Problem of Geometry proposed for the MMC 2017

Let ABC be an equilateral triangle, and let P be some point in its circumcircle. Determine, with reasons, all the numbers $n \in \mathbb{N}^*$ such that the sum

$$S_n(P) = |PA|^n + |PB|^n + |PC|^n$$

is independent of the choice of the point P.

Solution

We will take an orthonormal coordinate system, with origin in the point O (center of the circumcircle of ABC), taking moreover the point A on the Ox axis, and $|OA| = 1$. If the complex numbers z_A, z_B, z_C and z are respectively the affixes of the points A, B, C, P, we have

$$|z_A| = |z_B| = |z_C| = |z| = 1,$$

and therefore the first three are the roots of $z^3 = 1$, that is

$$z_A = 1;\ z_B = -\frac{1}{2} + i\frac{\sqrt{3}}{2};\ z_C = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

For another hand, $z = a + ib$, with $a^2 + b^2 = 1$. Then we have

$$S_n(P) = |PA|^n + |PB|^n + |PC|^n = |z - z_A|^n + |z - z_B|^n + |z - z_C|^n \qquad (*)$$

But as

$$|z - z_A| = \sqrt{2} \cdot \sqrt{1 - a};\ \ |z - z_B| = \sqrt{2 + a - b\sqrt{3}};\ \ |z - z_C| = \sqrt{2 + a + b\sqrt{3}},$$

we get from (*)

$$S_n(P) = 2^{n/2}(1 - a)^{n/2} + \left(2 + a - b\sqrt{3}\right)^{n/2} + \left(2 + a + b\sqrt{3}\right)^{n/2} \qquad (**)$$

If $P = A$, then $S_n(A) = 3^{n/2} + 3^{n/2} = 2 \cdot 3^{n/2}$. If $P_1\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, entonces $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ that is, $a = \frac{1}{2}, b = \frac{\sqrt{3}}{2}$ and from (**) we get

$$S_n(P_1) = 2^{n/2} \cdot 2^{-n/2} + \left(2 + \frac{1}{2} - \frac{3}{2}\right)^{n/2} + \left(2 + \frac{1}{2} + \frac{3}{2}\right)^{n/2} = 2 + 2^n.$$

Then, if $S_n(P)$ must to be independent of P, we get $S_n(A) = S_n(P_1) \iff 2 \cdot 3^{n/2} = 2 + 2^n \iff n = 2$ or $n = 4$.

Problem

Determine the smallest integer n, for which there exist integers $x_1, \ldots, x_n$ and positive integers $a_1, \ldots, a_n$ so that

$$x_1 + \cdots + x_n = 0, \qquad a_1x_1 + \cdots + a_nx_n > 0, \qquad a_1^2x_1 + \cdots + a_n^2x_n < 0.$$

Solution

The answer is $n = 3$. One possible example for $n = 3$ is $x_1 = 2$ and $x_2 = x_3 = -1$, with $a_1 = 4$, $a_2 = 1$, $a_3 = 6$.

For $n = 1$, the first constraint enforces $x_1 = 0$; this is in contradiction with the other two constraints. For $n = 2$, the first constraint enforces $x_2 = -x_1$. Then the second constraint is equivalent to $a_1x_1 - a_2x_1 > 0$. If we multiply this inequality by the positive value a_1+a_2, we get $a_1^2x_1-a_2^2x_1 > 0$; this is equivalent to $a_1^2x_1+a_2^2x_2 > 0$ and contradicts the third constraint.

Problem

A set S of integers is Balearic, if there are two (not necessarily distinct) elements $s, s' \in S$ whose sum $s + s'$ is a power of two; otherwise it is called a non-Balearic set.

Find an integer n such that $\{1, 2, \ldots, n\}$ contains a 99-element non-Balearic set, whereas all the 100-element subsets are Balearic.

Solution

Let $f(n)$ denote the largest cardinality of a non-Balearic set in $\{1, 2, \ldots, n\}$. One easily verifies that $f(0) = f(1) = 0$. Now consider an integer $n \geq 2$ and write it in the form $n = 2^a + b$ with $0 \leq b \leq 2^a - 1$. We want to show that

$$f(n) \;=\; f(2^a + b) \;=\; f(2^a - b - 1) + b.$$

Partition $\{1, 2, \ldots, n\}$ into $X = \{1, 2, \ldots, 2^a - b - 1\}$ and $Y = \{2^a - b, \ldots, 2^a + b\}$. A non-Balearic-subset S of $\{1, 2, \ldots, n\}$ contains at most $f(2^a - b - 1)$ elements from X (by definition of f) and at most b elements from Y (as it cannot contain 2^a altogether, and as it contains at most one of the two numbers $2^a - x$ and $2^a + x$). This establishes the first inequality $f(n) \leq f(2^a - b - 1) + b$.

Next consider a non-Balearic set $T \subseteq X$ of cardinality $f(2^a - b - 1)$. We claim that also $S = T \cup \{2^a + 1, \ldots, 2^a + b\}$ is a non-Balearic set. Suppose for the sake of contradiction that the sum $s + s'$ of some $s, s' \in S$ is a power of two. Then $s, s' \in T$ is impossible, as T itself is a non-Balearic set. Also $s, s' \in \{2^a + 1, \ldots, 2^a + b\}$ is impossible, as

$$2^{a+1} \;<\; (2^a + 1) + (2^a + 1) \;\leq\; s + s' \;\leq\; (2^a + b) + (2^a + b) \;<\; 2^{a+2}.$$

Hence one of s and s' must be in T and the other one in $\{2^a + 1, \ldots, 2^a + b\}$, which yields the final contradiction

$$2^a \;<\; s + s' \;\leq\; (2^a - b - 1) + (2^a + b) \;<\; 2^{a+1}.$$

Since the constructed non-Balearic set S is of cardinality $f(2^a - b - 1) + b$, we have established the second inequality $f(n) \geq f(2^a - b - 1) + b$. The two established inequalities together imply the desired recursive equation $f(n) = f(2^a - b - 1) + b$ displayed above.

The rest is computation.
It is easy to see (or to determine through the recursive equation) that $f(4) = 1$.
For $2^a = 8$ and $b = 3$, the recursion yields $f(11) = f(4) + 3 = 4$.
For $2^a = 32$ and $b = 20$, the recursion yields $f(52) = f(11) + 20 = 24$.
For $2^a = 128$ and $b = 75$, the recursion yields $f(203) = f(52) + 75 = 99$.
Hence an answer to the problem is $n = 203$ with $f(203) = 99$.

(Similar computations yield $f(202) = 98$ and $f(204) = 100$. Hence $n = 203$ constitutes the unique possible answer for the problem.)

MEDITERRANEAN MATHEMATICAL COMPETITION

Proposal

Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$(x^2+y^2+z^2)\left(\frac{a^3}{x^2+2y^2}+\frac{b^3}{y^2+2z^2}+\frac{c^3}{z^2+2x^2}\right)\geq\frac{1}{9}$$

holds for all positive reals x, y, z.

Solution. On account of the constrain $a+b+c=1$ we will prove that it holds the equivalent inequality

$$(x^2+y^2+z^2)\left(\frac{a^3}{x^2+2y^2}+\frac{b^3}{y^2+2z^2}+\frac{c^3}{z^2+2x^2}\right)\geq\frac{(a+b+c)^3}{9}$$

Indeed, Hölder's inequality claims that

$$\prod_{i=1}^{3}\left(a_i^3+b_i^3+c_i^3\right)^{1/3}\geq a_1a_2a_3+b_1b_2b_3+c_1c_2c_3$$

for all positive reals a_i, b_i, c_i, $1\leq i\leq 3$. Putting in the preceding

$$(a_1,a_2,a_3)=\left(\frac{a}{\sqrt[3]{x^2+2y^2}},1,\sqrt[3]{x^2+2y^2}\right),$$

$$(b_1,b_2,b_3)=\left(\frac{b}{\sqrt[3]{y^2+2z^2}},1,\sqrt[3]{y^2+2z^2}\right),$$

and

$$(c_1,c_2,c_3)=\left(\frac{c}{\sqrt[3]{z^2+2x^2}},1,\sqrt[3]{z^2+2x^2}\right)$$

yields

$$\sqrt[3]{3}\left(\frac{a^3}{x^2+2y^2}+\frac{b^3}{y^2+2z^2}+\frac{c^3}{z^2+2x^2}\right)^{1/3}(3x^2+3y^2+3z^2)^{1/3}\geq a+b+c$$

Cubing both sides and dividing both sides by $9\,(x^2+y^2+z^2)$ we obtain

$$\frac{a^3}{x^2+2y^2}+\frac{b^3}{y^2+2z^2}+\frac{c^3}{z^2+2x^2}\geq\frac{(a+b+c)^3}{9(x^2+y^2+z^2)}$$

from which claimed inequality follows. Equality holds when $a=b=c=x=y=z=1/3$, and the proof is complete.