8^{TH} European Mathematical Cup

14^{th} December 2019 - 22^{th} December 2019

Junior Category

MLADI NADARENI MATEMATIČARI
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Problems and Solutions

Problem 1. Every positive integer is marked with a number from the set $\{0, 1, 2\}$, according to the following rule:

if a positive integer k is marked with j, then the integer $k + j$ is marked with 0.

Let S denote the sum of marks of the first 2019 positive integers. Determine the maximum possible value of S.

(Ivan Novak)

First Solution. Consider an arbitrary marking scheme which follows the given rule.

Let a denote the number of positive integers from the set $\{1, \ldots, 2019\}$ which are marked with a 2, b the number of those marked with a 1, and c the number of those marked with a 0.

1 point.

We have $S = 2a + b$.

1 point.

For every positive integer $j \in \{1, \ldots, 2017\}$ which is marked with a 2, the number $j + 2$ is marked with a 0. This implies that the number of positive integers less than 2017 marked with 2 is less than or equal to c.

1 point.

Hence, this implies $a \leqslant c + 2$. We then have

$$S = 2a + b \leqslant a + b + c + 2 = 2019 + 2 = 2021.$$

3 points.

Consider the following marking scheme:

$$210|210|210|\underbrace{2200|2200|2200\ldots2200}_{\text{502 blocks of 2200}}|22|0000\ldots.$$

Here the i-th digit in the sequence denotes the mark of positive integer i. For this marking, $S = 2021$, and therefore 2021 is the maximum possible value of S.

4 points.

Second Solution. The marking scheme for which $S = 2021$ is the same as in the first solution.

4 points.

Let S_n denote the sum of marks of first n positive integers, and let a_k denote the mark of k. Without loss of generality we may assume $a_j = 0$ for all integers $j \leqslant 0$. We'll prove the following claim by strong mathematical induction:

$$\text{for every positive integer } n,\ S_n \leqslant n+2 \text{ and if equality holds, then } a_n = 2.$$

1 point.

The base cases for $n \in \{1, 2\}$ trivially hold. Suppose the claim is true for all $k \leqslant n$ for some $n \geqslant 2$.
Suppose there exists a marking scheme for which $S_{n+1} \geqslant n + 4$. Then if $a_{n+1} < 2$, we have $S_n \geqslant n + 3$, which is a contradiction. Hence, $a_{n+1} = 2$.

1 point.

This implies that $a_n \in \{0, 2\}$. If $a_n = 0$, then $S_{n-1} \geqslant n + 2$, which is a contradiction. So, $a_n = 2$.

1 point.

Now $a_{n-1} = 0$ because both a_n and a_{n+1} are nonzero. We now have $S_{n-2} \geqslant n$, and by the induction hypothesis, it must hold that $S_{n-2} = n$ and $a_{n-2} = 2$. However, this is in contradiction with a_n being nonzero. Hence, $S_{n+1} \leqslant n + 3$.

1 point.

Suppose $S_{n+1} = n + 3$ and $a_{n+1} \neq 2$. If $a_{n+1} = 0$, then $S_n \geqslant n + 3$, which is a contradiction. Thus, $a_{n+1} = 1$.

1 point.

Then $S_n = n + 2$, which implies $a_n = 2$. Then we must have $a_{n-1} = 0$, and then $S_{n-2} = n$, which implies $a_{n-2} = 2$, but a_n is nonzero, which is a contradiction. Therefore, the claim is true for $n + 1$, which implies it is true for all positive integers. In particular, $S_{2019} \leqslant 2021$, which combined with the construction implies that the maximum value of S is 2021.

1 point.

Notes on marking:

- If a student forgets to write additional zeros beyond the first 2019 digits in his construction, but the construction is otherwise valid, he should be awarded all **4 points** for this part.
- There are many different optimal marking schemes. For example, $2200|210|210|\ldots|210|22|000\ldots$, where the block $|210|$ repeats 671 times.
- In the **Second Solution**, if the student writes only the first part of the induction hypothesis without the assumption that $a_n = 2$ in the case of equality: he should be awarded **0 points**, unless he reaches additional conclusions which lead to the solution.
- In the **Second Solution**, if the student doesn't comment on the base case/cases at all, he should be deducted **1 point**.
- If the student proves any nontrivial lemma useful for any of the solutions, but the lemma itself isn't worth any points and the student wouldn't otherwise get any of the **6 points** given for proving the bound, he should get **1 point** for this part.

Problem 2. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence defined recursively such that $x_1 = \sqrt{2}$ and

$$x_{n+1} = x_n + \frac{1}{x_n} \text{ for } n \in \mathbb{N}.$$

Prove that the following inequality holds:

$$\frac{x_1^2}{2x_1x_2 - 1} + \frac{x_2^2}{2x_2x_3 - 1} + \ldots + \frac{x_{2018}^2}{2x_{2018}x_{2019} - 1} + \frac{x_{2019}^2}{2x_{2019}x_{2020} - 1} > \frac{2019^2}{x_{2019}^2 + \frac{1}{x_{2019}^2}}.$$

(Ivan Novak)

First Solution. Notice that by squaring the assertion $x_{n+1} = x_n + \frac{1}{x_n}$ we obtain the equality $x_{n+1}^2 = x_n^2 + \frac{1}{x_n^2} + 2 \implies x_n^2 + \frac{1}{x_n^2} = x_{n+1}^2 - 2$, which implies that the right hand side equals $\dfrac{2019^2}{x_{2020}^2 - 2}$.

1 point.

On the other hand, we have

$$2x_nx_{n+1} - 1 = 2x_n(x_n + \frac{1}{x_n}) - 1 = 2x_n^2 + 1.$$

1 point.

This implies that the sum on the left hand side can be written as

$$\frac{1}{2 + \frac{1}{x_1^2}} + \frac{1}{2 + \frac{1}{x_2^2}} + \ldots + \frac{1}{2 + \frac{1}{x_{2019}^2}}$$

1 point.

By squaring the given assertion, we get the equality $2 + \frac{1}{x_n^2} = x_{n+1}^2 - x_n^2$. This implies that the left hand side equals

$$\frac{1}{x_2^2 - x_1^2} + \frac{1}{x_3^2 - x_2^2} + \ldots + \frac{1}{x_{2019}^2 - x_{2018}^2} + \frac{1}{x_{2020}^2 - x_{2019}^2}.$$

1 point.

Using the inequality between arithmetic and harmonic mean, we find that the left hand side is greater than or equal to

$$\frac{2019^2}{(x_2^2 - x_1^2) + (x_3^2 - x_2^2) + \ldots + (x_{2020}^2 - x_{2019}^2)}.$$

4 points.

We now notice that the denominator is a telescoping sum and it equals $x_{2020}^2 - x_1^2$, which implies the right hand side equals

$$\frac{2019^2}{x_{2020}^2 - x_1^2} = \frac{2019^2}{x_{2020}^2 - 2},$$

which is exactly equal to the right hand side.

1 point.

The equality cannot hold because $x_2^2 - x_1^2 \neq x_3^2 - x_2^2$.

1 point.

Second Solution. As in the first solution, we obtain that the left hand side equals

$$\frac{1}{2+\frac{1}{x_1^2}}+\frac{1}{2+\frac{1}{x_2^2}}+\ldots+\frac{1}{2+\frac{1}{x_{2018}^2}}+\frac{1}{2+\frac{1}{x_{2019}^2}}.$$

2 points.

Using the inequality between arithmetic and harmonic mean, we get that the left hand side is greater than or equal to

$$\frac{2019^2}{2\cdot 2019+\frac{1}{x_1^2}+\frac{1}{x_2^2}+\ldots+\frac{1}{x_{2019}^2}}.$$

4 points.

We now prove by mathematical induction that

$$2\cdot n+\frac{1}{x_1^2}+\frac{1}{x_2^2}+\ldots+\frac{1}{x_{n-1}^2}=x_n^2$$

holds for every $n\in\mathbb{N}$.

1 point.

For $n=1$, we have $2\cdot 1=\sqrt{2}^2$. Suppose the claim is true for some $n\in\mathbb{N}$. Then

$$x_{n+1}^2=2+x_n^2+\frac{1}{x_n^2}=2+2n+\frac{1}{x_1^2}+\frac{1}{x_2^2}+\ldots+\frac{1}{x_{n-1}^2}+\frac{1}{x_n^2},$$

where we used the induction hypothesis for the last equality. This proves the claim.

2 points.

In particular, for $n=2019$, we have that

$$\frac{2019^2}{2\cdot 2019+\frac{1}{x_1^2}+\frac{1}{x_2^2}+\ldots+\frac{1}{x_{2019}^2}}=\frac{2019^2}{x_{2019}^2+\frac{1}{x_{2019}^2}},$$

which proves the inequality.

The equality cannot hold because $\frac{1}{x_1^2}+2\neq\frac{1}{x_2^2}+2$.

1 point.

Third Solution. We prove by mathematical induction that for every $n \geqslant 2$ the following inequality holds:

$$\frac{x_1^2}{2x_1x_2-1}+\frac{x_2^2}{2x_2x_3-1}+\ldots+\frac{x_n^2}{2x_nx_{n+1}-1} > \frac{n^2}{x_n^2+\frac{1}{x_n^2}}.$$

For $n=2$, the left hand side equals $\frac{2}{5}+\frac{4.5}{10}=\frac{17}{20}$, and the right hand side equals $\frac{4}{\frac{9}{2}+\frac{2}{9}}=\frac{72}{85}<\frac{17}{20}$, which proves the base case.

Suppose the claim was true for some $n \in \mathbb{N}$. Then by the induction hypothesis, we know that

$$\frac{x_1^2}{2x_1x_2-1}+\frac{x_2^2}{2x_2x_3-1}+\ldots+\frac{x_n^2}{2x_nx_{n+1}-1}+\frac{x_{n+1}^2}{2x_{n+1}x_{n+2}-1} > \frac{n^2}{x_n^2+\frac{1}{x_n^2}}+\frac{x_{n+1}^2}{2x_{n+1}x_{n+2}-1}.$$

It suffices to prove that

$$\frac{n^2}{x_n^2+\frac{1}{x_n^2}}+\frac{x_{n+1}^2}{2x_{n+1}x_{n+2}-1} \geqslant \frac{(n+1)^2}{x_{n+1}^2+\frac{1}{x_{n+1}^2}}.$$

1 point.

We now prove that $2x_{n+1}x_{n+2}-1=2x_{n+1}^2+1$ as in the first solution.

1 point.

We then have

$$\frac{n^2}{x_n^2+\frac{1}{x_n^2}}+\frac{x_{n+1}^2}{2x_{n+1}x_{n+2}-1}=\frac{n^2}{x_n^2+\frac{1}{x_n^2}}+\frac{x_{n+1}^2}{2x_{n+1}^2+1}=\frac{n^2}{x_n^2+\frac{1}{x_n^2}}+\frac{1}{2+\dfrac{1}{x_{n+1}^2}}.$$

1 point.

By the inequality of arithmetic and harmonic mean, this is greater than or equal to

$$\frac{(n+1)^2}{x_n^2+\frac{1}{x_n^2}+2+\frac{1}{x_{n+1}^2}}.$$

5 points.

Notice that squaring the assertion $x_{n+1}=x_n+\frac{1}{x_n}$, we obtain

$$x_n^2+\frac{1}{x_n^2}+2=x_{n+1}^2.$$

1 point.

This implies that

$$\frac{(n+1)^2}{x_n^2+\frac{1}{x_n^2}+2+\frac{1}{x_{n+1}^2}}=\frac{(n+1)^2}{x_{n+1}^2+\frac{1}{x_{n+1}^2}},$$

which is exactly equal to the right hand side. Therefore, the claim is proven by the principle of mathematical induction. In particular, the claim is true for $n=2019$, which proves the inequality.

1 point.

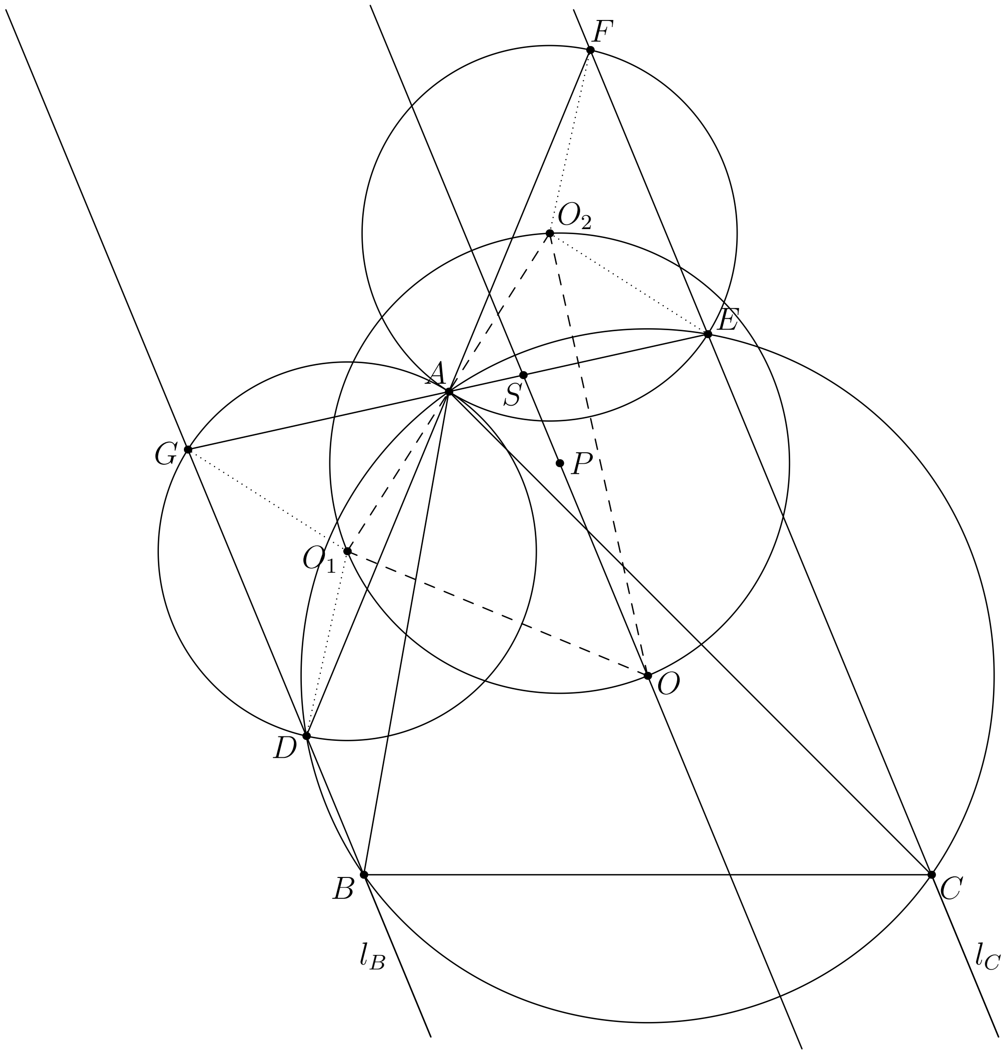
Notes on marking:

- Points from separate solutions can not be added. The student should be awarded the maximum of the points scored in the 3 presented solutions, or an appropriate number of points on an alternative solution.
- The third solution gives **5 points** for the use of AM-HM inequality as opposed to **4 points** in the first solution because in the third solution it is not necessary to comment the equality case. However, if a student has $n=1$ as a basis of induction and doesn't comment the equality case, he should be deducted **1 point** out of possible **5**.
- The point for proving that the equality cannot be achieved is only awarded if the student has proved the non-strict version of inequality.

Problem 3. Let ABC be a triangle with circumcircle ω. Let l_B and l_C be two lines through the points B and C, respectively, such that $l_B \parallel l_C$. The second intersections of l_B and l_C with ω are D and E, respectively. Assume that D and E are on the same side of BC as A. Let DA intersect l_C at F and let EA intersect l_B at G. If O, O_1 and O_2 are circumcenters of the triangles ABC, ADG and AEF, respectively, and P is the circumcenter of the triangle OO_1O_2, prove that $l_B \parallel OP \parallel l_C$.

(Stefan Lozanovski)

Sketch.



Solution. Let us write $\angle BAC = \alpha, \angle ABC = \beta, \angle ACB = \gamma$.

Lemma. Triangles AGD and AEF are similar to the triangle ABC.
Proof. As $DBCAE$ is a cylic pentagon we have

$$\angle GDA = \angle BCA = \gamma.$$

1 point.

Now from $l_B \parallel l_C$ we get that

$$\angle DBA = \angle DBC - \beta = 180^\circ - \angle BCE - \beta = \alpha + \gamma - \angle BCE = \alpha - \angle ACE$$

1 point.

so from the cyclicity

$$\angle BCD = \angle BAD = 180^\circ - \angle DBA - \angle ADB = 180^\circ - (\alpha - \angle ACE) - (180^\circ - \gamma) = \gamma - \alpha + \angle ACE$$

1 point.

Hence

$$\angle DAG = \angle DCE = \angle BCA - \angle BCD + \angle ACE = \alpha$$

1 point.

Therefore AGD is similar to the triangle ABC, and similarly for AEF. □

Now as G, A and E are collinear and F, A and D are collinear, using *Lemma* we get that O, O_1 and O_2 are collinear.

1 point.

As O_1 is the circumcenter of the triangle ADG and O_1D is the bisector of the chord $\overline{AD}$ we get that

$$\angle AO_1O = \frac{1}{2}\angle AO_1D = \angle AGD = \beta$$

and similarly $\angle AO_1O = \gamma$, so the triangle OO_1O_2 is similar to the triangle ABC.

2 points.

Now as P is the circumcenter of the triangle OO_1O_2 from the previous similarity we get that

$$\angle O_1OP = \angle BAO$$

1 point.

Hence

$$\angle DOP = \angle DOO_1 + \angle O_1OP = \angle DBA + \angle BAO = \angle DBA + \angle ABO = \angle DBO = \angle ODB$$

so $l_B \parallel OP \parallel l_C$.

2 points.

Notes on marking:

- If a student has a partial solution with analytic methods, only points for proving facts that can be expressed in geometric ways and lead to a compete solution can be awarded.

Problem 4. Let u be a positive rational number and m be a positive integer. Define a sequence $q_1, q_2, q_3, \ldots$ such that $q_1 = u$ and for $n \geqslant 2$:

$$\text{if } q_{n-1} = \frac{a}{b} \text{ for some relatively prime positive integers } a \text{ and } b, \text{ then } q_n = \frac{a+mb}{b+1}.$$

Determine all positive integers m such that the sequence $q_1, q_2, q_3, \ldots$ is eventually periodic for any positive rational number u.

Remark: A sequence $x_1, x_2, x_3, \ldots$ is *eventually periodic* if there are positive integers c and t such that $x_n = x_{n+t}$ for all $n \geqslant c$.

(Petar Nizić-Nikolac)

Solution. We will prove that the sequence is eventually periodic if and only if m is odd.

Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be sequences of numerators and denumerators of $q_1, q_2, q_3, \ldots$ respectively when written in the irreducible form, i.e. for $n \in \mathbb{N}$:

$$q_n = \frac{a_n}{b_n} \qquad gcd(a_n, b_n) = 1$$

Say that there was *reduction in the* n^{th} *step* if $\gcd(a_n + mb_n, b_n + 1) > 1$.

Case 1. m is even

Set $u = \frac{1}{1}$. Assume for the sake of contradiction that $q_1, q_2, q_3, \ldots$ is eventually periodic. Then $(b_n)_{n\in\mathbb{N}}$ is bounded so there is $r > 1$ (pick the smallest one) such that there was reduction in the r^{th} step. Easy to see that

$$q_1 = \frac{1}{1}, q_2 = \frac{m+1}{2}, q_3 = \frac{3m+1}{3}, q_4 = \frac{6m+1}{4}, q_5 = \frac{10m+1}{5}, \ldots, q_r = \frac{\binom{r}{2}m+1}{r}$$

2 points.

Now as m is even we have

$$\gcd(a_r + mb_r, b_r + 1) = \gcd\left(\binom{r}{2}m + 1 + mr, r+1\right) = \gcd\left(\binom{r+1}{2}m + 1, r+1\right) = \gcd\left((r+1)r\frac{m}{2} + 1, r+1\right) = 1$$

so this is a contradiction, and hence it is not eventually periodic for any positive rational number u.

1 point.

Case 2. m is odd

Assume that there is $r \in \mathbb{N}$ such that there was no reduction in the steps r, $r+1$, $r+2$ and $r+3$. Then for $i \in \{1, 2\}$:

$$(a_{r+i+2}, b_{r+i+2}) \equiv (a_{r+i} + mb_{r+i} + mb_{r+i+1}, b_{r+i} + 1 + 1) \equiv (a_{r+i} + 2mb_{r+i} + m, b_{r+i} + 2) \equiv (a_{r+i} + 1, b_{r+i}) \pmod 2$$

so at least one of the following pairs: $(a_{r+1}, b_{r+1}), (a_{r+2}, b_{r+2}), (a_{r+3}, b_{r+3}), (a_{r+4}, b_{r+4})$ has both even entries which is impossible (as they are coprime). Hence there was at least one reduction in the steps r, $r+1$, $r+2$ and $r+3$.

2 points.

Therefore for all $n \geqslant 1$:

$$\max\{b_{n+1}, b_{n+2}, b_{n+3}, b_{n+4}\} \leqslant \min\{b_{n+1}, b_{n+2}, b_{n+3}, b_{n+4}\} + 3 \leqslant \frac{1}{2}\max\{b_n, b_{n+1}, b_{n+2}, b_{n+3}\} + 3$$

so there exists $C \geqslant 1$ such that $b_n \leqslant 6$ for all $n \geqslant C$.

2 points.

Similarly for all $n \geqslant C$:

$$\max\{a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4}\} \leqslant \min\{a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4}\} + 3 \cdot 6m \leqslant \frac{1}{2}\max\{a_n, a_{n+1}, a_{n+2}, a_{n+3}\} + 18m$$

so there exists $D \geqslant C$ such that $a_n \leqslant 36m$ for all $n \geqslant D$.

2 points.

We conclude that for all $n \geqslant D$ there are finitely many pairs ($6 \cdot 36m = 216m$) that (a_n, b_n) attains so it becomes eventually periodic for any positive rational number u.

1 point.

Notes on marking:

- **Case 1** and **Case 2** are always worth **3 points** and **7 points** respectively.

8$^{\text{TH}}$ European Mathematical Cup

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Senior Category

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Problems and Solutions

Problem 1. For positive integers a and b, let $M(a, b)$ denote their greatest common divisor. Determine all pairs of positive integers (m, n) such that for any two positive integers x and y such that $x \mid m$ and $y \mid n$,

$$M(x + y, mn) > 1.$$

(Ivan Novak)

First Solution. We will prove that there are no solutions. Let m and n be any positive integers.

Let P denote the product of all primes which divide n and don't divide m. Then m is a divisor of m and P is a divisor of n, but we'll prove that $m + P$ and mn are relatively prime.

4 points.

Let p be any prime divisor of mn.
If p divides m, then p doesn't divide P and therefore p doesn't divide $m + P$.

3 points.

If p doesn't divide m, then p divides n, and then p divides P by definition of P, which implies that p doesn't divide $m + P$.

3 points.

Hence, $m + P$ and mn have no common prime factors, which implies they are relatively prime. Hence, there are no solutions.

Second Solution. We will prove that there are no solutions. Assume for the sake of contradiction that (m, n) was a solution. We will recursively construct an infinite unbounded sequence of pairs of positive integers $(x_k, y_k)_{k\in\mathbb{N}}$ such that $x_k \mid m$, $y_k \mid n$ and $M(x_k, y_k) = 1$.

1 point.

Then either $(x_k)_{k\in\mathbb{N}}$ or $(y_k)_{k\in\mathbb{N}}$ will be unbounded, but $x_k \leqslant m$ and $y_k \leqslant n$ for all $k \in \mathbb{N}$, which will yield a contradiction.

1 point.

Let $(x_1, y_1) = (1, 1)$. Let $k \in \mathbb{N}$. Suppose we have constructed (x_k, y_k) satisfying all of the above conditions. Then since (m, n) is a solution, there exists a prime divisor p of both mn and $x_k + y_k$.

1 point.

If p divides m, then let $(x_{k+1}, y_{k+1}) = (px_k, y_k)$.

2 points.

If p divides n and doesn't divide m, let $(x_{k+1}, y_{k+1}) = (x_k, py_k)$.

2 points.

In both cases x_{k+1} divides m and y_{k+1} divides n.

1 point.

Also, $M(x_{k+1}, y_{k+1}) = 1$ because p does not divide neither x_k nor y_k (as x_k and y_k are relatively prime and p divides $x_k + y_k$). Hence, the construction is valid.

2 points.

Notes on marking:

- In the First solution, there are different choices for pairs of divisors whose sum is relatively prime with mn. For example, one can take $(\text{rad}(m), \frac{\text{rad}(mn)}{\text{rad}(m)})$, where $\text{rad}(x)$ denotes the product of all prime divisors of x. If a student finds such a pair and claims that it is a solution without proving that their sum is relatively prime with mn, and if the proof is as straightforward as in the official solution, he should still get **4 points** from the first part of the solution.

Problem 2. Let n be a positive integer. An $n \times n$ board consisting of n^2 cells, each being a unit square coloured either black or white, is called *convex* if for every black coloured cell, both the cell directly to the left of it and the cell directly above it are also coloured black. We define the *beauty* of a board as the number of pairs of its cells (u, v) such that u is black, v is white and u and v are in the same row or column. Determine the maximum possible beauty of a convex $n \times n$ board.

(Ivan Novak)

First Solution. We colour the board so that in the i-th row, the leftmost $n+1-i$ cells are black. We'll call this board *the Unicorn.*

1 point.

The beauty of this board equals

$$2\sum_{k=1}^{n-1} k(n-k) = 2(\sum_{k=1}^{n-1} nk - \sum_{k=1}^{n-1} k^2) = 2\left(\frac{n^2(n-1)}{2} - \frac{n(n-1)(2n-1)}{6}\right) = \frac{n^3-n}{3}.$$

1 point.

We'll call any pair (u, v) such that u is white, v is black and u and v are in the same row or column a pretty pair. Now we will prove that the beauty of every convex board is less than or equal to the beauty of the Unicorn. We will do this by performing an algorithm which turns an arbitrary board into the Unicorn in finitely many steps.

Consider an arbitrary convex board. Let a_i be the number of black coloured cells in the i-th row. We perform the following algorithm:

If the board is equal to the Unicorn, we are done. Otherwise, find the first row in which $a_i \neq n+1-i$. Then, we consider two cases:

1. $a_i < n+1-i$. We colour the $a_i + 1$-th cell in the i-th row black.

   1 point.

   We claim that the beauty of the board didn't decrease.
   We now count the number of black/white cells which are in the same row or column as the cell we colored and which are distinct from it.
   The number of black cells in the same row is equal to a_i, and the number of black cells in the same column is $i-1$. On the other hand, the number of white cells in the same row is $n-1-a_i$ and the number of white cells in the same column is $n-i$.

   1 point.

   Therefore, the difference of beauties of the board before and after coloring the $a_i + 1$-th cell of i-th row black is $a_i + (i-1) - (n-1-a_i) - (n-i) = 2(a_i + i - n) \leqslant 0$, which implies that the new board's beauty is not smaller.

   1 point.

2. $a_i > n+1-i$. Let $j \geqslant i$ be the biggest index such that $a_j = a_i$. We colour the a_i-th cell of the j-th column white.

   1 point.

   We claim that the beauty of the board didn't decrease.
   As in the first case, we count the number of black/white cells which are in the same row or column as the cell we colored and which are distinct from it.
   The number of white cells in the same row equals $n - a_j$, and the number of white cells in the same column equals $n-j$. On the other hand, the number of black cells in the same row equals $a_j - 1$, and the number of black cells in the same column equals $j-1$.

   1 point.

   Therefore, the difference of beauties of the board before and after coloring the a_j-th cell of j-th row white is $(n-a_j) + (n-j) - (a_j - 1) - (j-1) = 2(n+1-j-a_j) \leqslant 2(n+1-i-a_i) < 0$, which implies that the new board's beauty is bigger.

   1 point.

The algorithm terminates because after each step, the number of positions where the board differs from the Unicorn decreases by 1. Therefore, the maximum beauty is achieved for the Unicorn.

1+1 points.

Second Solution. Consider an arbitrary convex board. Let a_i denote the number of black cells in the i-th row. Furthermore, we define $a_0 = n$ and $a_{n+1} = 0$. Then the number of pretty pairs (u, v) such that u and v are in the same row equals

$$\sum_{i=1}^{n} a_i(n - a_i).$$

1 point.

The number of columns with at least i black cells equals a_i.

1 point.

This implies that the number of columns with exactly i black cells equals the difference between the number of columns with at least i black cells and the number of columns with at least $i + 1$ black cells. Therefore, the number of columns with exactly i black cells equals $a_i - a_{i+1}$.

2 points.

This implies that the number of pretty pairs (u, v) such that u and v are in the same column equals

$$\sum_{i=1}^{n} (a_i - a_{i+1})i(n - i).$$

1 point.

Therefore, the beauty of the board equals

$$\sum_{i=1}^{n} a_i(n - a_i) + (a_i - a_{i+1})i(n - i) = \sum_{i=1}^{n} a_i(n - a_i + i(n - i) - (i - 1)(n + 1 - i)) = \sum_{i=1}^{n} a_i(2n + 1 - 2i - a_i).$$

1 point.

For a fixed $i \in \{1, \dots, n\}$, $a_i(2n + 1 - 2i - a_i)$ is a quadratic function of a_i, which is increasing for $a_i \in [0, n - i + \frac{1}{2}]$ and decreasing for $a_i \in [n - i + \frac{1}{2}, n]$, and the maximum among all integer a_i is then achieved if $a_i \in \{n - i, n - i + 1\}$.

1 point.

Therefore, the whole sum is maximised if $a_i \in \{n - i, n - i + 1\}$ for all $i \in \{1, \dots, n\}$.

1 point.

Any board with $a_i \in \{n - i, n - i + 1\}$ is convex since then $a_i \geqslant a_{i+1}$ for any of the possible choices.

1 point.

In this case, the sum equals

$$\sum_{i=1}^{n} (n - i)(n - i + 1) = \sum_{i=1}^{n} i(i - 1) = \sum_{i=1}^{n} i^2 - i = \frac{n(n + 1)(2n + 1) - 3n(n + 1)}{6} = \frac{n^3 - n}{6}.$$

1 point.

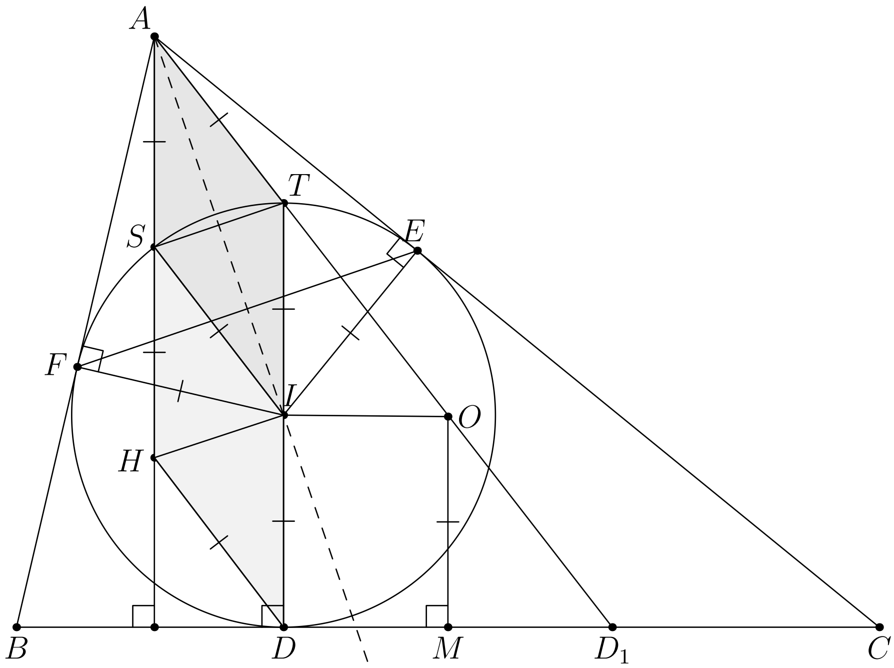
Notes on marking:

- A student is awarded the maximum of the two scores he gets by following either of the two marking schemes. Points from different solutions are not additive.
- If the student produces an optimal board, writes down its beauty, but does not simplify the expression to a closed form, then:
  a) if the student does not prove the optimality of the board (a "0+" solution), he is awarded **0 points** for this part;
  b) if the student proves the optimality of the board (a "10-" solution), he is awarded **1 point** for this part.
- In the **First Solution**, the last **2 points** are only awarded if he gives a correct algorithm.
- In the **First Solution**, if the student has a correct algorithm, but fails to prove that it terminates, he should be deducted **1 point**.
- In the **Second Solution**, the "other direction" is implicit in the last part of the solution. This is because the Unicorn configuration is covered by the given equality cases. If the student gives an optimal board as in the other solutions, and then shows that his optimal board is contained in the equality case, his solution is complete.
  However, if the student does not in any way show that his lower and upper bounds match, he should be deducted **1 point**.

Problem 3. In an acute triangle ABC with $|AB| \neq |AC|$, let I be the incenter and O the circumcenter. The incircle is tangent to $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ in D, E and F respectively. Prove that if the line parallel to EF passing through I, the line parallel to AO passing through D and the altitude from A are concurrent, then the point of concurrence is the orthocenter of the triangle ABC.

(Petar Nizić-Nikolac)

Sketch.



Solution. Let H be that concurrence point. We shall prove that H is the orthocenter of the triangle ABC. Firstly we observe that $AFIE$ is a deltoid (because $|AE| = |AF|$ and $|IE| = |IF|$), so $AI \perp EF \parallel HI$.

1 point.

Using the fact that AI is the bisector of $\angle OAH$ and $AH \parallel ID$ we conclude that

$$\angle DIH = \angle AID - 90^\circ = 180^\circ - \angle IAH - 90^\circ = 90^\circ - \frac{\angle OAH}{2} = 90^\circ - \frac{\angle HDI}{2}$$

so triangle IHD is an isoscales one.

1 point.

Denote by T the second intersection of the line DI and the incircle and S as the point such that $SHDI$ is a rhombus. It follows that S lies on AH, but also that triangle ISH is an isoscales one, so

$$\angle SIA = 90^\circ - \angle SIH = 90^\circ - \angle SHI = 90^\circ - \angle HID = \angle AIT = \angle IAS.$$

Hence $|AS| = |SI| = |ID| = |IT|$ (we used that I is the midpoint of $\overline{TD}$), so $ASIT$ is a rhombus.

3 points.

Lemma. A, T, O and D_1 are collinear, where D_1 is the point where A-excircle is tangent to BC.
Proof. Firstly, A, T and O are collinear as $AT \parallel SI \parallel HD \parallel AO$.

1 point.

Secondly, A, T and D_1 are collinear as there is homothety from A sending incircle to A-excircle, so the "highest" points (w.r.t. $\overline{BC}$) of these circles (T and D_1) and the center of homothety (A) are collinear. Therefore, A, T, O and D_1 are collinear. □

1 point.

Denote by M the midpoint of $\overline{BC}$. We know that $|BD| = \frac{|AB|+|BC|-|AC|}{2} = |CD_1|$, so M is the midpoint of $\overline{DD_1}$.

1 point.

As TDD_1 is a right triangle and $\angle OMD_1 = 90^\circ$ we conclude that $\overline{OM}$ is a D_1-midline in the triangle TDD_1, hence

$$2|OM| = |TD| = |TI| + |ID| = |AS| + |SH| = |AH|.$$

1 point.

Now we can conclude in various ways (for example, using the Euler line argument) that H is the orthocenter of the triangle ABC.

1 point.

Notes on marking:

- Essentially, **5 points** are awarded for proving that $AHDT$ is a parallelogram with longer side being twice the size of the shorter side, next **4 points** are awarded for proving that $2|OM| = |AH|$ is true, and **1 point** is awarded for deduction that H is indeed an orthocenter.
- If a student states that A, T, D_1 are collinear in a general triangle without using it to prove the problem (for example, by introducing the point O and stating that it should be on the line), it should be awarded **0 points**. On the other hand, if a student uses this fact to prove the problem, it does not have to prove this fact and it is enough to state it. In that case it is awarded **1 point**.
- If a student states that $2|OM| = |AH|$ in a general triangle without using it to prove the problem (for example, by noting that $|OM| = |ID|$), it should be awarded **0 points**. On the other hand, if a student uses this fact to prove the problem, it does not have to prove this fact and it is enough to state it. In that case it is awarded **1 point**.
- If a student has a partial solution with analytic methods, only points for proving facts that can be expressed in geometric ways and lead to a compete solution can be awarded.

Problem 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) + f(yf(x) + f(y)) = f(x + 2f(y)) + xy$$

for all $x, y \in \mathbb{R}$.

(Adrian Beker)

Solution. It is easily checked that $f(x) = x+1$ is a valid solution. We will prove that it is the only solution. Let $P(x, y)$ denote the assertion

$$f(x) + f(yf(x) + f(y)) = f(x + 2f(y)) + xy$$

and let $a = f(0)$. We will first prove the following claim:

Claim. f is injective

Proof: Suppose that $f(x) = f(y) = t$ for some $x, y \in \mathbb{R}$. We have:

$$P(x, x) \implies t + f(xt + t) = f(x + 2t) + x^2$$

$$P(x, y) \implies t + f(yt + t) = f(x + 2t) + xy$$

Subtracting the last two equations yields $f(xt+t)-f(yt+t) = x(x-y)$. Similarily, we have $f(yt+t)-f(xt+t) = y(y-x)$ which implies $(x - y)^2 = 0 \implies x = y$, hence f is injective. □

4 points.

We have:

$$P(x, 0) \implies f(x) + f(a) = f(x + 2a) \qquad (1)$$

Setting $x = -a$ yields $f(-a) = 0$. Now we have:

$$P(x, -a) \implies f(-af(x)) = -ax \qquad (2)$$

Again, setting $x = -a$ yields $a = a^2$, hence $a \in \{0, 1\}$.

1 point.

Case 1. $a = 0$

$$P(0, y) \implies f(f(y)) = f(2f(y))$$

Since f is injective, we have $f(y) = 2f(y) \implies f(y) = 0$ for all $y \in \mathbb{R}$, which is clearly impossible.

1 point.

Case 2. $a = 1$

Now (2) implies $f(-f(x)) = -x$ for all $x \in \mathbb{R}$ This means that f is bijective. On the other hand, (1) implies that $f(x) + f(1) = f(x + 2)$ for all $x \in \mathbb{R}$.

$$P(x+2, y) \implies f(x+2) + f(yf(x+2) + f(y)) = f(x+2+2f(y)) + (x+2)y$$

$$f(x) + f(y) + f(yf(x) + f(y) + yf(1)) = f(x + 2f(y)) + f(1) + xy + 2y$$

1 point.

By subtracting the initial equation from this one, we obtain:

$$f(yf(x) + f(y) + yf(1)) = f(yf(x) + f(y)) + 2y$$

If $y \neq 0$, we can choose $x \in \mathbb{R}$ such that $f(x) = -\frac{f(y)}{y}$ because f is surjective, hence the last equation yields:

$$f(yf(1)) = 2y + 1$$

2 points.

for all $y \neq 0$, but it is also true for $y = 0$. In particular, setting $y = -\frac{1}{2}$ yields $f(-\frac{f(1)}{2}) = 0$. Since f is injective and $f(-1)$ =0, it follows that $f(1) = 2 \implies f(2y) = 2y + 1$ for all $y \in \mathbb{R}$. Finally, we deduce that $f(x) = x + 1$ for all $x \in \mathbb{R}$, as desired. □

1 point.

Notes on marking:

- The case $a = 1$ can be finished without injectivity. If a student deduces that f is linear and checks that the only option for f is $f(x) = x + 1$, he should get **1 point**.
- If a student manages to prove that f is injective in the case $a = 0$, he should get **4 points** from the first part of the solution since in the case $a = 1$ the proof can be finished without injectivity.
- If a student doesn't check that $f(x) = x + 1$ is indeed a solution or at least mention that it can be easily checked, he should lose **1 point**.