XIV Asian Pacific Mathematics Olympiad
March 2002

Time allowed: 4 hours
No calculators are to be used
Each question is worth 7 points

Problem 1.

Let $a_1, a_2, a_3, \ldots, a_n$ be a sequence of non-negative integers, where n is a positive integer. Let

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \ .$$

Prove that

$$a_1!a_2!\ldots a_n! \geq \left(\lfloor A_n \rfloor !\right)^n ,$$

where $\lfloor A_n \rfloor$ is the greatest integer less than or equal to A_n, and $a! = 1 \times 2 \times \cdots \times a$ for $a \geq 1$ (and $0! = 1$). When does equality hold?

Problem 2.

Find all positive integers a and b such that

$$\frac{a^2 + b}{b^2 - a} \quad \text{and} \quad \frac{b^2 + a}{a^2 - b}$$

are both integers.

Problem 3.

Let ABC be an equilateral triangle. Let P be a point on the side AC and Q be a point on the side AB so that both triangles ABP and ACQ are acute. Let R be the orthocentre of triangle ABP and S be the orthocentre of triangle ACQ. Let T be the point common to the segments BP and CQ. Find all possible values of $\angle CBP$ and $\angle BCQ$ such that triangle TRS is equilateral.

Problem 4.

Let x, y, z be positive numbers such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

Show that

$$\sqrt{x + yz} + \sqrt{y + zx} + \sqrt{z + xy} \geq \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}.$$

Problem 5.

Let $\mathbf{R}$ denote the set of all real numbers. Find all functions f from $\mathbf{R}$ to $\mathbf{R}$ satisfying:

(i) there are only finitely many s in $\mathbf{R}$ such that $f(s) = 0$, and

(ii) $f(x^4 + y) = x^3 f(x) + f(f(y))$ for all x, y in $\mathbf{R}$.