29th BALKAN MATHEMATICAL OLYMPIAD
XXIX
βmo
2012 ANTALYA TURKEY

16th
25-29 JUNE 2012 VERIA GREECE
JUNIOR BALKAN MATHEMATICAL OLYMPIAD

BMO 26 09
26th BALKAN MATHEMATHICAL OLYMPIAD
April 28th - May 4th, Kragujevac, Serbia

BMO
27th
2010
May 2-8
Republic of Moldova
Balkan Mathematical Olympiad

Balkan Mathematical Olympiad
2011
Romania
1984
BMO
4 - 8 May 2011 - Iasi

M M
2013
O
20-та Македонска
математичка олимпијада

M M
J
2013
O
17-та JMMO

BMO 2013
CYPRUS

ANTALYA 2013 TURKEY
17th
JUNIOR BALKAN MATHEMATICAL OLYMPIAD

20-та Македонска
математичка олимпијада

17-та JMMO

Library Olympiads-Armaganka

Mathematical Olympiads

Macedonian Mathematical Olympiads
Balkan Mathematical Olympiads

Aleksa Malcheski, Ph.D.
Velinov Daniel, Ph.D.
Sanja Atanasova, Ph.D.
Risto Malcheski, Ph.D.
Slagjana Brsakoska, Ph.D.
Samoil Malcheski, Ph.D.
Bojan Prangoski, Ph.D.
p Pavel Dimovski, M.Sc.
Tomi Dimovski, M.Sc.
Vesna Andova, Ph.D.
Methodi Glavche, Ph.D.
Jasmina Angeleska

Macedonian Mathematical Society-Armaganka
Republic of Macedonia, Skopje 2021

President: Aleksa Malcheski
Publisher: Macedonian Mathematical Society-Armaganka
Street:. Ul.2.br.107A
Nas. Vizbegovo, Butel, 1000, Skopje, Macedonia

Foreword

This year in the Macedonia competitions of all levels in primary/secondary and high-school were held: school, regional, state and Olympiads.

After rigorous selection processes the BMO and JBMO teams were formed. The latter Olympiads took place in a pleasant atmosphere in the Republic of Bulgaria and the Republic of Macedonia, respectively.

After the IMO team selection test, the team to the IMO 2014 was formed. This year the IMO is taking place in Cape Town, the SAU.

The content of this book consists of the mathematical competitions that already took place in Macedonia and the Balkan region, as well as the solutions.

XVIII Macedonian Junior Mathematical Olympiad 2014

Faculty of Electrical Engineering and Information Technologies-Skopje, 31.05.2014

18-t a JMMO

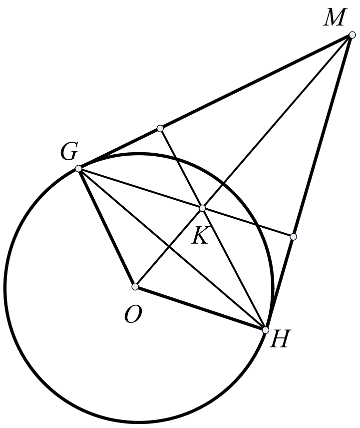
1. Prove that $\frac{1}{1\cdot 2013}+\frac{1}{2\cdot 2012}+\frac{1}{3\cdot 2011}+...+\frac{1}{2012\cdot 2}+\frac{1}{2013\cdot 1}<1$.

Solution 1. For arbitrary natural numbers $n\neq 1\neq m$ the inequality $nm\geq n+m$ holds, since $(n-1)(m-1)\geq 1\Rightarrow nm-n-m+1\geq 1\Rightarrow nm-n-m\geq 0\Rightarrow nm\geq n+m$, with equality only when $n=m=2$. Then for $n\geq 2$, we have $\frac{1}{n(2014-n)}<\frac{1}{n+2014-n}=\frac{1}{2014}$ and hence:

$$\frac{1}{1\cdot 2013}+\frac{1}{2\cdot 2012}+\frac{1}{3\cdot 2011}+...+\frac{1}{2012\cdot 2}+\frac{1}{2013\cdot 1}<\frac{2}{2013}+\frac{2011}{2014}<\frac{3}{2014}+\frac{2011}{2014}=1.$$

Solution 2.

$$\begin{aligned}
&\frac{1}{1\cdot 2013}+\frac{1}{2\cdot 2012}+\frac{1}{3\cdot 2011}+...+\frac{1}{2012\cdot 2}+\frac{1}{2013\cdot 1}=\\
&=\frac{1}{2014}\left(\frac{1+2013}{1\cdot 2013}+\frac{2+2012}{2\cdot 2012}+\frac{3+2011}{3\cdot 2011}+...+\frac{2012+2}{2012\cdot 2}+\frac{2013+1}{2013\cdot 1}\right)=\\
&=\frac{1}{2014}\left(\left(\frac{1}{1}+\frac{1}{2013}\right)+\left(\frac{1}{2}+\frac{1}{2012}\right)+...+\left(\frac{1}{2012}+\frac{1}{2}\right)+\left(\frac{1}{2013}+\frac{1}{1}\right)\right)=\\
&=\frac{1}{2014}2\left(\frac{1}{1}+\frac{1}{2}+...+\frac{1}{2012}+\frac{1}{2013}\right)=\frac{1}{2014}\left(3+\frac{2}{3}+\frac{2}{4}+...+\frac{2}{2012}+\frac{2}{2013}\right)<\\
&<\frac{1}{2014}\left(3+\underbrace{1+1...+1+1}_{2011}\right)=1
\end{aligned}$$



2. Two tangents are drawn from a point M to circle k ,that touch it at points G and H. If O is the center of k and K is the orthocenter of the triangle MGH prove that $\angle GMH=\angle OGK$.

Solution. Let us notice that K must lie on OM. From $HK\perp GM$ and $OG\perp GM$, it follows that $HK\parallel OG$. Analogously $OH\parallel GK$.(1p) From $\overline{OG}=\overline{OH}$, it follows that $OHKG$ is a rhombus. (1p) Let us notice that O, H, M and G lie on the circle with diameter OM. Hence $\angle OGH=\angle OMH$. Now the statement of the exercise follows from $\angle OGK=2\angle OGH$ and $\angle GMH=2\angle OMH$.

3.Find all $n\in\mathbb{N}$ divisible by 11, such that all numbers that can be obtained from n by an arbitrary rearrangement of its digits are again divisible by 11.

Solution. From the condition $11|n$, the number n must have at least two digits. Let $n=\overline{a_k a_{k-1}...a_0}$ where $a_i, 0\leq i\leq k$ are digits and $a_k\neq 0$. From the former discussion we have $k\geq 1$.

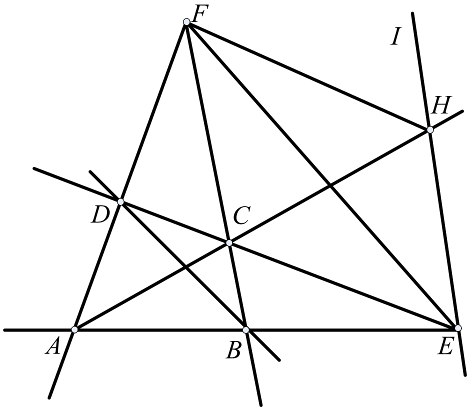
We will show that all digits in the number n are equal. Namely, from the condition of the exercise, the number $n'=\overline{a_k a_{k-1}...a_{i+1}a_{i-1}a_i a_{i-2}...a_0}$ (n' is obtained from n by exchanging the positions of the digits a_{i-1} and a_i) is also divisible by 11. Therefore $11|n-n'$, i.e. $11|10^{i-1}\left(\overline{a_i a_{i-1}}-\overline{a_{i-1}a_i}\right)$ or $11|10^{i-1}\cdot 9\left(a_i-a_{i-1}\right)$, and hence $a_i=a_{i-1}$.

It follows that $n=a\cdot\underbrace{11...11}_{k+1}$. We easily check that $11|n$ if and only if k is an odd number.

4. A convex quadrilateral $ABCD$ is given. Let E be the intersection of AB and CD, F be the intersection of AD and BC, and G be the intersection of AC and EF. Prove that the following two statements are equivalent:

(i) BD and EF are parallel

(ii) G is the midpoint of the segment $\overline{EF}$



Solution. We draw a line l through E which is parallel to BC. Let H be the intersection of l and AG. Now we have that G is the intersection of the diagonals in the trapezoid $EHFC$.

$(i)\Rightarrow(ii)$: Let the lines BD and EF be parallel. Then, from Thales' theorem for parallel segments we have the equalities:

$$\frac{\overline{AC}}{\overline{AH}}=\frac{\overline{AB}}{\overline{AE}} \text{ and } \frac{\overline{AB}}{\overline{AE}}=\frac{\overline{AD}}{\overline{AF}}.$$

It follows that $\frac{\overline{AC}}{\overline{AH}}=\frac{\overline{AD}}{\overline{AF}}$, and therefore from the same Thales' theorem we conclude that the lines HF and ED are parallel. Therefore $EHFC$ is a parallelogram and its diagonals bisect each other in the intersecting point G.

$(ii)\Rightarrow(i)$: Let G be the midpoint of the segment $\overline{EF}$. Then $\Delta EGH\cong\Delta FGC$, so that $EHFC$ is a parallelogram and we conclude that HF and ED are parallel. Therefore the equalities

$$\frac{\overline{AC}}{\overline{AH}}=\frac{\overline{AB}}{\overline{AE}} \text{ and } \frac{\overline{AC}}{\overline{AH}}=\frac{\overline{AD}}{\overline{AF}}$$

hold.

It follows that $\frac{\overline{AB}}{\overline{AE}}=\frac{\overline{AD}}{\overline{AF}}$, and therefore from the same Thales' theorem we conclude that BD and EF are parallel.

5. Prove that there exist pairwise disjoint sets A_1, A_2 ,..., A_{2014} whose union is the set of natural numbers and for which the following condition holds:

For arbitrary natural numbers a and b, at least two of the numbers $a, b, \gcd(a,b)$ belong to one of the sets A_1, A_2 ,..., A_{2014}.

Solution. Let $v_2(n)$ greatest integer for which $2^{v_2(n)}$ is a divisor of n. Then $v_2(\gcd(a,b))=\min\{v_2(a),v_2(b)\}$. Therefore at least two of the numbers $v_2(a)$, $v_2(b)$ and $v_2(\gcd(a,b))$ are equal.

We define sets $A_{i+1}=\{n \mid v_2(n)\equiv i\pmod{2014}\}$ for $0\le i\le 2013$.

Obviously, the sets A_1, A_2 ,..., A_{2014} are pairwise disjoint, their union is N and two of the numbers $a, b, \gcd(a,b)$ belong to the set A_{i+1}, where i is the residue produced by division of $v_2(\gcd(a,b))$ by 2014.

Macedonian Mathematical Olympiad 2014
Faculty of Natural Sciences and Mathematics-Skopje
12.04.2014

1. 2014 lines are given in a plane, arranged in three groups of pairwise parallel lines. What is the greatest possible number of triangles formed by the lines (each side from such a triangle lies on one of the lines)?.

Solution. Let $a \ge b \ge c$ be the numbers of the lines in the three groups for which the greatest possible number of triangles is attained. Then $a+b+c=2014$, and the greatest possible number of triangles is abc (when no three lines have a common point). We will show that $a \le c+1$. Let us suppose the opposite, i.e. $a > c+1$. Then $abc < b(ac+a-c-1) = b(a-1)(c+1)$, which is contradictory to the choice of a, b and c. It cannot be that $a=c$, because in that case $a=b=c=\frac{2014}{3}$ is not an integer. In order for a, b and c to be integers, it must be that $a=672$ and $b=c=671$ and so the number of triangles is $672 \cdot 671^2$.

2. Give all integer solutions of the equation:

$$3^{2a+1}b^2+1=2^c.$$

Solution.

Case 1. $a \ge 0$.

Clearly $c \ge 0$ where $c=0$ implies $b=0$. We get that $(a,0,0)$ is a solution for an arbitrary non-negative integer a. From the equality $3^{2a+1}b^2+1=2^c$ it follows that b is an odd integer. We can write the left-hand side in the following form

$$3^{2a+1}b^2+1=(3^{2a+1}+1)b^2-(b-1)(b+1).$$

For the right-hand side of the last equality we notice that $(b-1)(b+1)$ is divisible by 8, while $(3^{2a+1}+1)b^2$ is divisible by 4 but not by 8. Therefore $2^c=4$ i.e. $c=2$. But then $3^{2a+1}b^2=3$, so that $a=0$ and $b=\pm 1$.

Case 2. $a<0$.

Again $c \ge 0$ where $c=0$ implies $b=0$ and then a can be an arbitrary negative integer. Therefore we restrict ourselves to the case $c>0$. It is enough to consider the case $b>0$. Putting $d=-a$, the Diophantine equation from the statement of the exercise gets the form

$$(2^c-1)3^{2d-1}=b^2,$$

where b, c and d are natural numbers. Therefore b is divisible by 3, and hence c is an even number. Hence we have $b=3^d x$, $c=2y$ for some natural numbers x and y. The Diophantine equation gets the form

$$4^{y-1}+4^{y-2}+\cdots+1=x^2.$$

This implies $x=y=1$. Namely, for $y \ge 2$ we would get that $x^2 \equiv 5 \pmod 8$, which is impossible. Therefore in this case the only solutions are $(a,3^{-a},2)$, where a is an arbitrary negative integer.

The set M of all solutions to the Diophantine equation from the statement of the exercise is:

$$M=\{(a,0,0)|\ a \in \mathrm{Z}\} \cup \left\{(a,\pm 3^{-a},2)\,\middle|\, a \in \mathrm{Z}^- \cup \{0\}\right\}.$$

3. Let k_1, k_2 and k_3 be three circles with centers O_1, O_2 and O_3 respectively, such that none of the centers lies inside any of the two other circles. The circles k_1 and k_2 intersect in A and P, k_1 and k_3 intersect in C and P and k_2 and k_3 intersect in B and P. Let X be a point on k_1 such that the intersection of the line XA with the circle k_2 is

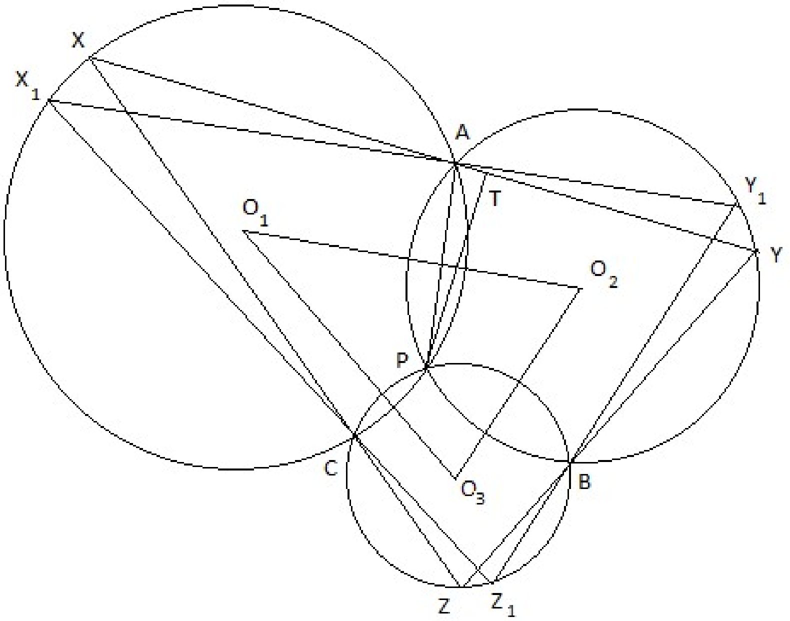
Y and the intersection of the line XC with k_3 is Z and so that Y belongs neither inside k_1 nor inside k_3 and Z belongs neither inside k_1 nor inside k_2.

a) Prove that the triangles XYZ and $O_1O_2O_3$ are similar.

б) Prove that the area of the triangle XYZ is not greater than four times the area of the triangle $O_1O_2O_3$. Is the maximum attainable?

Solution. We will first show that the points Y, B and Z are collinear. Since the quadrilateral $BYAP$ is inscribed we have $\angle PBY = \angle PAX$. Since the quadrilateral $AXCP$ is inscribed we have $\angle PAX = \angle PCZ$. Since the quadrilateral $CPBZ$ is inscribed we obtain $\angle PBZ + \angle PCZ = 180^\circ$. Therefore $\angle YBZ = \angle YBP + \angle PBZ = 180^\circ$.

Let us notice that $\angle CO_1O_3 = \angle PO_1O_3$ and $\angle AO_1O_2 = \angle PO_1O_2$, from where it follows that $\angle O_2O_1O_3 = \frac{1}{2}\angle AO_1C = \angle AXC$. Similarly $\angle O_1O_2O_3 = \angle AYB$ and $\angle O_1O_3O_2 = \angle CZB$. It follows that $\Delta XYZ \sim \Delta O_1O_2O_3$, with which we've proven the statement under a).



Let the line X_1Y_1 be parallel to O_1O_2 and pass through A, where X_1 lies on k_1 and Y_1 lies on k_2. Let Z_1 be the intersection of the line X_1C with the circle k_3. From the afore-proven, the points Y_1, B and Z_1 are collinear and $\Delta X_1Y_1Z_1 \sim \Delta O_1O_2O_3$. Furthermore, $\angle PXA = \angle PX_1A$ and $\angle PYA = \angle PY_1A$. Therefore $\Delta PXY \sim \Delta PX_1Y_1$. Let PT be the altitude dropped from the vertex P to the side XY. PA is the altitude of the triangle PX_1Y_1. Since PA is a hypotenuse in the right-angled triangle PAT we get $\overline{PT} \le \overline{PA}$. Therefore $P_{PXY} \le P_{PX_1Y_1}$ and analogously $P_{PYZ} \le P_{PY_1Z_1}$ and $P_{PXZ} \le P_{PX_1Z_1}$. From this we get $P_{XYZ} \le P_{X_1Y_1Z_1}$. The points P, O_1 and X_1 are collinear since $\angle PAX_1 = 90^\circ$. Similarly P, O_2 and Y_1 are collinear and P, O_3 and Z_1 are collinear. We get that O_1O_2, O_1O_3 and O_2O_3 are midsegments in the triangles X_1Y_1P, X_1Z_1P and Y_1Z_1P respectively, and so $P_{X_1Y_1Z_1} = 4P_{O_1O_2O_3}$. This gives us the required inequality. Equality is attained when the points X and X_1 coincide, and with that the points Y and Y_1 as well as the points Z и Z_1 coincide.

4. Let a,b,c be real numbers for which $a+b+c=4$ and $a,b,c>1$. Prove that

$$\frac{1}{a-1}+\frac{1}{b-1}+\frac{1}{c-1}\geq 8\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right).$$

Solution. Since it holds that $\frac{1}{a-1}-\frac{8}{b+c}=\frac{1}{a-1}-\frac{8}{4-a}=\frac{12-9a}{(a-1)(4-a)}=\frac{3(4-3a)}{(a-1)(4-a)}$ the given inequality is equivalent to

$$3\left(\frac{4-3a}{(a-1)(4-a)}+\frac{4-3b}{(b-1)(4-b)}+\frac{4-3c}{(c-1)(4-c)}\right)\geq 0.$$

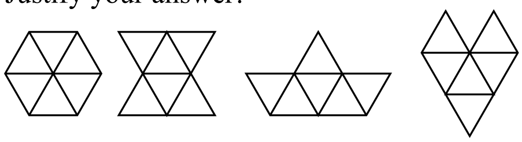
Without loss of generality we can assume that $a\geq b\geq c.$ Then clearly it holds $4-3a\leq 4-3b\leq 4-3c.$ From $1<a,b,c<4$ it follows that $\frac{1}{(a-1)(4-a)},\frac{1}{(b-1)(4-b)},\frac{1}{(c-1)(4-c)}$ are positive real numbers. We will prove that $(a-1)(4-a)\geq(b-1)(4-b)$.

We have $(a-1)(4-a)\geq(b-1)(4-b)\Leftrightarrow 5a-a^2\geq 5b-b^2\Leftrightarrow(a-b)(5-a-b)\geq 0$. Analogously $(b-1)(4-b)\geq(c-1)(4-c)$. Hence $\frac{1}{(a-1)(4-a)}\leq\frac{1}{(b-1)(4-b)}\leq\frac{1}{(c-1)(4-c)}$. Since $4-3a\leq 4-3b\leq 4-3c$ we can use Chebyshev's inequality to obtain:

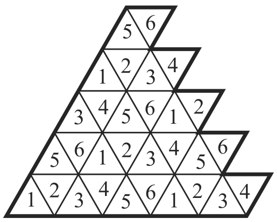
$$\frac{4-3a}{(a-1)(4-a)}+\frac{4-3b}{(b-1)(4-b)}+\frac{4-3c}{(c-1)(4-c)}\geq\frac{4-3a+4-3b+4-3c}{3}\cdot$$
$$\cdot\left(\frac{1}{(a-1)(4-a)}+\frac{1}{(b-1)(4-b)}+\frac{1}{(c-1)(4-c)}\right)=0.$$

Equality holds for $4-3a=4-3b=4-3c$ i.e. $a=b=c=\frac{4}{3}$.

5. Out of an equilateral triangle with side 2014 an equilateral triangle with side 214 is cut out, such that the two triangles have one vertex in common and two of the sides of the cut-out triangle lie on two of the sides of the initial one. Can this figure be covered by the figures shown below without overlap (rotation is allowed), if the triangles in the figures are equilateral with side 1? Justify your answer!



Solution. First we cut the given figure into equilateral triangles with side 1. We label the triangles in the given figure by numbers from 1 to 6, as on the picture to the right. (in the first row successively from 1 to 6, then the numbers repeat, in the second we start from 5, in the third from 3, then from 1 and the procedure repeats). It can easily be noticed that each of the given figures covers exactly one of the numbers 1 to 6. Therefore, in order for the figure to be coverable by the given figures, each of the numbers has to appear an equal number of times. If we compare how often the number 1 and the number 2 appear we will notice that in the first, fourth and each row of the form $3k+1$ there is one more 1 than 2's, and in the remaining rows the number of 1's and 2's is equal, therefore, it follows that the number of 1's and 2's is unequal, and therefore not every number can appear an equal number of times. It follows that the figure cannot be covered in the required way.



Balkan Mathematical Olympiads 2014
02.05-07.05.2014, Pleven, Bulgaria

Problem 1. Let x, y and z be positive real numbers such that $xy + yz + zx = 3xyz$. Prove that

$$x^2y + y^2z + z^2x \geq 2(x + y + z) - 3$$

Solution. The given condition can be rearranged to $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3$. Using this, we obtain:

$$x^2y + y^2z + z^2x - 2(x + y + z) - 3 = x^2y - 2x + \frac{1}{y} + y^2z - 2y + \frac{1}{z} + z^2x - 2x + \frac{1}{y} =$$

$$= y\left(x - \frac{1}{y}\right)^2 + z\left(y - \frac{1}{z}\right)^2 + x\left(z - \frac{1}{x}\right)^2 \geq 0$$

Equality holds if and only if we have $xy + yz + zx = 1$, or, in other words, $x = y = z = 1$.

Alternative solution. It follows from $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3$ and Cauchy-Schwarz inequality that

$$3(x^2y + y^2z + z^2x) = \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)(x^2y + y^2z + z^2x) =$$

$$= \left(\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right)((x\sqrt{y})^2 + (y\sqrt{z})^2 + (z\sqrt{x})^2) \geq$$

$$= (x + y + z)^2$$

Therefore, $x^2y + y^2z + z^2x) \geq \frac{(x + y + z)^2}{3}$ and if $x + y + z = t$ it suffices to show that $\frac{t^2}{3} \geq 2t - 3$. The latter is equaivalent to $(t - 3)^2 \geq 0$. Equality holds when

$$x\sqrt{y}\sqrt{y} = y\sqrt{z}\sqrt{z} = z\sqrt{x}\sqrt{x},$$

i.e. $xy = yz = zx$, and $t = x = y + z = 3$. Hence, $x = y = z = 1$.

Comment. The inequality is true with the condition $xy + yz + zx \leq 3xyz$.

Problem 2. A special number is a positive integer n for which there exist positive integer a, b, c and d with

$$n = \frac{a^3 + 2b^3}{c^3 + 2d^3}.$$

Prove that

(a) There are infinitely many special numbers;

(b) 2014 is not a special number.

Solution. (a) Every perfect cube k^3 of a positive integer is spcial because we can write

$$k^3 = k^3 \frac{a^3 + 2b^3}{a^3 + 2b^3} = \frac{(ka)^3 + 2(kb)^3}{a^3 + 2b^3},$$

for some positive integers a, b.

(b) Observe that $2014 = 2 \cdot 19 \cdot 53$. If 2014 is special, then we have,

$$x^3 + 2y^3 = 2014(u^3 + 2v^3) \qquad (1)$$

for some positive integers x, y, u, v. We may assume that $x^3 + 2y^3$ is minimal with this property. Now, we will use the fact that of 19 divides $x^3 + 2y^3$, then it divides both x and y. Indeed, if 19 does not divide x then it does not divide y too. The relation $x^3 \equiv -2y^3 \pmod{19}$ implies $(x^3)^6 \equiv (-2y^3)^6 \pmod{19}$. The latter congruence is equivalent to $x^{18} \equiv 2^6 y^{18} \pmod{19}$. Now, according to the Fermat's Little Theorem, we obtain $1 \equiv 2^6 \pmod{19}$, that is 19 divides 63 , not possible.

It follows $x = 19x_1, y = 19y_1$, for some positive integers x_1 and y_1 . Replacing in (1) we get

$$19^2(x_1^3 + 2y_1^3) = 2 \cdot 53(u^3 + 2v^3) \qquad (2)$$

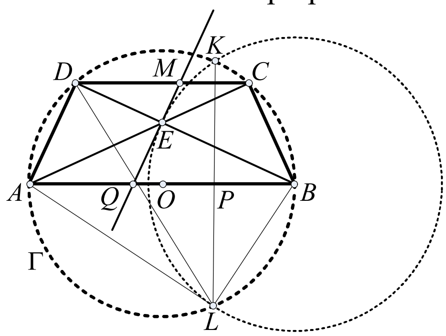
i.e. $19 \mid u^3 + 2v^3$. It follows $u = 19u_1$ and $v = 19v_1$, and replacing in (2) we get

$$x_1^3 + 2y_1^3 = 2014(u_1^3 + 2v_1^3) .$$

Clearly, $x_1^3 + 2y_1^3 \le x^3 + 2y^3$, contradicting the minimality of $x^3 + 2y^3$.

Problem 3. Let $ABCD$ be a trapezium inscribed in a circle Γ with diameter AB . Let E be the intersection point of the diagonals AC and BD. The circle with center B and radius BE meets Γ at the points K and L , where K is on the same side of AB as C . The line perpendicular to BD at E intersects CD at M .

Prove that KM is perpendicular to M .



Solution. Since $AB \parallel CD$, we have that $ABCD$ is isosceles trapezium. Let O be the center of k and EM meets AB at point Q . Then, from the right angled triangle BEQ , we have $BE^2 = BO \cdot BQ$. Since $BE = BK$, we get

$$BK^2 = BO \cdot BQ . \qquad (1)$$

Suppose that KL meets AB at P . Then, from the right angled triangle BAK , we have

$$BK^2 = BP \cdot BA . \qquad (2)$$

From (1) and (2) we get $\frac{BP}{BQ} = \frac{BO}{BA} = \frac{1}{2}$, and therefore

P is the midpoint of BQ . (3)

However, $DM \parallel AD$ (both are perpendicular to DB). Hence, $AQMD$ is parallelogram and thus $MQ = AD = BC$. We conclude that $QBCM$ is isosceles trapezium. It follows from (3) that KL is the perpendicular bisector of BQ and CM , that is, M is symmetric to C with respect to KL . Finally, we get that M is orthocenter of thew triangle DLK by using the well-known result that the reflection of the orthocenter of a triangle to every side belongs to the circumcircle of the triangle and vice versa.

Problem 4. Let n be a positive integer. A regular hexagon with side with length n is divided into equilateral triangles with side length 1 by lines parallel to its sides.

Find the number of regular hexagons all of whose vertices are aong the vertices of the equlateral triangles.

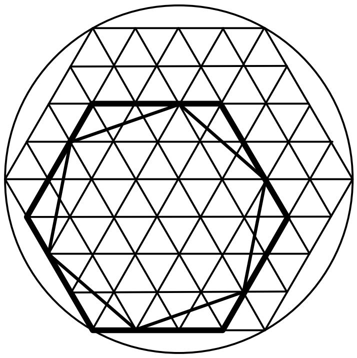
Solution. By a lattice hexagon we will mean a regular hexagon whose sides run along edges of the lattice. Given any regular hexagon H , we construct a lattice hexagon whose edges pass through the vertices of H , as shown in the figure, which we will call enveloping lattice hexagon of H . Given a lattice hexagon G of side length m , the number of regular hexagons whose enveloping lattice hexagon is G is exactly m .

Yet also there are precisely $3(n-m)(n-m+1)+1$ lattice hexagons of side length m in our lattice:they are those with centers lying at most $n-m$ steps from the centre of the lattice. In particular, the total number of regular equals

$$N=\sum_{m=1}^{n}(3(n-m)(n-m+1)+1)m=(3n^2+3n)\sum_{m=1}^{n}m-3(2n+1)\sum_{m=1}^{n}m^2+3\sum_{m=1}^{n}m^3 .$$

Since $\sum_{m=1}^{n}m=\frac{n(n+1)}{2}$, $\sum_{m=1}^{n}m^2=\frac{n(n+1)(2m+1)}{6}$ and $\sum_{m=1}^{n}m^3=\left(\frac{n(n+1)}{2}\right)^2$ it is easily checked that

$$N=\left(\frac{n(n+1)}{2}\right)^2 .$$



Junior Balkan Mathematical Olympiad 2014
21.06-26.06.2014, Ohrid, Republika Makedonija

Problem 1. *Find all distinct prime numbers p, q and r such that*

$$3p^4 - 5q^4 - 4r^2 = 26 .$$

Solution. First notice that if both primes q and r differ from 3, then $q^2 \equiv r^2 \equiv 1 (\mathrm{mod}\ 3)$, hence the left hand side of the given equation is congruent to zero modulo 3, which is impossible since 26 is not divisible by 3. Thus, $q = 3$ or $r = 3$. We consider two cases.

Case 1. $q = 3$.

The equation reduces to $3p^4 - 4r^2 = 431 \qquad (1)$.

If $p \neq 5$, by Fermat's little theorem, $p^4 \equiv 1\ (\mathrm{mod}\ 5)$, which yields $3 - 4r^2 \equiv 1\ (\mathrm{mod}\ 5)$, or equivalently, $r^2 + 2 \equiv 0\ (\mathrm{mod}\ 5)$. The last congruence is impossible in view of the fact that a residue of a square of a positive integer belongs to the set $\{0, 1, 4\}$. Therefore $p = 5$ and $r = 19$.

Case 2. $r = 3$.

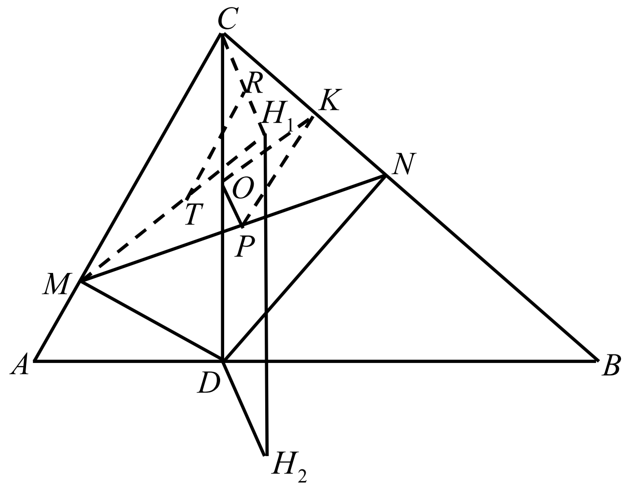
The equation becomes $3p^4 - 5q^4 = 62 \qquad (2)$.

Obviously $p \neq 5$. Hence, Fermat's little theorem gives $p^4 \equiv 1\ (\mathrm{mod}\ 5)$. But then $5q^4 \equiv 1\ (\mathrm{mod}\ 5)$, which is impossible.

Hence, the only solution of the given equation is $p = 5$, $q = 3$, $r = 19$.

Remark. Reduction of the equation to an equation with two variables brings 4 points. A further reduction to an equation in one variable brings additional 4 points. Completion brings the final 2 points.

Problem 2. Consider an acute triangle ABC with area S. *Let $CD \perp AB$ ($D \in AB$), $DM \perp AC$ ($M \in AC$) and $DN \perp BC$ ($N \in BC$). Denote by H_1 and H_2 the orthocentres of the triangles MNC and MND respectively. Find the area of the quadrilateral AH_1BH_2 in terms of S.*



Solution 1. Let *O*, *P*, *K*, *R* and *T* be the mid-points of the segments *CD*, *MN*, *CN*, CH_1 and MH_1, respectively. From ΔMNC we have that $\overline{PK} = \frac{1}{2}\overline{MC}$ and $PK \parallel MC$.

Analogously, from ΔMH_1C we have that $\overline{TR} = \frac{1}{2}\overline{MC}$ and $TR \parallel MC$. Consequently, $\overline{PK} = \overline{TR}$ and $PK \parallel TR$. Also $OK \parallel DN$ (from ΔCDN) and since $DN \perp BC$ and $MH_1 \perp BC$, it follows that $TH_1 \parallel OK$. Since *O* is the circumcenter of ΔCMN, $OP \perp MN$. Thus, $CH_1 \perp MN$ implies $OP \parallel CH_1$. We conclude $\Delta TRH_1 \cong \Delta KPO$ (they have parallel sides and $\overline{TR} = \overline{PK}$), hence $\overline{RH_1} = \overline{PO}$, i.e. $\overline{CH_1} = 2\overline{PO}$ and $CH_1 \parallel PO$.

Analogously, $\overline{DH_2} = 2\overline{PO}$ and $DH_2 \parallel PO$. From $\overline{CH_1} = 2\overline{PO} = \overline{DH_2}$ and $CH_1 \parallel PO \parallel DH_2$ the quadrilateral CH_1H_2D is a parallelogram, thus $\overline{H_1H_2} = \overline{CD}$ and $H_1H_2 \parallel CD$. Therefore the area of the quadrilateral AH_1BH_2 is $\frac{\overline{AB} \cdot \overline{H_1H_2}}{2} = \frac{\overline{AB} \cdot \overline{CD}}{2} = S$.

Solution 2. Since $MH_1 \parallel DN$ and $NH_1 \parallel DM$, $MDNH_1$ is a parallelogram. Similarly, $NH_2 \parallel CM$ and $MH_2 \parallel CN$ imply $MCNH_2$ is a parallelogram. Let P be the midpoint of the segment $\overline{MN}$. Then $\sigma_P(D) = H_1$ and $\sigma_P(C) = H_2$, thus $CD \parallel H_1H_2$ and $\overline{CD} = \overline{H_1H_2}$ (4points). From $CD \perp AB$ we deduce $A_{AH_1BH_2} = \frac{1}{2}\overline{AB} \cdot \overline{CD} = S.$

Remark. Just proving that $H_1H_2 \perp AB$ is worth 5 points. Just proving that $\overline{H_1H_2} = \overline{CD}$ brings 5 points. Proving both $H_1H_2 \perp AB$ and $\overline{H_1H_2} = \overline{CD}$ brings 9 points. The last point is obtained by deducing from this that the area of AH_1BH_2 is equal to S.

Problem 3. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq 3(a+b+c+1).$$

When does equality hold?

Solution1. By using AM-GM ($x^2+y^2+z^2 \geq xy+yz+zx$) we have

$$\begin{aligned}\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 &\geq \left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)+\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)+\left(c+\frac{1}{a}\right)\left(a+\frac{1}{b}\right)\\ &=\left(ab+1+\frac{a}{c}+a\right)+\left(bc+1+\frac{b}{a}+b\right)+\left(ca+1+\frac{c}{b}+c\right)\\ &=ab+bc+ca+\frac{a}{c}+\frac{c}{b}+\frac{b}{a}+3+a+b+c\end{aligned}$$

Notice that by AM-GM we have $ab+\frac{b}{a} \geq 2b,\ bc+\frac{c}{b} \geq 2c,$ and $ca+\frac{a}{c} \geq 2a$

Thus ,

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq \left(ab+\frac{b}{a}\right)+\left(bc+\frac{c}{b}\right)+\left(ca+\frac{a}{c}\right)+3+a+b+c \geq 3(a+b+c+1)$$

The equality holds if and only if $a=b=c=1$.

Solution2. From QM-AM we obtain

$$\sqrt{\frac{\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2}{3}} \geq \frac{a+\frac{1}{b}+b+\frac{1}{c}+c+\frac{1}{a}}{3} \Leftrightarrow$$

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq \frac{\left(a+\frac{1}{b}+b+\frac{1}{c}+c+\frac{1}{a}\right)^2}{3} \qquad (1)$$

From AM-GM we have $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 3\sqrt[3]{\frac{1}{abc}} = 3$ (1point) , and substituting in (1) we get

$$\begin{aligned}&\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq \frac{\left(a+\frac{1}{b}+b+\frac{1}{c}+c+\frac{1}{a}\right)^2}{3} \geq \frac{(a+b+c+3)^2}{3} =\\ &=\frac{(a+b+c)(a+b+c)+6(a+b+c)+9}{3} \geq \frac{(a+b+c)3\sqrt[3]{abc}+6(a+b+c)+9}{3} =\\ &=\frac{9(a+b+c)+9}{3} = 3(a+b+c+1)\end{aligned}$$

The equality holds if and only if $a=b=c=1$

Solution 3. By using $x^2+y^2+z^2 \geq xy+yz+zx$

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2=a^2+b^2+c^2+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2}+\frac{2a}{b}+\frac{2b}{c}+\frac{2c}{a}\geq$$
$$\geq ab+ac+bc+\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}+\frac{2a}{b}+\frac{2b}{c}+\frac{2c}{a}.$$

Clearly

$$\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{abc}{bc}+\frac{abc}{ca}+\frac{abc}{ab}=a+b+c$$

$$ab+\frac{a}{b}+bc+\frac{b}{c}+ca+\frac{c}{a}\geq 2a+2b+2c$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}=3$$

Hence

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2\geq\left(ab+\frac{a}{b}\right)+\left(ac+\frac{c}{a}\right)+\left(bc+\frac{b}{c}\right)+a+b+c+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq$$
$$\geq 2a+2b+2c+a+b+c+3=3\left(a+b+c+1\right)$$

The equality holds if and only if $a=b=c=1$

Solution4. $a=\frac{x}{y}$, $b=\frac{y}{z}$, $c=\frac{z}{x}$

$$\left(\frac{x}{y}+\frac{z}{y}\right)^2+\left(\frac{y}{z}+\frac{x}{z}\right)^2+\left(\frac{z}{x}+\frac{y}{x}\right)^2\geq 3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+1\right)$$

$$(x+z)^2x^2z^2+(y+x)^2y^2x^2+(z+y)^2z^2y^2\geq 3xyz(x^2z+y^2x+z^2y+xyz)$$

$$x^4z^2+2x^3z^3+x^2z^4+x^2y^4+2x^3y^3+x^4y^2+y^2z^4+2y^3z^3+y^4z^2\geq$$
$$\geq 3x^3yz^2+3x^2y^3z+3xy^2z^3+3x^2y^2z^2$$

1)$x^3y^3+y^3z^3+z^3x^3\geq 3x^2y^2z^2$.

$$\left.\begin{array}{l}2)x^4z^2+z^4x^2+x^3y^3\geq 3x^3z^2y\\3)x^4y^2+y^4x^2+y^3z^3\geq 3y^3x^2z\\4)z^4y^2+y^4z^2+x^3z^3\geq 3z^3y^2x\end{array}\right\}$$

Equality holds when $x=y=z$, i.e., $a=b=c=1$.

Solution 5. $\sum\limits_{cyc}(a+\frac{1}{b})^2\geq 3\sum\limits_{cyc}a+3$

$$\Leftrightarrow 2\sum_{cyc}\frac{a}{b}+\sum_{cyc}\left(a^2+\frac{1}{a^2}-3a-1\right)\geq 0$$

$$2\sum_{cyc}\frac{a}{b}\geq 6\sqrt[3]{\frac{a}{b}\frac{b}{c}\frac{c}{a}}=6 \qquad (1)$$

$$\forall a>0,\ a^2+\frac{1}{a^2}-3a\geq\frac{3}{a}-4$$
$$\Leftrightarrow a^4-3a^3+4a^2-3a+1\geq 0$$
$$\Leftrightarrow\left(a-1\right)^2\left(a^2-a+1\right)\geq 0(4\text{points})$$

$$\sum_{cyc}\left(a^2+\frac{1}{a^2}-3a-1\right)\geq 3\sum_{cyc}\frac{1}{a}-15\geq 9\sqrt[3]{\frac{1}{abc}}-15=-6 \qquad (2)$$

Using (1) and (2) we obtain

$$2\sum_{cyc}\frac{a}{b}+\sum\left(a^2+\frac{1}{a^2}-3a-1\right)\geq 6-6=0$$

Equality holds when $a=b=c=1$.

Remark. Just stating a known inequality does not worth any points.

Problem 4. For a positive integer n, two players A and B play the following game: Given a pile of s stones, the players take turn alternatively with A going first. On each turn the player is allowed to take either one stone, or a prime number of stones, or a multiple of n stones. The winner is the one who takes the last stone. Assuming both A and B play perfectly, for how many values of s the player A cannot win?

Solution. Denote by k the sought number and let $\{s_1,s_2,...,s_k\}$ be the corresponding values for s. We call each s_i a losing number and every other nonnegative integer a winning numbers.

(I) **Clearly every multiple of n is a winning number.**

Suppose there are two different losing numbers $s_i>s_j$, which are congruent modulo n. Then, on his first turn of play, player A may remove s_i-s_j stones (since $n|s_i-s_j$), leaving a pile with s_j stones for B. This is in contradiction with both s_i and s_j being losing numbers.

(II) Hence, there are at most $n-1$ losing numbers, i.e. $k\leq n-1$.

Suppose there exists an integer $r\in\{1,2,...,n-1\}$, such that $mn+r$ is a winning number for every $m\in\mathbb{N}_0$. Let us denote by u the greatest losing number (if $k>0$) or 0 (if $k=0$), and let $s=LCM(2,3,...,u+n+1)$. Note that all the numbers $s+2$, $s+3$, …, $s+u+n+1$ are composite. Let $m'\in\mathbb{N}_0$, be such that $s+u+2\leq m'n+r\leq s+u+n+1$. In order for $m'n+r$ to be a winning number, there must exist an integer p, which is either one, or prime, or a positive multiple of n, such that $m'n+r-p$ is a losing number or 0, and hence lesser than or equal to u. Since $s+2\leq m'n+r-u\leq p\leq m'n+r\leq s+u+n+1$, p must be a composite, hence p is a multiple of n (say $p=qn$). But then $m'n+r-p=(m'-q)n+r$ must be a winning number, according to our assumption. This contradicts our assumption that all numbers $mn+r$, $m\in\mathbb{N}_0$ are winning.

(III) Hence, each nonzero residue class modulo n contains a loosing number.

(IV) There are exactly $n-1$ losing numbers (one for each residue $r\in\{1,2,...,n-1\}$).

Similar proof of **(III) :**

Lemma: No pair (u,n) of positive integers satisfies the following property:

(*) In $\mathbb{N}$ exists an arithmetic progression $(a_t)_{t=1}^{\infty}$ with difference n such that each segment $[a_i-u,a_i+u]$ contains a prime.

Proof of the lemma: Suppose such a pair (u,n) and a corresponding arithmetic progression $(a_t)_{t=1}^{\infty}$ exist. In $\mathbb{N}$ exist arbitrarily long patches of consecutive composites. Take such a patch P of length $3un$. Then, at least one segment $[a_i-u,a_i+u]$ is fully contained in P, a contradiction.

Suppose such a nonzero residue class modulo n exists (hence $n>1$). Let $u\in\mathbb{N}$ be greater than every loosing number. Consider the members of the supposed residue class which are greater than u. They form an arithmetic progression with the property (*) , a contradiction (by the lemma).