices to explore various property of Moore-Penrose Hermitian matrices. In this paper, we use a different approach, exploiting the structure of rings with involution to investigate Moore-Penrose Hermitian elements. We give several characterizations, and the proofs are based on ring theory only.

## 2 Results

E. Boasso proved the following result in [3].

**Proposition 2.1** *Consider a $C^*$-algebra $\mathcal{A}$ and an element $a \in \mathcal{A}$. Then the following statements hold:*

(i) *Necessary and sufficient for $a$ to be a Moore-Penrose Hermitian is $a = a^3$ and $(a^2)^* = a^2$,*

(ii) *If $a$ is a Moore-Penrose Hermitian, then $a^n$ also is, $n \in \mathcal{N}$.*

(iii) *The element $a$ is a Moore-Penrose Hermitian if and only if $a^*$ is.*

(iv) *If $a$ is a Moore-Penrose Hermitian, then $\sigma(a) \subseteq \{0, -1, 1\}$, where $\sigma(a)$ denotes the spectrum of $a$.*

Observe that the previous proposition holds on Moore-Penrose Hermitian elements in rings with involution. Some new characterizations of Moore-Penrose Hermitian elements in rings with involution are given in the following results.

**Theorem 2.1** *Let $\mathcal{R}$ be a ring with involution and $a \in \mathcal{R}^\dagger$. Then $a$ is a Moore-Penrose Hermitian if and only if one of the following equivalent conditions holds:*

(i) $a^\dagger$ *is a Moore-Penrose Hermitian,*

(ii) $a^* = (a^*)^3$ *and* $(a^2)^* = a^2$,

(iii) $a^\dagger = (a^\dagger)^3$ *and* $((a^\dagger)^2)^* = (a^\dagger)^2$,

(iv) $a = a^3$ *and $a$ is an EP element,*

(v) $a^* = (a^*)^3$ *and $a^*$ is an EP element,*

(vi) $a^\dagger = (a^\dagger)^3$ *and $a^\dagger$ is an EP element,*

(vii) $a = a^3$ *and* $a^\dagger aaa^\dagger = aa^\dagger a^\dagger a$,

(viii) $a^\dagger = (a^\dagger)^3$ *and* $a^\dagger aaa^\dagger = aa^\dagger a^\dagger a$,

(ix) $a^* = aaa^*$ *(or* $a^* = a^*aa$*),*

(x) $a = a^*a^*a$ *(or* $a = aa^*a^*$*)*

(xi) $a = a^3$ *and* $a^\dagger = aa^*a^*$ *(or* $a = a^3$ *and* $a^\dagger = a^*a^*a$*),*

(xii) $a^\dagger a = aa$ *(or* $aa^\dagger = aa$*),*

(xiii) $aa^\dagger = a^\dagger a^\dagger$ *(or* $a^\dagger a = a^\dagger a^\dagger$*),*

(xiv) $a^\dagger = a^\dagger aa$ *(or* $a^\dagger = aaa^\dagger$*),*

(xv) $a = aa^\dagger a^\dagger$ *(or* $a = a^\dagger a^\dagger a$*).*

**Proof.** If $a$ is is a Moore-Penrose Hermitian, then it commutes with its Moore-Penrose inverse and $a^\dagger = a$. It is not difficult to verify that conditions (iii)-(xiii) hold.

Conversely, to conclude that $a$ is is a Moore-Penrose Hermitian, we show that the condition $a^\dagger = a$ is satisfied, or that the element is subject to one of the preceding already established conditions of this theorem.

(i) From $(a^\dagger)^\dagger = a^\dagger$ and $(a^\dagger)^\dagger = a$ follows $a^\dagger = a$. Hence, the element $a$ is a Moore-Penrose Hermitian.

(ii) The conclusion follows by (i) of Proposition 2.1 and (iii) of Proposition 2.1.

(iii) The conclusion follows by (i) of Proposition 2.1 and (i).

(iv) Suppose that $a = a^3$ and $aa^\dagger = a^\dagger a$. Then

$$a^\dagger = (a^\dagger)^2 a = (a^\dagger)^2 a^3 = ((a^\dagger)^2 a)a^2 = a^\dagger a^2 = a.$$

(v) The conclusion follows by (iii) of Proposition 2.1 and (iv).

(vi) The conclusion follows by (i) and (iv).

(vii) From $a = a^3$ and $a^\dagger aaa^\dagger = aa^\dagger a^\dagger a$, we get

$$aa^\dagger = a^3 a^\dagger = aaa^\dagger aaa^\dagger = a^3 a^\dagger a^\dagger a = aa^\dagger a^\dagger a$$

and, similar,

$$a^\dagger a = a^\dagger a^3 = a^\dagger aaa^\dagger aa = aa^\dagger a^\dagger a^3 = aa^\dagger a^\dagger a.$$

Thus, $aa^\dagger = a^\dagger a$ and the condition (iv) is satisfied.

(viii) The conclusion follows by (i) and (vii).

(ix) If $a^* = aaa^*$, then

$$a^2 = a(a^*)^* = a(a^2a^*)^* = a^2(a^2)^* = (a^2a^*)a^* = a^*a^* = (a^2)^*.$$

Multiplying $a^2 = (a^2)^*$ by $a$, we obtain $a^3 = a(a^*)^2$. Now,

$$a = (a^*)^* = (a^2a^*)^* = a(a^*)^2 = a^3.$$

Hence, the condition (i) of Proposition 2.1 holds.

If the equality $a^* = a^*aa$ applies, then the proof is analogous.

(x) The conclusion follows by (iii) of Proposition 2.1 and (ix).

(xi) Applying $a^\dagger = aa^*a^*$ and $a = a^3$, we have

$$a^* = a^\dagger aa^* = a(a^*)^2aa^* = a^3(a^*)^2aa^* = a^2(a(a^*)^2)aa^* = a^2a^\dagger aa^* = a^2a^*.$$

Therefore, the condition (ix) is satisfied.

If the equalities $a = a^3$ and $a^\dagger = a^*a^*a$ hold, then the proof is analogous.

(xii) The equality $a^\dagger a = aa$ gives

$$a^* = a^*aa^\dagger = a^*a^2.$$

Hence, the condition (ix) holds.

If the equality $aa^\dagger = aa$ applies, then the proof is analogous.

(xiii) The conclusion follows by (i) and (xii).

(xiv) Suppose that $a^\dagger = a^\dagger aa$. Then

$$aa^\dagger = aa^\dagger aa = aa.$$

Thus, the condition (xii) is satisfied.

If the equality $a^\dagger = aaa^\dagger$ holds, then the proof is analogous.

(xv) The conclusion follows by (i) and (xiv). □

In the following theorem, we assume that the element $a$ is both Moore-Penrose invertible and group invetible. Then, we study the conditions involving $a^\dagger$, $a^\sharp$ and $a$ to ensure that $a$ is a Moore-Penrose Hermitian element.

**Theorem 2.2** *Let $\mathcal{R}$ be a ring with involution and $a \in \mathcal{R}^\dagger$. Then $a$ is a Moore-Penrose Hermitian if and only if $a \in \mathcal{A}^\sharp$ and one of the following equivalent conditions holds:*

(i) $a^\dagger a = a^\sharp a^\dagger$.

(ii) $a = a^\dagger aa^\sharp$.

(iii) $a^{\dagger}a^{\sharp} = a^{\sharp}a$.

(iv) $a^{\sharp} = a^{\dagger}a^{\sharp}a^{\sharp}$.

**Proof.** (i) Multiplying $a^{\dagger}a = a^{\sharp}a^{\dagger}$ by $a^2$ from the left side, we get $aa = aa^{\dagger}$. Hence, the condition (xii) of Theorem 2.1 is satisfied.

(ii) Using $a = a^{\dagger}aa^{\sharp}$, we obtain $aa = a^{\dagger}(aa^{\sharp}a) = a^{\dagger}a$. Thus, the condition (xii) of Theorem 2.1 holds.

(iii) When we multiplying $a^{\dagger}a^{\sharp} = a^{\sharp}a$ by $a^2$ from the right side, we have $a^{\dagger}a = aa$. Hence, $a$ satisfies the condition (xii) of Theorem 2.1.

(iv) Multiplying the equality $a^{\sharp} = a^{\dagger}a^{\sharp}a^{\sharp}$ by $a$ from the right side, we obtain the condition (iii):

$$a^{\dagger}a^{\sharp} = a^{\sharp}a.\square$$

Finally, we prove the result involving Moore-Penrose Hermitian elements in a ring with involution.

**Theorem 2.3** *Let $\mathcal{R}$ be a ring with involution and let $a \in \mathcal{R}$. If $a$ is Moore-Penrose Hermitian and star-dagger, then $a$ is normal.*

**Proof.** Using $a^{\dagger} = a$ and $a^{\sharp}a^{\dagger} = a^{\dagger}a^{\sharp}$, we get

$$a^*a = a^*a^3 = a^*(a^{\dagger})^3 = (a^{\dagger})^3a^* = a^{\dagger}a^* = aa^*.$$

Thus, $a$ is normal. $\square$

# 3 Conclusions

In this paper, we consider Moore-Penrose invertible or both Moore-Penrose invertible and group invertible elements in rings with involution to characterize Moore-Penrose Hermitian elements. In the theory of complex matrices various authors used an elegant respresentation of complex matrices to investigate Moore-Penrose Hermitian matrices. In this paper, we applied a purely algebraic technique, involving some characterizations of the Moore-Penrose inverse.

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# KNOT BENDING

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**Abstract.** In this paper we point out the geometric aspect of knots. By the term knot we mean a closed self-avoiding curve in a 3-dimensional Euclidean space. We consider infinitesimal bending of knots and examine the behavior of torus knots under this type of deformation.

## 1. INTRODUCTION

A knot is a closed, self-avoiding curve. From the topological point of view, we have more general definition. In order to give a topological definition, some basic terms will be introduced.

A **homeomorphism** between two topological spaces is continuous bijection $h: X \to Y$ whose inverse is also continuous. If such a map exists, then $X$ and $Y$ are **homeomorphic** or **topology equivalent**.

Let be given a smooth closed curve $C$ in $\mathbb{R}^3$. The **homeomorphic** curves $C$, $C'$ are called **isotopic** if there exists a continuous family of curves $C_t$ depending on $t$, $(0 \leq t \leq 1)$, such that $C_t$ is homeomorphic to $C$, $C_0 = C$ and $C_1 = C'$. We say that $C$ is a **knot** if it is homeomorphic to a circle but is not isotopic to a circle.

According to this definition, another knot $C'$ is equivalent to $C$ if it can be continuously deformed into $C$ without crossing itself during the process. Equivalent knots are considered the same in the topological sense and determine an equivalence class of knots named **knot type**. It is therefore possible to think of a knot as a curve with small but positive thickness which allows us to present it as a tube. But in geometrical sense we can observe a particular representative of a knot type as a closed self-avoiding curve in 3-dimensional space.

The simplest knot is the **unknot** also known as the trivial knot which can be deformed to a geometric circle in $\mathbb{R}^3$. Two other rather easy knots are the **trefoil** and the **figure-eight knots**.

In classical knot theory, mathematicians are often interested in knot classes and how to distinguish between them. Contrary to this approach, geometric knot theory deals with the specific shape of knots and how to find or compute particularly nice representatives of a given knot class. In this sense knot theory

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studies relations between the geometry of space curves and the knot types they represent. Therefore, in geometric sense we could observe small deformations of knots as the family of different curves.

Knots are interesting not only from a mathematical point of view, but also from the aspect of other sciences such as physics, chemistry, biology, computer graphics, etc. (see [2], [3], [5], [10]).

The study of deformations dates from the ancient times and stems from purely practical problems. In physics, a deformation means a change of a shape of body under the influence of the external or internal forces. In the case when the body returns to its original shape after the termination of the effect of forces, it is said to be elastic, otherwise, if it is deformed permanently, we say it is plastic. Testing of deformations of different materials is of great importance in civil engineering. Measurement of deformations of high buildings allows us to provide an adequate level of safety and security from potential damage and disaster. Deformations are in close connection with thin elastic shell and have a huge application from the mechanical point of view. In biology, the notion of deformation has also found his place. We are talking about the elasticity of the cell membrane in connection with the fluidity that allows the proteins to move along the membrane.

In geometry, the problem of deformations is covered by so-called the **surface bending theory**. The surface bending theory considers the bending of surfaces, ie. the isometric deformations, as well as the infinitesimal bending of surfaces. Under bending, surface is included in continuous family of isometric surfaces, so that any curve on the surface preserves its arc length. The angles are also preserved. On the other hand, **infinitesimal bending** of surfaces is not an isometric deformation, or roughly speaking it is an isometric deformation with appropriate precision. Arc length is stationary under infinitesimal bending with a given precision. In the case of infinitesimal bending the surface is deformed so that in the initial moment of a deformation, the arc length on the surface is stationary, i. e. initial velocity of its change is zero.

In addition to infinitesimal bending of surfaces, infinitesimal bending of curves and manifolds is also considered in bending theory.

In the present article we consider infinitesimal bending of a knot as a closed simple curve. Some papers related to this topic are [4], [6], [8], [9].

## 2. INFINITESIMAL BENDING

Let us consider a regular curve

$$C: \mathbf{r} = \mathbf{r}(u), \quad u \in J \subseteq \mathbb{R}^3 \tag{1}$$

of a class $C^k$, $k \geq 3$, included in a family of the curves

$$C_\epsilon: \mathbf{r}_\epsilon(u) = \mathbf{r}(u) + \epsilon \mathbf{z}^{(1)}(u) + \epsilon^2 \mathbf{z}^{(2)}(u) + \cdots + \epsilon^m \mathbf{z}^{(m)}(u), \quad m \geq 1, \tag{2}$$

where $\epsilon \geq 0, \epsilon \to 0$, and we get $C$ for $\epsilon = 0$ ($C = C_0$). The fields $\mathbf{z}^{(j)}(u) \in C^k, k \geq 3, j = 1, \ldots, m$, are vector functions defined in the points of $C$.

**Definition 1.** [1] *A family of curves* $C_\epsilon$ (2) *is an* **infinitesimal bending of the order** $m$ **of the curve** $C$ (1) *if*

$$ds_\epsilon^2 - ds^2 = o(\epsilon^m). \tag{3}$$

*The field* $\mathbf{z}^{(j)} = \mathbf{z}^{(j)}(u)$ *is the* **infinitesimal bending field of the order** $j$, $j = 1, \dots, m$, *of the curve* $C$.

Based on the definition of the infinitesimal bending of the order $m$ and Eq. (3), the next relation is valid

$$d\mathbf{r}_\epsilon^2 - d\mathbf{r}^2 = (d\mathbf{r}_\epsilon - d\mathbf{r}) \cdot (d\mathbf{r}_\epsilon + d\mathbf{r}) = o(\epsilon^m).$$

From here we have

$$d\left(\sum_{j=1}^{m} \epsilon^j \mathbf{z}^{(j)}\right) \cdot d\left(2\mathbf{r} + \sum_{j=1}^{m} \epsilon^j \mathbf{z}^{(j)}\right) = o(\epsilon^m),$$

or more precise

$$\left(\epsilon d\mathbf{z}^{(1)} + \epsilon^2 d\mathbf{z}^{(2)} + \cdots + \epsilon^m d\mathbf{z}^{(m)}\right) \cdot \left(2d\mathbf{r} + \epsilon d\mathbf{z}^{(1)} + \epsilon^2 d\mathbf{z}^{(2)} + \cdots + \epsilon^m d\mathbf{z}^{(m)}\right) = o(\epsilon^m).$$

The necessary and sufficient condition for the left side to be infinitesimal value with respect to $\epsilon^m$ is to be

$$d\mathbf{r} \cdot d\mathbf{z}^{(1)} = 0,\ 2d\mathbf{r} \cdot d\mathbf{z}^{(j)} + \sum_{l=1}^{j-1} d\mathbf{z}^{(l)} \cdot d\mathbf{z}^{(j-l)} = 0,\ j = 2, \dots, m. \tag{4}$$

According to that, the next theorem states.

**Theorem 1.** [1] *Necessary and sufficient condition for the curves* $C_\epsilon$, (2), *to be infinitesimal bending of the* $m$ *−th order of the curve* $C$, (1), *is to be valid* (4).

If infinitesimal bending is reduced to rigid motion of the curve, without internal deformations, we say it is **trivial** infinitesimal bending. The corresponding bending field is also called trivial.

Specially, infinitesimal bending of the first order, or shorter, infinitesimal bending, is a family of curves

$$C_\epsilon: \mathbf{r}_\epsilon(u) = \mathbf{r}(u) + \epsilon \mathbf{z}(u),$$

where $\mathbf{z}(u)$ is an infinitesimal bending field (of the first order).

The following theorem is related to determination of the infinitesimal bending field of a curve $C$.

**Theorem 2.** [12] *The infinitesimal bending field for the curve* $C$ *is*

$$\mathbf{z}(u) = \int [p(u)\mathbf{n}_1(u) + q(u)\mathbf{n}_2(u)]\, du, \tag{5}$$

*where* $p(u)$ *and* $q(u)$ *are arbitrary integrable functions, and vectors* $\mathbf{n}_1(u)$ *and* $\mathbf{n}_2(u)$ *are respectively unit principal normal and binormal vector fields of a curve* $C$.

Similarly, for the infinitesimal bending of the second order we have the following theorem.

**Theorem 3.** [7] The infinitesimal bending fields of the first and the second order of the curve $C$ are respectively

$$\mathbf{z}^{(1)} = \int [p(u)\mathbf{n}_1(u) + q(u)\mathbf{n}_2(u)]\, du,$$
$$\mathbf{z}^{(2)} = \int \left[-\frac{p^2(u)+q^2(u)}{2\|\dot{\mathbf{r}}\|}\mathbf{t} + r(u)\mathbf{n}_1 + g(u)\mathbf{n}_2\right] du,$$

where $p(u), q(u), r(u), g(u)$ are arbitrary integrable functions and vectors $\mathbf{t}(u)$, $\mathbf{n}_1(u)$, $\mathbf{n}_2(u)$ are unit tangent, principal normal and binormal vector fields, respectively, of the curve $C$.

Under infinitesimal bending, geometric magnitudes of the curve are changed which is described with variations of these geometric magnitudes.

**Definition 2.** [11] Let $A = A(u)$ be a magnitude which characterizes a geometric property on the curve $C$ and $A_\epsilon(u)$ the corresponding magnitude on the curve $C_\epsilon$ being infinitesimal bending of the curve $C$, and set

$$\Delta A = A_\epsilon - A = \epsilon\delta A + \epsilon^2\delta^2 A + \cdots + \epsilon^n\delta^n A + \cdots$$

The coefficients $\delta A$, $\delta^2 A, \ldots, \delta^n A$ are **the first, the second**, ..., **the n-th variation** of the geometric magnitude $A$, respectively, under the infinitesimal bending $C_\epsilon$ of the curve $C$.

## 3. INFINITESIMAL BENDING OF KNOTS

In geometric sense, small deformations of a knot (a particular curve from the corresponding knot type) can be considered as the family of different curves. We are going to demonstrate this fact through examples.

Let us consider the trefoil knot

$$\mathbf{r}(u) = (4\cos 2u + 2\cos u, 4\sin 2u - 2\sin u, \sin 3u).$$

The field

$$\mathbf{z} = \int \frac{d\mathbf{r}}{du} \times \frac{d^2\mathbf{r}}{du^2} du = (60\cos u - 6\cos 2u - \frac{3}{2}\cos 4u + \frac{12}{5}\cos 5u,$$
$$-60\sin u - 6\sin 2u + \frac{3}{2}\sin 4u + \frac{12}{5}\sin 5u, 124u - \frac{16}{3}\sin 3u),$$

$u \in [0, 2\pi]$, is corresponding infinitesimal bending field. This field is obtained, according to (5), for $p(u) = 0$ and $q(u) = \left\|\frac{d\mathbf{r}}{du} \times \frac{d^2\mathbf{r}}{du^2}\right\|$. Obviously, $\mathbf{z}(0) \neq \mathbf{z}(2\pi)$, and the knot "is torn" under this infinitesimal bending, see Figs. 1.

If we want to get a family of closed curves under infinitesimal bending of a knot, we must specify the condition $\mathbf{z}(0) = \mathbf{z}(2\pi)$ for the infinitesimal bending field.

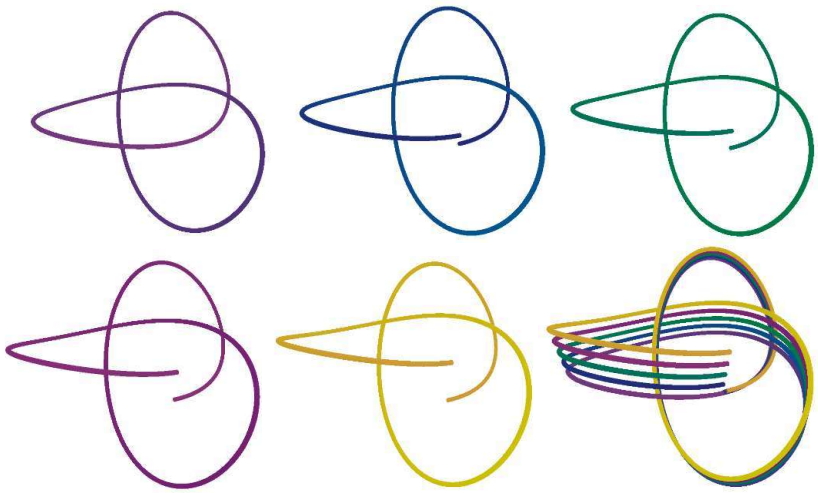

**Figure 1:** The trefoil knot and its infinitesimal bending for $\epsilon = 0.0005, 0.001, 0.0015, 0.002$.

For $p(u) = \left\|\left(\frac{d\mathbf{r}}{du} \times \frac{d^2\mathbf{r}}{du^2}\right) \times \frac{d\mathbf{r}}{du}\right\|$, $q(u) = 0$, we obtain

$$\mathbf{z} = \int \left(\frac{d\mathbf{r}}{du} \times \frac{d^2\mathbf{r}}{du^2}\right) \times \frac{d\mathbf{r}}{du} du$$

$$= \left(303 \sin u - 540 \sin 2u - \frac{53}{2} \sin 4u + \frac{46}{5} \sin 5u + \frac{9}{7} \sin 7u + \frac{9}{4} \sin 8u, 303 \cos u + 540 \cos 2u - \frac{53}{2} \cos 4u - \frac{46}{5} \cos 5u + \frac{9}{7} \cos 7u - \frac{9}{4} \cos 8u, -12(-17 \cos 3u + \cos 6u)\right).$$

All bent curves are also closed, see Figs. 2.

## 4. FRENET-SERRET FRAME UNDER INFINITESIMAL BENDING

Let us consider an infinitesimal bending of the second order of the knot $C: \mathbf{r} = \mathbf{r}(s) = \mathbf{r}(u(s)), s \in I \subseteq \mathbb{R}$, parameterized by arc length $s$:

$$C_\epsilon: \mathbf{r}_\epsilon(s) = \mathbf{r}(s) + \epsilon \mathbf{z}^{(1)}(s) + \epsilon^2 \mathbf{z}^{(2)}(s).$$

Since the vector fields $\mathbf{z}^{(1)}$ and $\mathbf{z}^{(2)}$ are defined in the points of $C$, they can be presented in the form

$$\mathbf{z}^{(j)} = z^{(j)}\mathbf{t} + z_1^{(j)}\mathbf{n}_1 + z_2^{(j)}\mathbf{n}_2, \quad j = 1,2,$$

where $z^{(j)}\mathbf{t}$ is tangential and $z_1^{(j)}\mathbf{n}_1 + z_2^{(j)}\mathbf{n}_2$ is normal component, $z^{(j)}$, $z_1^{(j)}$, $z_2^{(j)}$ are the functions of $s$; $\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2$ are unit vectors of Frenet-Serret frame which form an orthonormal basis spanning $\mathbb{R}^3$.

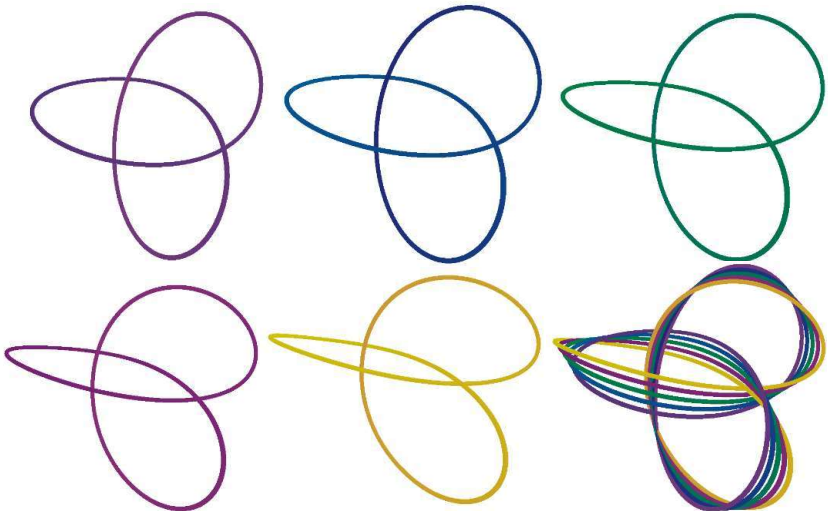

**Figure 1:** The trefoil knot and its infinitesimal bending for $\epsilon = 0.00025, 0.0005, 0.00075, 0.001$.

Under infinitesimal bending of the second order of knots, unit vectors of Frenet-Serret frame make changes which is described by their variations of the first and the second order. Most of them have already been determined. Thus, $\delta\mathbf{t}, \delta\mathbf{n}_1, \delta\mathbf{n}_2, \delta^2\mathbf{t}$ are obtained in [7], and $\delta^2\mathbf{n}_1$ in [8]. It remains to examine $\delta^2\mathbf{n}_2$, which we will do below.

**Theorem 4.** Under second order infinitesimal bending of the knot $C$, the second variation of the binormal vector is

$$\delta^2\mathbf{n}_2 = f_t\mathbf{t} + f_{n_1}\mathbf{n}_1 + f_{n_2}\mathbf{n}_2,$$

where

$$f_t = -\left(z_2^{(2)'} + \tau z_1^{(2)}\right) + \frac{1}{k}\left(kz^{(1)} + z_1^{(1)'} - \tau z_2^{(1)}\right)\left(k\tau z^{(1)} + 2\tau z_1^{(1)'} + \tau' z_1^{(1)} + z_2^{(1)''} - \tau^2 z_2^{(1)}\right),$$

$$f_{n_1} = \frac{1}{\tau}\Big\{kg_t + g_{n_1}' - \tau g_{n_2} + k\left(kz^{(2)} + z_1^{(2)'} - \tau z_2^{(2)}\right) + \left(k'z^{(1)} + z_1^{(1)''} + (k^2 - \tau^2)z_1^{(1)} - 2\tau z_2^{(1)'} - \tau' z_2^{(1)}\right)\left(kz^{(1)} + z_1^{(1)'} - \tau z_2^{(1)}\right) + \frac{1}{k}\Big[\tau' z^{(1)} + 2k\tau z_1^{(1)} + kz_2^{(1)'} + \Big(\frac{1}{k}(2\tau z_1^{(1)'} + \tau' z_1^{(1)} + z_2^{(1)''} - \tau^2 z_2^{(1)})\Big)'\Big]\left(k\tau z^{(1)} + 2\tau z_1^{(1)'} + \tau' z_1^{(1)} + z_2^{(1)''} - \tau^2 z_2^{(1)}\right)\Big\},$$

$$f_{n_2} = -\frac{1}{2}\left[\left(z_2^{(1)'} + \tau z_1^{(1)}\right)^2 + \frac{1}{k^2}\left(k\tau z^{(1)} + 2\tau z_1^{(1)'} + \tau' z_1^{(1)} + z_2^{(1)''} - \tau^2 z_2^{(1)}\right)^2\right],$$

$k$ is the curvature, $\tau$ is the torsion, $g_t, g_{n_1}, g_{n_2}$ are the tangent, the normal and the binormal component of $\delta^2\mathbf{n}_1$, respectively.

*Proof.* Applying the second variation of the equation $\mathbf{n}_2 \cdot \mathbf{t} = 0$ , we obtain

$$\mathbf{t} \cdot \delta^2\mathbf{n}_2 = -\mathbf{n}_2 \cdot \delta^2\mathbf{t} - \delta\mathbf{n}_2 \cdot \delta\mathbf{t}.$$

We used here the following property of the second variation:

$$\delta^2 AB = A\delta^2 B + B\delta^2 A + \delta A \delta B,$$

whereby $A$ and $B$ are some magnitudes which characterize some geometric properties of the knot. If we use the expressions for $\delta\mathbf{t}$, $\delta^2\mathbf{t}$ and $\delta\mathbf{n}_2$ from [7], we obtain $f_t = \mathbf{t} \cdot \delta^2\mathbf{n}_2$.

Further, if we take the second variation of the well-known second Frenet-Serret equation $\mathbf{n}_1' = -k\mathbf{t} + \tau\mathbf{n}_2$ and dot with $\mathbf{n}_1$, we obtain

$$\mathbf{n}_1 \cdot \delta^2\mathbf{n}_2 = \frac{1}{\tau}(\mathbf{n}_1 \cdot \delta^2\mathbf{n}_1{}' + k\mathbf{n}_1 \cdot \delta^2\mathbf{t} + \delta k\, \mathbf{n}_1 \cdot \delta\mathbf{t} - \delta\tau\mathbf{n}_1 \cdot \delta\mathbf{n}_2).$$

Since $\mathbf{n}_1 \cdot \delta^2\mathbf{n}_1' = \mathbf{n}_1 \cdot (\delta^2\mathbf{n}_1)'$, after using the Frenet-Serret equations we obtain

$$\mathbf{n}_1 \cdot \delta^2\mathbf{n}_1' = kg_t + g_{n_1}' - \tau g_{n_2},$$

where $g_t, g_{n_1}, g_{n_2}$ are the tangent, the normal and the binormal component of $\delta^2\mathbf{n}_1$, respectively. Also, using the expressions for $\delta k$ and $\delta\tau$ [7], we obtain $\mathbf{n}_1 \cdot \delta^2\mathbf{n}_2 = f_{n_1}$.

Finally, we will determine the binormal component of $\delta^2\mathbf{n}_2$. Using the second variation of the equation $\mathbf{n}_2 \cdot \mathbf{n}_2 = 1$, we obtain

$$\mathbf{n}_2 \cdot \delta^2\mathbf{n}_2 = -\frac{1}{2}\delta\mathbf{n}_2 \cdot \delta\mathbf{n}_2.$$

Applying $\delta\mathbf{n}_2$ from [7] we obtain $f_{n_2} = \mathbf{n}_2 \cdot \delta^2\mathbf{n}_2$. □

## 5. TORUS KNOT BENDING

A **torus knot** is a special kind of knots that lies on the surface of a torus. The parametric equations of $(p, q)$-torus knot are:

$$\begin{aligned} x(t) &= (c + a\cos pt)\cos qt, \\ y(t) &= (c + a\cos pt)\sin qt, \\ z(t) &= a\sin pt, \end{aligned} \tag{6}$$

$t \in [0,2\pi)$, $a, c \in \mathbb{R}$. The parametric equations of a torus are:

$$\begin{aligned} x(u,v) &= (c + a\cos v)\cos u, \\ y(u,v) &= (c + a\cos v)\sin u, \\ z(u,v) &= a\sin v, \end{aligned} \tag{7}$$

whereby the $c$ is the distance from the center of the tube to the center of the torus (major radius), and the $a$ is the radius of the tube (minor radius).

Let us determine an infinitesimal bending field so that all bent torus knots are on the same surface of the torus with a given precision, i.e.

$$F\big(x(t), y(t), z(t)\big) = 0,$$
$$F(x_\epsilon(t), y_\epsilon(t), z_\epsilon(t)) = o(\epsilon),$$

where $F(x, y, z) = 0$ is the implicit torus equation.

**Theorem 5.** *Infinitesimal bending field* $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$ *satisfying the condition*

$$z_3(t) = -(z_1(t)\cos qt + z_2(t)\sin qt)\cot pt \tag{8}$$

*includes the torus knot* (6) *under infinitesimal bending of the first order into the family of deformed curves on the torus* (7).

*Proof.* An implicit equation in Cartesian coordinates for a torus radially symmetric about the $z$-axis is

$$\left(c - \sqrt{x^2 + y^2}\right)^2 + z^2 = a^2, \tag{9}$$

or the solution of $F(x, y, z) = 0$, where

$$F(x, y, z) = \left(c - \sqrt{x^2 + y^2}\right)^2 + z^2 - a^2.$$

Let $C$ be a torus knot that bends infinitesimally so that all deformed curves are on the same torus with a given precision and let (6) be its parametric equation. The family of deformed knots $C_\epsilon$ under infinitesimal bending is

$$C_\epsilon: \begin{cases} (c + a\cos pt)\cos qt + \epsilon z_1(t) \\ (c + a\cos pt)\sin qt + \epsilon z_2(t) \\ a\sin pt + \epsilon z_3(t). \end{cases}$$

Since the curves $C_\epsilon$ should be on the torus (7) ie. (9), it must be valid

$$\left(c - \sqrt{((c + a\cos pt)\cos qt + \epsilon z_1(t))^2 + ((c + a\cos pt)\sin qt + \epsilon z_2(t))^2}\right)^2 + (a\sin pt + \epsilon z_3(t))^2 = a^2,$$

after neglecting the terms of order higher than 1with respect to $\epsilon$. From the last equation, after a little calculation, we obtain the condition (8) for the infinitesimal bending field. □

By examining the condition (8) we check whether the deformed knot remains on the torus. Also, using this condition we can determine the infinitesimal bending field $\mathbf{z}$ taking into account the condition $d\mathbf{r} \cdot d\mathbf{z} = 0 \Leftrightarrow \dot{\mathbf{r}}(t) \cdot \dot{\mathbf{z}}(t) = 0$. Thus, we obtain the infinitesimal bending field under which all deformed curves are on the torus with a given precision.

## 6. CONCLUSIONS

The geometric knot theory includes the consideration of knots as space curves which allows us to consider them from the point of view of geometry. In this regard it is possible to distinguish different representatives from the same knot type as an equivalence class. In this paper we pointed out to this fact considering the infinitesimal bending of particular representatives of knots. We examined the change of the unit vectors of Frenet-Serret frame under this type of deformation. Finally, we investigated the special kind of knots, so called torus knot, and its infinitesimal bending on the torus.

**COMPETING INTERESTS**

Authors have declared that no competing interests exist.

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# MAXIMUM RELIABILITY *K*-CENTER LOCATION PROBLEM

UDC: 519.872/.874:519.178

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**Abstract.** This paper presents an approach for solving the maximum reliability *k*-center location problem. We are modifying the well-known *p*-center problem in order to determine the location of the observed objects and maximize the reliability of supply system coverage. The problem is defined as a stochastic problem of multi-center location on a graph. As a solution, two new algorithms have been proposed. The first is modified Dijkstra's algorithm for determination of the most reliable paths between nodes in the graph. The output of this algorithm is used as an input for the second algorithm designed to find the reliability of node coverage from a predetermined number of nodes.

## 1. INTRODUCTION

There are numerous mathematical models formulated in order to solve complex location problems. Some of these models are described in [1]-[3]. In this paper, we are observing the location model with stochastic input data. We present an algorithm for calculating the performances of a system as a basis for finding the optimal location. Most of the models described in the existing literature are deterministic but the practice showed that it is necessary to include stochasticity in the facility's location planning. Therefore, in this paper, a stochastic model of *k*-centre location on the graph has been formulated to determine the location of a given number of facilities to maximize the reliability of the system. The problem refers to a network structure that is determined by a graph whose nodes contain the locations of demand and potential facilities, while the weight of the branches represents reliability, i.e. the probability that an appropriate branch is available (operational). In the end, a new algorithm has been formulated to determine the reliability of covering a node from *k* nodes (*k* - covering reliability).

The location model with stochastic inputs has been studied in various situations. A special case of a stochastic set-covering location problem was studied in [4], while Alegre et al. [5] solved a stochastic facility location problem for determining the best locations of health resources for patients who have suffered a diabetic coma. The location problem with stochastic demand was considered in [6]-[7]. Some of the researchers used the scenario planning in order to include stochasticity in location planning [8]-[9]. Classical facility location models assume that once constructed, the chosen facilities will always operate as planned. In reality, facilities fail from time to time.

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## 2. PROBLEM DESCRIPTION

In this paper we are discussing the location problems on networks. Actually, we are observing the weighted graph in which each branch is joined with the number from the interval (0, 1). That number presents the reliability of the branch i.e. the probability that the information will be transmitted or the flow of the vehicle through a traffic line will be achieved during a certain period without failure. So, we assume that the failure of the branches are jointly statistically independent events and based on that, the reliability of the path between two nodes can be defined as the product of reliability of branches of which this path consists. Considering that there can be several paths between two nodes in a graph, the probability of jointly reaching a pair of nodes is defined as the probability of the most reliable path between the observed nodes. This probability can alternatively be called the reliability of covering the terminal node from the given initial node. The probability of mutual reaching defined in such a manner differs from the commonly adopted definition of reliability between two nodes of a network structure, which takes into account all of the possible paths between them.

In order to select the optimal location for the facility, we set up the classic problem of the maximum coverage of the remaining nodes from the selected one. Furthermore, the nodes which location needs to be determined, we will denote as warehouses, while for the nodes that need to be reached form the warehouses, we will use the term - consumers. So, a max-min optimization task is formulated and the lowest value of calculated probabilities is used as a criterion.

In order to solve the problem of the facilities (warehouse) location, we assumed that the consumer is covered from at least one facility and coverage (reaching) of the consumer from a specific warehouse is achieved along the path of maximum reliability from that particular warehouse to the consumer.

Based on the given assumptions, it follows that the probability of covering a consumer should be calculated as the probability of a union of events that the consumer will be covered from the observed warehouses. Starting from this definition, analogous to the location problem of a single warehouse, it is again possible to calculate the probability of covering every consumer and to adopt the maximal as the criterion of optimization.

Mathematically, this problem can be described as:

$$(\max) \sum_{(i,j)\in E} r_{ij}x_{ij}$$

s.t.

$$\sum_{j\in\Gamma(c)} x_{cj} = 1, \qquad \sum_{j\in\Gamma^{-1}(t)} x_{jt} = 1$$

$$\sum_{j\in\Gamma^{-1}(i)} x_{ji} = \sum_{j\in\Gamma(i)} x_{ij}, \quad i\in V\setminus\{c,t\}, x_{ij}\in\{0,1\}, \quad (i,j)\in E$$

$r_{ij}, (i,j) \in V$ are reliabilities assigned to each branch of the observed weighted graph and $x \in \{0,1\}^E$. Constraints should ensure that the obtained solution is indeed the elementary path of maximum reliability between start node (consumer) *c* and end node *t*.

## 3. PROBLEM SOLVING

In order to determine the reliability of covering a node from *k* location nodes $R_i$ an algorithm given below is defined. The paths of maximum reliability $R_{iw}$ between consumer and each of the *k* warehouses need to be found. All those most reliable paths make a set $V_{iw}$ and form a subgraph (oriented graph with a direction from its root towards its leaves). The locations of the warehouses are randomly determined and for each consumer, the path of maximal reliability between it and each of the *k* warehouses has been found.

First, we randomly determine the *k* location (of nodes) in which the facilities (warehouses) will be located. For every consumer, it is necessary to find the elementary path of maximal reliability between it and each of the *k* warehouses. To achieve that we are using the modified Dijkstra's algorithm as follows:

ALGORITHM 1 - ALGORITHM FOR DETERMINING THE PATH OF MAXIMUM RELIABILITY

1. Initial labels are joined to nodes as follows:
   - The first node *c* is denoted $p^+(c) = 1$ (it gets a permanent label);
   - Remaining nodes get temporary labels: $p^-(j) = 0, \quad \forall j \in V \setminus \{c\}$;
   - *i* gets a value: $i = c$
2. We determine a set $A_i$ of the nodes following node *i*, which do not have a permanent label:
$$A_i = \left\{ j \,\middle|\, j \in \Gamma(i) \wedge p(j) = p^-(j) \right\}.$$
3. For each $j \in A_i$, new temporary labels are determined as:
$$p^-(j) = \max\left\{ p^-(j), p^+(i)^* r_{ij} \right\}.$$
4. Only one node $j^*$ receives a permanent label (out of all temporarily labelled), and it is the one for which:
$$p(j^*) \to p^+\left(j^*\right); \; j^* : p^-(j) = \max_{j \in N}\left\{ p^-(j) \right\}$$
so $p^+\left(j^*\right) = p^-\left(j^*\right)$.

5. If $j^* = t$, i.e. the ending node is marked with a permanent label. If not, then $i = j^*$ and return to step 2. If it is, the reliability of the path $p^*(t)$ has been determined.
6. Moving backwards for node $t$ towards node $c$: $t \to j_k \to j_{k-1} \to ... \to c$ the path of maximal reliability $p_{\max} = (c, j_1, j_2, ..., j_k, t)$ is determined:

$$j_k : p^+(t) / p^+(j_k) = r_{j_k t}$$
$$j_{k-1} : p^+(j_k) / p^+(j_{k+1}) = r_{j_{k-1} j_k}$$
$$...$$
$$c : p^+(j_1) / p^+(c) = r_{c j_1}$$

7. It is evident that:

$$p^+(t) = r_{cj_1} * r_{j_1 j_2} * ... * r_{j_{k-1} j_k} * r_{j_k t} .$$

By using this algorithm, all the paths $P_{iw}$, $w \in W$, $i \in I$ are determined, we should determine reliability $R_i$ of covering consumer $i$ from all $k$ warehouses. To solve this task for particular consumer $i$, we firstly form a tree structure graph of all of the paths $P_w$, $w \in W$, where root is a consumer $i$ and the leaves are warehouse nodes.

It can be noticed that one maximum reliability path from a warehouse $w_1$ to a particular customer $i$ can include a node that represents a different warehouse $w_2$. In this case, it is reasonable to exclude from consideration the sub-path between warehouses.

For the disjoint paths $P_{iw}$ the $k$-covering reliability is calculated based on a well-known formula:

$$R_i = \sum_{w=1}^{k} R_{iw} - \sum_{w=1}^{k-1} \sum_{j=w+1}^{k} R_{iw} R_{ij} + \sum_{w=1}^{k-2} \sum_{j=w+1}^{k-1} \sum_{z=j+1}^{k} R_{iw} R_{ij} R_{iz} - ... + (-1)^{k-1} \prod_{w=1}^{k} R_{iw}$$

Furthermore, the novel algorithm has been developed for calculating reliability $R_i$, which is based on Bellman's principle of optimality and concept of notation.

Let $V_i$ represent the set of all nodes of the tree. Each node from the set $V_i$ is joined with the notation $r_j$ which can be temporary $r_j^-$ or permanent $r_j^+$. Permanent notation of the node $j \in V_i$ represents the (maximum) reliability of covering node $j$ from the observed warehouses, i.e. leaves of the tree. (By definition, the reliability of covering of the leaves (warehouses) equals 1. $r_w = 1 \quad \forall w \in W$) The temporary notation of node $j$ is the temporary reliability

of covering node *j*. It is always less than or equal to the permanent notation. In addition to these, the following notations shall be used:

$A_j$ – set of the successors of node $j$ $A_j = \Gamma(j)$

$B_l$ – set of the predecessors of node $l$ $B_l = \Gamma^{-1}(l)$

$S$ – set of nodes with a permanent notation

$Q$ – set of nodes with a temporary notation

The sets $S$ and $Q$ are iteratively exchanged.

The algorithm for calculating the reliability of covering a node ($k$– covering reliability) consists of the following steps:

ALGORITHM 2 - ALGORITHM FOR DETERMINATION OF RELIABILITY OF COVERING A NODE FROM *K* NODE

1. Initialization

   $S = W$

   $Q = V_i \setminus S$

   $r_w = r_w^+ = 1 \quad \forall w \in W$

   $r_j = r_j^- = 0 \quad \forall j \in Q$

   $A_j = \left\{ l \middle| l \in \Gamma(j) \wedge l \in V_i \right\} \quad \forall j \in V_i$

2. Out of the set of unlabelled nodes $Q$, determine the set of nodes whose successors are all permanently labelled. First, determine the set of all nodes which precede the permanently labelled nodes:

   $$B = \left\{ l \middle| l \in \Gamma^{-1}(S) \wedge \forall l \in Q \right\},$$

   then out of those, take the nodes whose successors are all permanently labelled:

   $$A = \left\{ j \middle| j \in B_i \wedge \Gamma(j) \in S \right\}$$

3. Calculate the reliability of node $j$:

   $$r_j = \sum_{(l \in A_j)} r_{jl} r_l - \sum_{(l \in A_j)} \sum_{(k \in A_j)} r_{jl} r_l r_{jk} r_k +$$

   $$\sum_{(l \in A_j)} \sum_{(k \in A_j)} \sum_{(z \in A_j)} r_{jl} r_l r_{jk} r_k r_{lz} r_z - \ldots + (-1)^{M_i - 1} \prod_{(t \in A_j)}^{M_i} r_{jt} r_t \quad \forall j \in A$$

   Change the notations of these nodes into permanent ones: $r_j = r_j^+$

4. Update the set of permanently labelled nodes: $S = S \cup A$
5. Determine the set of unlabelled nodes: $Q = V_i \setminus S$
6. If $Q \neq \varnothing$, return to step 2. If it is equal, proceed to step

The end.

As we can see, this problem can be observed as a sub-problem of the problem of determining the location of $k$ warehouse in order to maximize system reliability. The reliability of the system is equal to the minimal reliability of covering nodes that comprise it. The procedure for calculation of system reliability remains the same if we chose some other $k$ location for the warehouses. If we were to examine all possible warehouse locations, then we would select the particular combination of $k$ warehouses for which the reliability of the system would be maximal.

## 5. CONCLUSION

As a result, we showed how it is possible to determine the most reliable paths between any two nodes of a graph and determine their reliability by using the modification of Dijkstra's algorithm. Also, system reliability can be calculated by determining the reliability of node coverage. Variables of this problem are integers, so this is a combinatorial problem. The objective function is not given analytically, so it has to be calculated numerically. Constraints arise from topological characteristics and define locations on the graph. The only constraint is a given number of nodes i.e. facilities. For further research, we will consider to formulate a relaxed problem and solve it by using some of the available software or to develop a heuristic algorithm or to apply some of the metaheuristics.

## NOTATION

$G=(V,E,R)$ – weighted graph, where:

$V=(1,...,i,...,n)$ – set of nodes

$E\subseteq V\times V=\{(i,j)|i\in V, j\in V\}$ - set of arcs

$R$ - function which joins to each arc $(i, j)$ the weight (reliability) $r_{ij}$ from the interval $[0,1]$ $R:E\to[0,1]$

$r_{ij}$ - reliability of arc $(i , j)$.

$W=\{j_1,...,j_w,...,j_k\}$ – set of warehouse indexes, $W\subset V$

$k$ – total number of warehouses to be located

$C=V\setminus W=\{j_{k+1},...,j_i,...,j_n\}$ – set of consumer nodes

For the purpose of simplicity, warehouse indexes will be abbreviated as $w$, $w\in W$ while consumer indexes as $i$, $i\in C$

$P_{iw}$ – path of maximum reliability from warehouse $w$ to consumer $i$. The path consists of an array of arcs $\left((w,v_{j1}),(v_{j1},v_{j2}),...,(v_{jl},i)\right)$ and is alternatively represented as an array of nodes $P_{iw}=(w,v_{j1},v_{j2},...,v_{jl},i)$.

$R_{iw}$ – reliability of covering consumer from warehouse w (reliability of path $P_{iw}$) $R_{iw} = \prod_{(j,l)\in P_{iw}} r_{jl}$

$R_i$ – reliability of covering consumer $i$ from all of the $k$ warehouses

$f(W)$ – reliability of the system, the objective function is defined as the minimum value of $R_i$ - $f(W) = \min\{R_i | i \in C\}$

**COMPETING INTERESTS**

Authors have declared that no competing interests exist.

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# USING FRACTAL ANALYSIS FOR DETECTING LEUKEMIA

UDC: 511.75-7:616.155.392

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**Abstract.** Leukemia is a cancer that affects the white blood cells or more specifically the lymphocytes. In this paper we give an overview of a technique that separates leukocytes from other blood cells, and then extracts lymphocytes from leukocytes. For these lymphocytes, fractal features, shape features, and other texture features are considered. Any classifier that determines presence of leukemia can use these characteristics. Then, we apply the technique of determining the dimension of box counting to the already isolated lymphocytes in order to determine whether the lymphocytes are normal or lymphoblast.

## 1. INTRODUCTION

White blood cells or leukocytes play an important role in diagnosing many diseases. Leukemia is a cancer that affects the blood cells or more specifically the lymphocytes as a subtype of white blood cells. There are two types of leukemia: acute leukemia (which develops very quickly) and chronic leukemia (which develops slowly). Lymphocytes are fundamental to the immune system because they determine the specificity of the immune response to infectious microorganisms and other foreign substances [2]. In humans, lymphocytes make up about 25 to 33% of the total leukocyte count. They are concentrated in the central lymphoid organs and tissues, such as the spleen, tonsils, and lymph nodes, where the immune response is most common.

Leukemia occurs when the body begins to accumulate overtly abnormal leukocytes in the blood or bone marrow. As this happens, the number and capacity of mature blood cells decreases. In people with leukemia, immature lymphocytes, called lymphoblastic or leukemic cells, accumulate in the body because they cannot die and cannot be used. The accumulation of leukemic cells occurs in the bone marrow, where all the normal white and red blood cells and platelets are being expelled without being regenerated [3].

The symptoms of leukemia are very common and similar to those of the flu, which makes it difficult for diagnose. Differential blood counts are not sufficient to confirm the disease. Therefore, microscopic examination is the most important diagnostic methodology. Several techniques are already being used around the world to classify lymphocytes as normal or lymphoblast. Some of them are automation segmentation classification [15], cell segmentation using active contour models [12], and intermediate shift cell segmentation [7].

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This paper presents a technique [1,11,14] that first separates leukocytes from other blood cells and then extracts lymphocytes from leukocytes. For these lymphocytes, fractal features, shape features, and other texture features are considered. Any classifier to determine the presence of leukemia can use these characteristics. Then, the technique of determining the Hausdorff dimension, i.e. the dimension of box counting was applied to the already isolated lymphocytes, in order to determine whether the lymphocytes are normal or lymphoblastic. The paper is overview of many papers cited accordingly, and it is a nice survey on the problematic in consideration.

## 2. METHODS FOR LYMPHOCYT CLASSIFICATION

The procedure for classifying lymphocytes in microscopic images consists of preprocessing, segmentation, feature extraction, and classification. The difference of this technique from other techniques for diagnosing leukemia is in the last two steps, i.e. in the extraction of specific features and then classification.

The microscopic blood count consists of red blood cells, white blood cells, and platelets. The method in this paper is based on the segmentation of the color image and the aim is to separate the white blood cells from the background and to obtain a separate nucleus as a region of interest [3, 11].

Pre-processing is necessary because in case of excessive staining of blood images and due to the process of improving the quality of images, there is always noise. Generally, the image is described using three colors. Images generated by digital microscopes are usually in RGB colors that are difficult to segment. Therefore, in the pre-processing step, it is necessary to convert the color image to $L*a*b$ color space. Color is expressed with three numerical values, $L$ for light where the darkest black is obtained for $L=0$, and the lightest white for $L=1$. The constants $a$ and $b$ are used for color coordinates, where the $a$-axis represents the green-red components, and the $b-$axis blue-yellow components. In practice, various reasons such as camera settings, different lighting, and old stain can cause the image of blood cells and the background of the image itself to vary greatly in color and intensity.

In order to make robust cell segmentation related to these variations, it is necessary to reduce the memory used and improve the computational time [3]. The purpose of image segmentation is to extract important information from the input image. The efficiency of extraction and classification functions largely depends on the accuracy of the segmentation. In the segmentation process, the so-called K-medium clustering is applied to segment the image into four regions. Using certain techniques, these clusters are then used to obtain sub-images. Using image morphology [10], only those subtypes containing lymphocytes are selected for feature extraction.

Functional feature extraction in image processing is a technique of redefining a large number of redundant data into a set of reduced size (or

functional vector). Three types of data are usually extracted from the lymphocyte image, namely fractal size, color, and shape characteristics, including contour uniqueness, shape, texture, and optical density (the number of dark spots in a given area).

In this paper, special attention will be paid to the last step of the overall procedure, which will consider the use of the Hausdorff dimension for the classification of lymphocytes.

## 3. HOUSDORFF DIMENSION

Mandelbrot first defined fractal as a term in 1983 as a way of classifying "irregular" objects whose dimensions are not integers. Most fractal objects are similar to themselves, although there are fractal objects that are not self-similar and also have infinite complexity. We will deal with fractal object that are similar to itself or self-similar and are composed of a finite number of reduced copies of itself. A more detailed definition of self-similar fractals can be found in [4, 20].

There are different dimensions useful for characterizing "irregular" objects: cube counting dimension (Minkowski dimension), fractal, Hausdorff (or Hausdorff-Besikovic) dimension, capacity dimension, information dimension, and others [16]. The fractal dimension is a statistical quantity that shows how completely the fractal fills the space. The Hausdorff dimension (HD) and the cube counting dimension are the most important theoretical fractal dimensions. The packing dimension is similar to the Hausdorff dimension, as "packing" small open balls into a given subset construct the packing dimension, while the Hausdorff dimension is constructed by covering a given subset with such small balls.

The HD is determined by the optimal coverage of the original object by a set of smaller objects. The Hausdorff dimension is most commonly used in theoretical mathematics, and its practical application is minimized. The reason for this lies in the complexity of the calculations. That is why boxes are most often used to determine the dimension, i.e. the dimension of counting boxes is used.

If boxes or squares were used as a model, it would mean that the minimum number of boxes with side $\delta$ (small enough) covering length could be used for the dimension, i.e. $N_\delta = \frac{\text{Lenght}}{\delta}$, or the minimum number of boxes covering area, $N_\delta = \frac{\text{Area}}{\delta^2}$, or the minimum number of boxes covering the body, $N_\delta = \frac{\text{Volume}}{\delta^3}$.

If $X$ is an object and $N_\delta(X)$ is the minimum number of boxes with side $\delta$ covering $X$, then the *box -counting dimension* is given by:

$$d = \lim_{\delta \to 0} \frac{\ln(N_\delta(X))}{-\ln \delta}.$$

In general, HD $\leq d$. The HD and the box-counting dimension have similar definitions, with the Hausdorff dimension minimizing the number of boxes by allowing different box sizes, [18]. This minimization gives the Hausdorff dimension its theoretical advantage because it excludes pathologies that may occur when using smaller boxes and covering isolated spots. The Hausdorff dimension and the box count dimension match for self-similar and compact fractals [8, 20], for fractals described by rapidly convergent scaling functions [18], and for Julia sets [19].

### 3.1. BOX COUNTING METHOD

In practical application, the box-counting dimension is widely used, due to the fact that it is easy to implement. This method does not change the size of the measured object, but the size of the element used for measuring (square, cube). The technique is usually implemented in software used to separate schemes from digital media, although the basic method can also be used to physically examine some models.

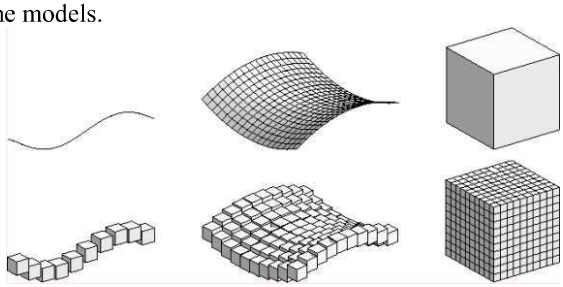

**Figure 1.** Hausdorff dimension with box counting method [10].

Theoretically, box counting aims to measure fractal scaling. This would mean that the length (size) of the side of the boxes or the scaling factor is known in advance. But in fractal analysis, the scaling factor is not always known, so box counting algorithms try to find an optimized way to divide the object under consideration to detect the scaling factor. The basic method for this starts with a set that contains a number of measuring elements-boxes. Each of these boxes has a side of length $a$. The algorithm must determine how to increase or decrease the side length of boxes (for example, linearly or exponentially), which can have a profound effect on overall results. These boxes are applied on the considered object and then counted.

In box counting algorithms, the number of boxes is a power function of the box. We estimate all fractal dimensions as a power indicator of such power

function, and they are real numbers that characterize the fractality (texture or roughness) of the objects. The perimeter roughness of the core can be used for differentiation (separation due to differences).

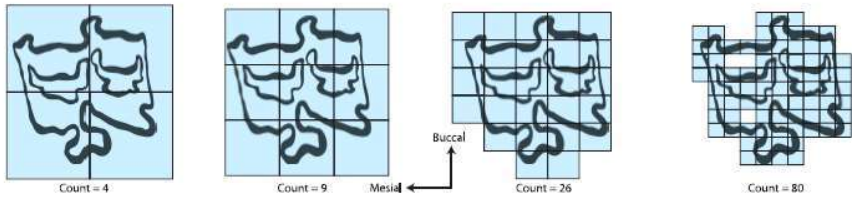


**Figure 2.** Box counting method [9]. In each subsequent step, the length of the sides of the boxes decreases. Only those boxes that contain part of the object under consideration are counted.

## 3.2. PROCEDURE FOR DETERMINING THE HAUSDORFF DIMENSION USING THE BOX COUNTING METHOD

The Hausdorff dimension is an essential feature of fractal geometry and will be used as a measure of the roughness of cell boundaries, allowing it to be classified as normal or lymphoblast. The procedure for determining the Hausdorff dimension using the box counting method [13] is presented below as an algorithm. In this procedure, the Hausdorff dimension and the box counting dimension do not match.

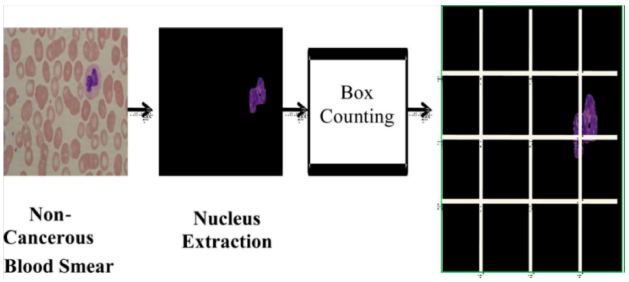


**Figure 3.** Covering the core with a grid of squares using the box counting method [2]

*Step 1*- Each color image of the nucleus (red blood cells) is converted to a binary image, i.e. two-color image as shown in Figure 4.

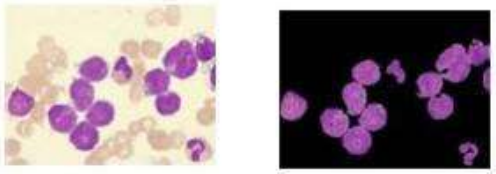

**Figure 4.** Converting a blood count into a binary image [10].

*Step 2*- Detect the core boundaries using an edge detection technique (Figure 5).

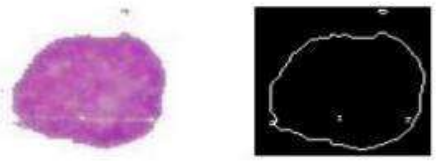

**Figure 5.** Core edge detection [11].

*Step 3*- The edges are covered with a grid of squares using the box counting method. Thereby, we especially pay attention on how the number of boxes changes, while we strive to make the finest network of boxes, i.e. a network of boxes with a smaller side. For each subject, the applied SCC software [5], counts the number of pixels occupied inside the blue line, including the pixels on the border (Figure 6 right). In the case of the cytoplasm (Figure 6 left), the software counts the number of pixels trapped between the two blue lines. The larger blue line corresponds to the cytoplasmic boundary and the smaller blue line to the nucleus boundary.

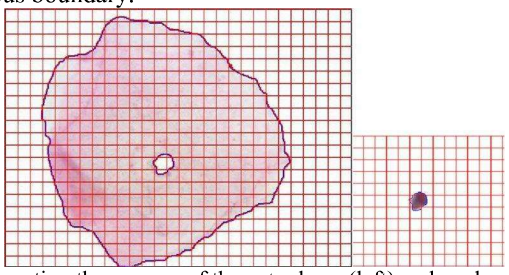

**Figure 6.** Counting the squares of the cytoplasm (left) and nucleus (right) [5].

*Step 4*- Then we calculate the Hausdorff dimension of the core using the following equation

$$\mathrm{HD}=\frac{\ln(N)}{\ln(N(S))}\,, \tag{1}$$

where $N$ is the total number of squares that cover the core network, and $N(S)$ is the number of squares occupied inside the blue line, including the border pixels. For illustrating the procedure, we use the results obtained in [17]. There, images of 40 healthy and 40 diseased (cancerous) lymphocytes are considered and analyzed. First, the images are processed, namely each image is converted to a black and white image. Then, by setting the appropriate threshold function, the image is converted to binary [1]. The edge of the lymphocyte nucleus is determined using a border detection technique proposed by Canny [6]. The Hausdorff dimension of the lymphocyte nucleus is determined from the resulting binary image. The HD result of normal cells and cancer cells is given in Figure 7 and Figure 8, respectively [2, 14]. By applying equation (1) one gets

that the Hausdorff dimension of a healthy nucleus is approximately 1.5501, and of a diseased one (carcinogenic) is approximately 1.7828. The Hausdorff dimension is approximately determined by the box counting method. The number of occupied boxes in the images of cancer cells (i.e. nuclei) is higher than that of healthy cells (nuclei). This will result in a comparable HD aspect ratio. The nucleus of a healthy cell has a lower HD value than that of a carcinogen. If it is an earlier stage of the disease HD on carcinogens (nuclei) it will be smaller, i.e. closer to HD on healthy nuclei. Therefore, HD is an important feature in the proposed system.

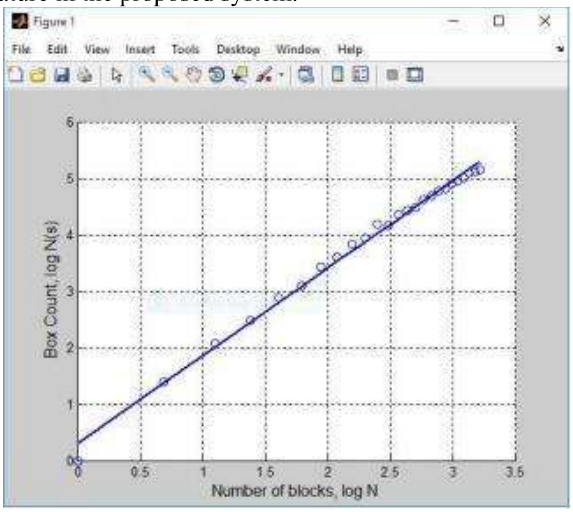


**Figure 7.** Healthy core - HD = 1.5501 [14]. From (1), the Hausdorff dimension is the coefficient of direction of the displayed line.

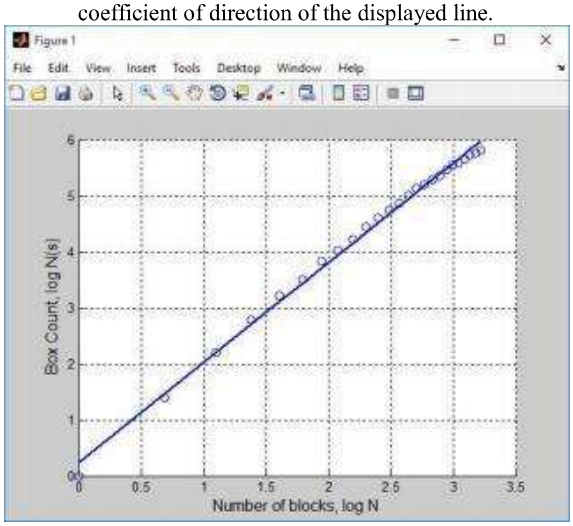

**Figure 8.** Cancer nucleus- HD = 1.7828 [14]. From (1), the Hausdorff dimension is the coefficient of direction of the displayed line.

## 4. CONCLUSIONS

Medical image processing is one of the fastest growing fields in medicine and clinical research. Image analysis helps in gathering information, detecting diseases, diagnosing diseases, controlling and treating, monitoring and evaluating. Blood disorders can be identified by visual inspection of microscopic blood cell images. This identification helps to classify certain blood-related diseases, including leukemia.

This paper is an overview of the work done in this area. The main topic is segmentation of white blood cells from colored blood images, followed by appropriate extraction to detect leukemia. Special attention is paid to measuring core irregularities using the Hausdorff dimension. The edge of the lymphocyte nucleus is determined using a border detection technique proposed by Canny [6], where edge detection and localization is done by numerical optimization. This technique is suitable for images in shades of gray. Note that other edge detectors can be applied depending on the used fractal dimension. For example, in the paper [21], the edge detection technique is suitable for binary images, using the operators Sobel, Roberts and Laplace.

This way of diagnosing leukemia is accurate enough and a good start for further research into other diseases. Compared to manual counting, the advantage is that images with many lymphocytes in the visual field can be viewed (shortens the analysis time), but the automated process does not require human intervention (minor error).

Also the simplicity of the Hausdorff dimension is in favor of experts from other fields outside mathematics, so in this way we have nice application of mathematics in other sciences including medicine. However, in addition to the Hausdorff dimension, there are other ways to detect leukemia and other diseases. More recently, imaging techniques, various statistical methods, and machine learning have been used, making them accurate in diagnosing the disease.

## COMPETING INTERESTS

Authors have declared that no competing interests exist.

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