

**4<sup>TH</sup> EUROPEAN MATHEMATICAL CUP**  
5<sup>th</sup> December 2015–13<sup>th</sup> December 2015  
Junior Category



**Problem 1.** We are given an  $n \times n$  board. Rows are labeled with numbers 1 to  $n$  downwards and columns are labeled with numbers 1 to  $n$  from left to right. On each field we write the number  $x^2 + y^2$  where  $(x, y)$  are its coordinates. We are given a figure and can initially place it on any field. In every step we can move the figure from one field to another if the other field has not already been visited and if at least one of the following conditions is satisfied:

- the numbers in those 2 fields give the same remainders when divided by  $n$ ,
- those fields are point reflected with respect to the center of the board.

Can all the fields be visited in case:

- $n = 4$ ,
- $n = 5$ ?

(Josip Papić)

**Problem 2.** Let  $m, n, p$  be fixed positive real numbers which satisfy  $mnp = 8$ . Depending on these constants, find the minimum of

$$x^2 + y^2 + z^2 + mxy + nxz + pyz,$$

where  $x, y, z$  are arbitrary positive real numbers satisfying  $xyz = 8$ . When is the equality attained?  
Solve the problem for:

- $m = n = p = 2$ ,
- arbitrary (but fixed) positive real numbers  $m, n, p$ .

(Stijn Cambie)

**Problem 3.** Let  $d(n)$  denote the number of positive divisors of  $n$ . For positive integer  $n$  we define  $f(n)$  as

$$f(n) = d(k_1) + d(k_2) + \dots + d(k_m),$$

where  $1 = k_1 < k_2 < \dots < k_m = n$  are all divisors of the number  $n$ . We call an integer  $n > 1$  *almost perfect* if  $f(n) = n$ . Find all almost perfect numbers.

(Paulius Ašvydis)

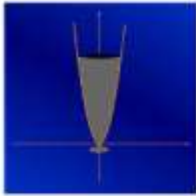
**Problem 4.** Let  $ABC$  be an acute angled triangle. Let  $B', A'$  be points on the perpendicular bisectors of  $AC, BC$  respectively such that  $B'A \perp AB$  and  $A'B \perp AB$ . Let  $P$  be a point on the segment  $AB$  and  $O$  the circumcenter of the triangle  $ABC$ . Let  $D, E$  be points on  $BC, AC$  respectively such that  $DP \perp BO$  and  $EP \perp AO$ . Let  $O'$  be the circumcenter of the triangle  $CDE$ . Prove that  $B', A'$  and  $O'$  are collinear.

(Steve Dinh)

*Time allowed: 240 minutes.*

*Each problem is worth 10 points.*

*Calculators are not allowed.*



## Problems and Solutions

**Problem 1.** We are given an  $n \times n$  board. Rows are labeled with numbers 1 to  $n$  downwards and columns are labeled with numbers 1 to  $n$  from left to right. On each field of the board we write the number  $x^2 + y^2$  where  $(x, y)$  are its coordinates. We are given a figure and can initially place it on any field. In every step we can move the figure from one field to another if the other field has not already been visited and if at least one of the following conditions is satisfied:

- the numbers in those 2 fields give the same remainders when divided by  $n$ ,
- those fields are point reflected with respect to the center of the board.

Can all the fields be visited in case:

- $n = 4$ ,
- $n = 5$ ?

(Josip PupiĆ)

**Solution.** a) The answer is NO.

	1	2	3	4
1	2	5	10	17
2	5	8	13	20
3	10	13	18	25
4	17	20	25	36

	1	2	3	4
1	2	1	2	1
2	1	0	1	0
3	2	1	2	1
4	1	0	1	0

On the left we have the board from the problem, on the right we have the same board, but with remainders of the values from the board instead of the values themselves.

We will denote *field*  $i$  for a field with number  $i$  written on it in the right table. Let's assume that we can visit all of the fields. That means that at some point we will visit a *field* 1. Obviously, when using the first type of move, we can visit any other *field* 1 which hasn't yet been visited. Also, it is easy to notice, that for *field* 1, the reflection of that field is also a *field* 1. That means that both types of moves lead to another *field* 1. Also, in the same fashion we conclude that for the each step, if the figure is on the *field* 1, then in the step after (if that wasn't the last one) and in the step before (if that wasn't the first one) should be *field* 1.

Now we conclude that the first visited *field* 1 must be the field visited in the first step. Same way we conclude that the last visited *field* 1 must be the field visited in the last step. But, we know that all of *fields* 1 are visited consecutively, in exactly 8 moves (because there are 8 *fields* 1), while there are exactly 16 moves that we have to make. This leads to contradiction.

- The answer is YES.

	1	2	3	4	5
1	2	5	10	17	26
2	5	8	13	20	29
3	10	13	18	25	34
4	17	20	25	36	41
5	26	29	34	41	50

	1	2	3	4	5
1	2	0	0	2	1
2	0	3	3	0	4
3	0	3	3	0	4
4	2	0	0	2	1
5	1	4	4	1	0

Again, on the left we have the board from the problem, on the right we have the same board, but with remainders of the values from the board instead of the values themselves.

We can move from any field to another with the same number written on the field in the right table by using the second move.

One idea to visit all the fields is the following:

- Find the 4 pairs of the fields of types *field i* and *field j*, such that all 8 fields are different, in each pair  $i \neq j$ , those two field in one pair are symmetric, and the second member of the  $n$ -th pair has the same value on the right board as the first member of the  $(n + 1)$ -th pair. Also, we want that all the values of the right table are mentioned through members of those pairs. For example:

$$((2, 2), (4, 4)), ((1, 4), (5, 2)), ((3, 5), (3, 1)), ((2, 1), (4, 5))$$

- Now, the algorithm is: after second member of  $n$ -th pair and before the first member of the  $(n + 1)$ -th pair visit all fields by using the first step. Of course, before first pair and after fourth pair move in similar way. Jump from the first member of the pair to the second member of the pair by using second step.

This is one of the ways to do it: We start with the field (3,3). Then we visit all of the *fields* 3, using the first move, in any way as long as the last visited field is (2, 2). Then, using the second move, we visit the field (4, 4). Again, using the first move we visit all *fields* 2 in any way as long as the last visited field is (1, 4). Using the second move we visit the field (5, 2). Then, using the first move we visit all *fields* 4 in any way as long as the last visited field is (3, 5). In same fashion, using the second move we visit the field (3, 1). After visiting all *fields* 0 in any way as long as the last visited field is (2, 1), we visit the field (4, 5) using the second move. We conclude by visiting all *fields* 1 in any way.

**Problem 2.** Let  $m, n, p$  be fixed positive real numbers which satisfy  $mnp = 8$ . Depending on these constants, find the minimum of

$$x^2 + y^2 + z^2 + mxy + nxz + pyz,$$

where  $x, y, z$  are arbitrary positive real numbers satisfying  $xyz = 8$ . When is the equality attained? Solve the problem for:

- a)  $m = n = p = 2$ ,
- b) arbitrary (but fixed) positive real  $m, n, p$ .

(Stijn Cambie)

**First Solution.** a) Use AM-GM and  $xyz = 8$  to get

$$x^2 + y^2 + z^2 + xy + xy + yz + yz + xz + xz \geq 9\sqrt[3]{x^6y^6z^6} = 36.$$

We have equality for  $x = y = z = 2$ .

- b) Using  $xyz = 8$ , we can transform the given expression:

$$x^2 + y^2 + z^2 + mxy + nxz + pyz = x^2 + \frac{8p}{x} + y^2 + \frac{8n}{y} + z^2 + \frac{8m}{z}$$

Since all numbers are positive reals, we can apply AM-GM inequality to get:

$$x^2 + \frac{8p}{x} = x^2 + \frac{4p}{x} + \frac{4p}{x} \geq 6\sqrt[3]{2p^2}$$

When we apply the same procedure for  $x, y, z$  and sum the inequalities, we get:

$$x^2 + y^2 + z^2 + mxy + nxz + pyz = x^2 + \frac{8p}{x} + y^2 + \frac{8n}{y} + z^2 + \frac{8m}{z} \geq 6\sqrt[3]{2}(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2}).$$

In order to get equality, we must have equality in all above inequalities and that happens for

$$\begin{aligned} x &= \sqrt[3]{4p}, \\ y &= \sqrt[3]{4n}, \\ z &= \sqrt[3]{4m}. \end{aligned}$$

Desired minimum is therefore

$$6\sqrt[3]{2}(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2}).$$

**Second Solution.** We only present solution for b) part here, marking scheme for a) part is the same as in first solution. We use weighted AM-GM:

$$\begin{aligned} & x^2 + y^2 + z^2 + mxy + nxz + pyz = \\ & \sqrt[3]{p^2} \frac{x^2}{\sqrt[3]{p^2}} + \sqrt[3]{n^2} \frac{y^2}{\sqrt[3]{n^2}} + \sqrt[3]{m^2} \frac{z^2}{\sqrt[3]{m^2}} + 2\sqrt[3]{m^2} \frac{mxy}{2\sqrt[3]{m^2}} + 2\sqrt[3]{n^2} \frac{nxz}{2\sqrt[3]{n^2}} + 2\sqrt[3]{p^2} \frac{pyz}{2\sqrt[3]{p^2}} \geq \\ & 3(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2}) \cdot \sqrt[3]{(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2})} \sqrt{\left(\frac{x^2}{\sqrt[3]{p^2}}\right)^{\sqrt[3]{p^2}} \left(\frac{y^2}{\sqrt[3]{n^2}}\right)^{\sqrt[3]{n^2}} \left(\frac{z^2}{\sqrt[3]{m^2}}\right)^{\sqrt[3]{m^2}}} \\ & \quad \sqrt[3]{(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2})} \sqrt{\left(\frac{\sqrt[3]{m}xy}{2}\right)^{2\sqrt[3]{m^2}} \left(\frac{\sqrt[3]{n}xz}{2}\right)^{2\sqrt[3]{n^2}} \left(\frac{\sqrt[3]{p}yz}{2}\right)^{2\sqrt[3]{p^2}}} \\ & = 3(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2}) \cdot \sqrt[3]{(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2})} \sqrt{\left(\frac{xyz}{2}\right)^{2(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2})}} \\ & = 3(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2}) \cdot \sqrt{\left(\frac{xyz}{2}\right)^2} = 3(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2}) \cdot \sqrt[3]{4} = 6\sqrt[3]{2}(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2}) \end{aligned}$$

We have shown that the minimum value the expression can take is  $6\sqrt[3]{2}(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2})$ . Equality can only be achieved when  $x = \sqrt[3]{4p}, y = \sqrt[3]{4n}, z = \sqrt[3]{4m}$ .

**Problem 3.** Let  $d(n)$  denote the number of positive divisors of  $n$ . For positive integer  $n$  we define  $f(n)$  as

$$f(n) = d(k_1) + d(k_2) + \dots + d(k_m),$$

where  $1 = k_1 < k_2 < \dots < k_m = n$  are all divisors of the number  $n$ . We call an integer  $n > 1$  *almost perfect* if  $f(n) = n$ . Find all almost perfect numbers.

(Paulius Ašvydis)

**First Solution.** Alternative way to define  $f(n)$  is

$$f(n) = \sum_{k|n, k \geq 1} d(k).$$

Let  $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$  be the prime factorisation of  $n$ . We have  $d(n) = \prod_{i=1}^r (a_i + 1)$ .

We prove the function  $f$  is multiplicative, in particular, given coprime  $n, m$  we have  $f(nm) = f(n)f(m)$ .

Using  $n, m$  are coprime for the second inequality and the fact that function  $d$  is multiplicative we get:

$$f(nm) = \sum_{k|nm} d(k) = \sum_{k_1|n, k_2|m} d(k_1 k_2) = \sum_{k_1|n, k_2|m} d(k_1) d(k_2) = \left( \sum_{k_1|n} d(k_1) \right) \left( \sum_{k_2|m} d(k_2) \right) = f(n)f(m)$$

If  $r = 1$  we have  $n = p_1^{a_1}$ . We note that divisors of  $n$  are  $1, p_1, p_1^2, \dots, p_1^{a_1}$  so  $f(n) = \sum_{i=0}^{a_1} (i+1) = \frac{(a_1+1)(a_1+2)}{2}$ .

Combining this with the multiplicativity result for  $f$  we deduce  $f(n) = \prod_{i=1}^r \frac{(a_i+1)(a_i+2)}{2}$ .

We now prove that for primes  $p \geq 5$  and  $p = 3$  provided  $a \geq 3$  we have  $f(p^a) = \frac{(a+1)(a+2)}{2} < \frac{3}{2}p^a$  by induction on  $a$ . As a basis  $3 < \frac{2^2}{2}$  for  $p \geq 5$  and  $6 < \frac{3}{2} \cdot 3^3$ . For the step it is enough to notice that  $\frac{a+3}{a+1} \leq 2 < p$  in both cases.

Similarly we can prove for  $p = 2$  that  $f(p^a) < p^a$  provided  $a \geq 4$ . By explicitly checking the remaining cases  $p = 2$  and  $a = 1, 2, 3$  and  $p = 3$   $a = 1, 2$  we conclude  $f(p^a) \leq \frac{3}{2}p^a$  for all  $p, a$  and  $f(p^a) \leq p^a$  for all  $p \geq 3$  and  $p = 2, a \geq 4$ .

Assuming  $f(n) = n$  we would have  $\prod_{i=1}^k \frac{f(p_i^{a_i})}{p_i^{a_i}} = 1$  so the above considerations imply that only possible prime divisors are 2, 3. If  $k = 1$  the only possible solution is  $n = 3$ . If  $k = 2$  we have  $p_1 = 2, p_2 = 3$  and  $1 \leq a_1 \leq 2$  and  $1 \leq a_2 \leq 2$  which give 4 cases to check giving the other 2 solutions  $n = 18, 36$ .

So, all almost perfect numbers are 3, 18, 36.

**Second Solution.** We hereby present one similar but different solution which does not use a lot of properties of the function  $f$ .

Firstly, we will prove the following lemma:

**Lemma:** For any positive integer  $n > 1$  and prime  $p$  we have

$$f(pn) \leq 3f(n).$$

The equality holds if and only if  $GCD(p, n) = 1$ . *Proof:* For every integer  $m$  we have that the set of divisors of the number  $pm$  is the union of the following two sets:

- set of divisors of  $m$ ,
- set of divisors of  $m$  multiplied by  $p$ .

Also, those two mentioned sets are disjoint if and only if  $GCD(p, m) = 1$  (if we have that  $p, m$  are disjoint, then it is obvious that none of the divisors of  $pm$  are in both sets; if they are not coprime, then the number  $p$  belongs to both sets).

This is why we have  $d(pm) \leq 2d(m)$  and

$$f(pn) = \sum_{k|pn} d(k) \leq \sum_{k|n} d(k) + \sum_{k|n} d(pk) \leq f(n) + \sum_{k|n} 2d(k) = 3f(n).$$

In both inequalities equality holds if and only if sets from before are disjoint, i.e. when  $GCD(p, n) = 1$ .

Also, we simply see that  $f(2^k) = d(1) + d(2) + \dots + d(2^k) = 1 + 2 + \dots + (k+1) = \frac{(k+1)(k+2)}{2}$ .

Notice that if for some positive integer  $n$  we have  $f(n) < n$ , then for every  $p \geq 3$  we have  $f(pn) \leq 3f(n) \leq pf(n) < pn$ . Consequently, if  $f(n) < n$ , then for every odd  $m$  we have  $f(mn) < mn$ . Because of this, we will introduce new terms. Number  $n$  is *nice multiple* of  $m$  if  $m \mid n$  and  $\frac{n}{m}$  is odd number. Analogously, we define *nice divisor*. Our statement from above is: if for some  $n$  we have  $f(n) < n$ , then neither of its nice multiples is almost perfect number. Our strategy will be the following: check the cases of the "small" numbers and see ratio of numbers  $n$  and  $f(n)$ . When we have that  $n > f(n)$ , conclude that there are not almost perfect numbers among their nice multiples. With formula for  $f(2^k)$  conclude that for sufficiently big  $k$  (when  $f(2^k) < 2^k$ ) this is enough to conclude that there are no more almost perfect numbers. By induction, it is simple to prove that  $f(2^k) < 2^k$  for  $k \geq 4$ . Thus, there are no almost perfect numbers of the form  $2^k \cdot m$ , where  $k \geq 4$  and  $m$  is odd, since they all have  $2^k$  as their nice divisor. We only have to check the numbers of the form  $2^k \cdot m$ , where  $k \leq 3$  and  $m$  is odd.

*First case:  $k = 0$*

For any odd prime  $p$  we have  $f(p) = d(1) + d(p) = 3 \leq p$ . From that we see that  $n = 3$  is solution. Moreover, we do not have any more solutions: if some odd number has a prime divisor different from 3, since  $f(p) < p$  this number can not be almost perfect number; if it is a power of 3 bigger than 3, since  $f(9) < 3f(3) = 9$ , there are no more solutions as well (9 is nice divisor of every power of 3 bigger than 3).

*Second case:  $k = 1$*

For any odd prime we have  $f(2p) = 3f(2) = 9$ . If  $p > 5$  then we have  $2p > f(2p)$ , so for all almost perfect numbers of the form  $2^1 \cdot m$  number  $m$  has to have prime divisors 3 and/or 5.

We directly see that neither 6 or 10 is almost perfect. So, in this case, almost perfect number has to have a nice divisor of the form  $2 \cdot 9$ ,  $2 \cdot 15$  or  $2 \cdot 25$ . For  $n = 18$  we have another solution, in other two cases we have inequality  $f(n) < n$ . If we want to seek new solution in this case, since they cannot be nice multiples of 30 and 50, the only possibility is that almost perfect number has nice divisor  $2 \cdot 27$ . But we have (equality case in lemma) that  $f(2 \cdot 27) < 3f(2 \cdot 9) = 2 \cdot 27$ . So, there are no more solutions in this case.

*Third case:  $k = 2$*

For any odd prime we have  $f(4p) = 3f(4) = 18$ . If  $p > 5$  then we have  $4p > f(4p)$ , so for all almost perfect numbers of the form  $2^2 \cdot m$  number  $m$  has to have prime divisors 3 and/or 5.

We directly see that neither 12 or 20 is almost perfect. So, in this case, almost perfect number has to have a nice divisor of the form  $4 \cdot 9$ ,  $4 \cdot 15$  or  $4 \cdot 25$ . For  $n = 36$  we have another solution, in other two cases we have inequality  $f(n) < n$ . If we want to seek new solution in this case, since they cannot be nice multiples of 60 and 100, the only possibility is that almost perfect number has nice divisor  $4 \cdot 27$ . But we have (equality case in lemma) that  $f(4 \cdot 27) < 3f(4 \cdot 9) = 4 \cdot 27$ . So, there are no more solutions in this case.

*Fourth case:  $k = 3$*

For any odd prime we have  $f(8p) = 3f(8) = 30$ . Similarly to other cases, we only observe candidates of the form  $8 \cdot 3^l$ . Number  $8 \cdot 3$  is not almost perfect, all other candidates have nice divisor  $8 \cdot 9$ . But, we have  $f(72) = 60 < 72$ . As we always concluded, we do not have any new solutions.

So, all almost perfect numbers are 3, 18, 36.



**Problem 4.** Let  $ABC$  be an acute angled triangle. Let  $B', A'$  be points on the perpendicular bisectors of  $AC, BC$  respectively such that  $B'A \perp AB$  and  $A'B \perp AB$ . Let  $P$  be a point on the segment  $AB$  and  $O$  the circumcenter of the triangle  $ABC$ . Let  $D, E$  be points on  $BC, AC$  respectively such that  $DP \perp BO$  and  $EP \perp AO$ . Let  $O'$  be the circumcenter of the triangle  $CDE$ . Prove that  $B', A'$  and  $O'$  are collinear.

(Steve Dinh)

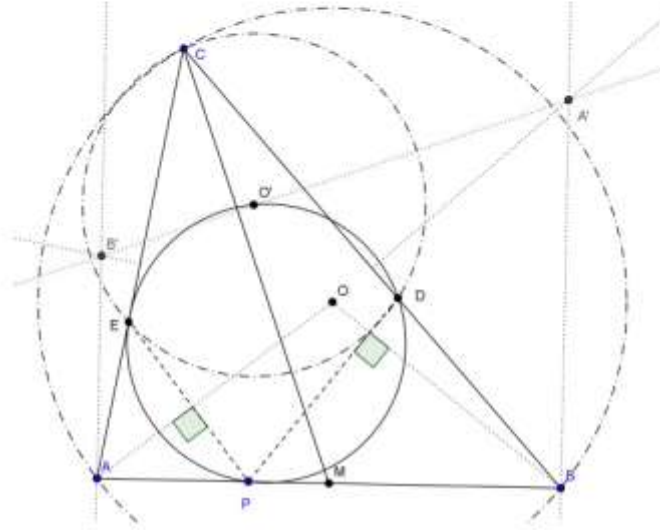
**Solution.** *Remark* We first start by giving some intuition on how the problem can be approached. We won't go into detail here but do give partial marks for correct ideas. We believe that any essentially correct solution should have them in the background so we don't require them to be written down explicitly.

We notice that if  $P \equiv A$  then  $O' \equiv B'$  while if  $P \equiv B$  we have  $O' \equiv A'$ . So the problem is equivalent to showing that as  $P$  varies on the segment  $AB$  respective  $O'$  map to a segment and we are now interested in identifying this segment.

It is hence natural to draw a picture not containing anything dependent on  $P$  and try to identify the line  $A'B'$ . Which turns out to be perpendicular to  $CM$  where  $M$  is the midpoint of  $AB$ .

Furthermore we note that  $B'M^2 - B'C^2 = AM^2 = A'M^2 - A'C^2$  and this defines the line uniquely (and shows  $A'B' \perp CM$ ).

The following sketch represents the problem setting when we do include the elements depending on  $P$ .

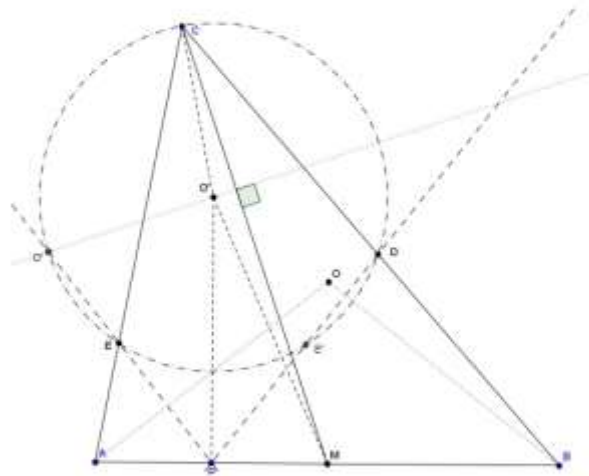


We now start with the formal proof.

It is enough to show that  $O'M^2 - O'C^2 = AM^2$  for all  $P$ , including  $P = A, B$ . Which allows us to draw the following sketch omitting  $B', C'$ .

We first prove that  $O'EPD$  is a cyclic quadrilateral. This follows as  $EO'D = 2ACB = APE + BPD = \pi - EPD$  as  $ACB = APE = BPD$ . This in turn implies  $PO'$  is an angle bisector of the angle  $EPD$  and  $PO' \perp AB$ .

We now have all the ingredients to show  $O'M^2 - O'C^2 = AM^2$ . The following sketch illustrates the last part of the proof.



We introduce the point  $D'$  as the second intersection of the line  $PE$  and the circumcircle of  $CDE$  so that  $O'P^2 - O'C^2 = PE \cdot PD'$ .

Now as  $PO'$  is the angle bisector of  $EPD$  we have  $PD = PD'$  by the extended  $S - S - K$  congruency theorem and the following observation. There is some care needed here, mainly the options we get by  $S - S - K$  are  $PD = PD'$  or

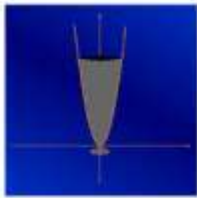
$PD = PE$  but if  $PD = PE$  triangles  $P'EO'$  and  $P'DO'$  are congruent by  $S-S-S$  congruency theorem so in particular  $EO'P = DO'P = CAB$  while  $EPO' = DPO' = \frac{\pi}{2} - CAB$  so  $PD$  and  $PE$  are tangents so in fact  $D' \equiv E$  so the above claim is still true.

Now noticing triangles  $APE$  and  $BPD$  are similar we get  $\frac{PE}{AP} = \frac{PD}{BP}$  implying  $AP \cdot BP = PE \cdot PD = PE \cdot PD'$

As  $PO' \perp AB$  by using pythagoras theorem we get  $O'M^2 - O'C^2 - AM^2 = O'P^2 - O'C^2 + PM^2 - AM^2 = PD' \cdot PE - AP \cdot BP = 0$ . Where we used  $O'P^2 - O'C^2 = PE \cdot PD'$  by the power of the point  $P$  to the circumcircle of  $CDE$  and  $AM^2 - PM^2 = (BM + PM)(AM - PM) = AP \cdot PB$ .

This completes the proof. ■





**4<sup>TH</sup> EUROPEAN MATHEMATICAL CUP**  
5<sup>th</sup> December 2015–13<sup>th</sup> December 2015  
Senior Category



**Problem 1.**  $A = \{a, b, c\}$  is a set containing three positive integers. Prove that we can find a set  $B \subset A$ ,  $B = \{x, y\}$  such that for all odd positive integers  $m, n$  we have

$$10 \mid x^m y^n - x^n y^m.$$

(Tomi Dimovski)

**Problem 2.** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove that

$$\frac{a+b+c+3}{4} \geq \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}.$$

(Dimitar Trenevski)

**Problem 3.** Circles  $k_1$  and  $k_2$  intersect in points  $A$  and  $B$ , such that  $k_1$  passes through the center  $O$  of the circle  $k_2$ . The line  $p$  intersects  $k_1$  in points  $K$  and  $O$  and  $k_2$  in points  $L$  and  $M$ , such that the point  $L$  is between  $K$  and  $O$ . The point  $P$  is orthogonal projection of the point  $L$  to the line  $AB$ . Prove that the line  $KP$  is parallel to the  $M$ -median of the triangle  $ABM$ .  
(Matko Ljulj)

**Problem 4.** A group of mathematicians is attending a conference. We say that a mathematician is  $k$ -content if he is in a room with at least  $k$  people he admires or if he is admired by at least  $k$  other people in the room. It is known that when all participants are in a same room then they are all at least  $3k+1$ -content. Prove that you can assign everyone into one of 2 rooms in a way that everyone is at least  $k$ -content in his room and neither room is empty. *Admiration is not necessarily mutual and no one admires himself.*

(Matija Bucić)

*Time allowed: 240 minutes.*

*Each problem is worth 10 points.*

*Calculators are not allowed.*



## Problems and Solutions

**Problem 1.**  $A = \{a, b, c\}$  is a set containing three positive integers. Prove that we can find a set  $B \subset A$ ,  $B = \{x, y\}$  such that for all odd positive integers  $m, n$  we have

$$10 \mid x^m y^n - x^n y^m.$$

(Tomi Dimovski)

**Solution.** Let  $f(x, y) = x^m y^n - x^n y^m$ . If  $n = m$ , the problem statement will be fulfilled no matter how we choose  $B$  so from now on, without loss of generality, we consider  $n > m$ . Since  $m$  and  $n$  are both odd, we have that  $n - m$  is even and we get

$$f(x, y) = x^m y^m (y^{n-m} - x^{n-m})$$

$$\implies f(x, y) = x^m y^m (y^2 - x^2) Q(x, y)$$

$$\implies f(x, y) = x^m y^m (y - x)(y + x) Q(x, y),$$

where  $Q(x, y) = y^{n-m-2} + y^{n-m-4} x^2 + \dots + x^{n-m-2}$ .

Now if one of  $x, y$  is even,  $f(x, y)$  is even. If both are odd, then  $f(x, y)$  is again even since  $x + y$  and  $x - y$  are even in that case. This shows that we only need to consider divisibility by 5. If  $A$  contains at least one element divisible by 5, we can put it in  $B$  and that will give us the solution easily. Now we consider the case when none of the elements in  $A$  is divisible by 5. If some two numbers in  $A$  give the same remainder modulo 5, we can choose them and then  $x - y$  will be divisible by 5 which solves the problem. Now we consider the case when all remainders modulo 5 in  $A$  are different. Take a look at the pairs  $(1, 4)$  and  $(2, 3)$ . Since we have three different remainders modulo 5 in  $A$ , by pigeonhole principle one of these pairs has to be completely in  $A$  (when elements are considered modulo 5). Then if we pick the numbers from  $A$  that correspond to those two remainders we get that  $x + y$  is divisible by 5 so the problem statement is fulfilled again. This completes the proof.

**Problem 2.** Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove that

$$\frac{a+b+c+3}{4} \geq \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}.$$

(Dimitar Trenevski)

**First Solution.** Rewrite the left hand side of inequality in following way:

$$\frac{a+b+c+3}{4} = \frac{a+b+c+3}{4\sqrt{abc}} = \frac{a+1}{4\sqrt{abc}} + \frac{b+1}{4\sqrt{abc}} + \frac{c+1}{4\sqrt{abc}}.$$

Rewrite denominators:

$$\frac{a+1}{4\sqrt{abc}} + \frac{b+1}{4\sqrt{abc}} + \frac{c+1}{4\sqrt{abc}} = \frac{a+1}{2\sqrt{ab \cdot c} + 2\sqrt{ac \cdot b}} + \frac{b+1}{2\sqrt{bc \cdot a} + 2\sqrt{ab \cdot c}} + \frac{c+1}{2\sqrt{ac \cdot b} + 2\sqrt{bc \cdot a}},$$

and then by arithmetic mean – geometric mean inequality, we have

$$= \frac{a+1}{2\sqrt{ab \cdot c} + 2\sqrt{ac \cdot b}} + \frac{b+1}{2\sqrt{bc \cdot a} + 2\sqrt{ab \cdot c}} + \frac{c+1}{2\sqrt{ac \cdot b} + 2\sqrt{bc \cdot a}} \geq \frac{a+1}{ab+c+ac+b} + \frac{b+1}{bc+a+ab+c} + \frac{c+1}{ac+b+bc+a}.$$

This problem is now solved, because

$$\begin{aligned} \frac{a+1}{ab+c+ac+b} + \frac{b+1}{bc+a+ab+c} + \frac{c+1}{ac+b+bc+a} &= \frac{a+1}{(a+1)(b+c)} + \frac{b+1}{(b+1)(a+c)} + \frac{c+1}{(c+1)(a+b)} = \\ &= \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \end{aligned}$$

**Second Solution.** We introduce change of variables:  $a = x^3$ ,  $b = y^3$ ,  $c = z^3$ . We now have the condition  $xyz = 1$ . We apply Schur inequality (with exponent  $r = 1$ ) to the numerator of the left hand side:

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + x^2z + y^2x + y^2z + z^2x + z^2y,$$

to obtain inequality

$$\frac{x^2y + x^2z + y^2x + y^2z + z^2x + z^2y}{4} \geq \frac{1}{x^3 + y^3} + \frac{1}{y^3 + z^3} + \frac{1}{z^3 + x^3}.$$

We apply arithmetic mean – geometric mean inequality for the denominators of the right hand side:

$$x^3 + y^3 \geq 2x^{3/2}y^{3/2} \implies \frac{1}{x^3 + y^3} \leq \frac{1}{2x^{3/2}y^{3/2}} = \frac{1}{2}z^2\sqrt{yz},$$

and similarly to the other terms. We now have to prove

$$\frac{x^2y + x^2z + y^2x + y^2z + z^2x + z^2y}{2} \geq x^2\sqrt{yz} + y^2\sqrt{xz} + z^2\sqrt{xy}.$$

We apply arithmetic mean – geometric mean inequality in pairs on the left hand side:

$$\frac{x^2y + x^2z}{2} \geq x^2\sqrt{yz},$$

$$\frac{y^2x + y^2z}{2} \geq y^2\sqrt{xz},$$

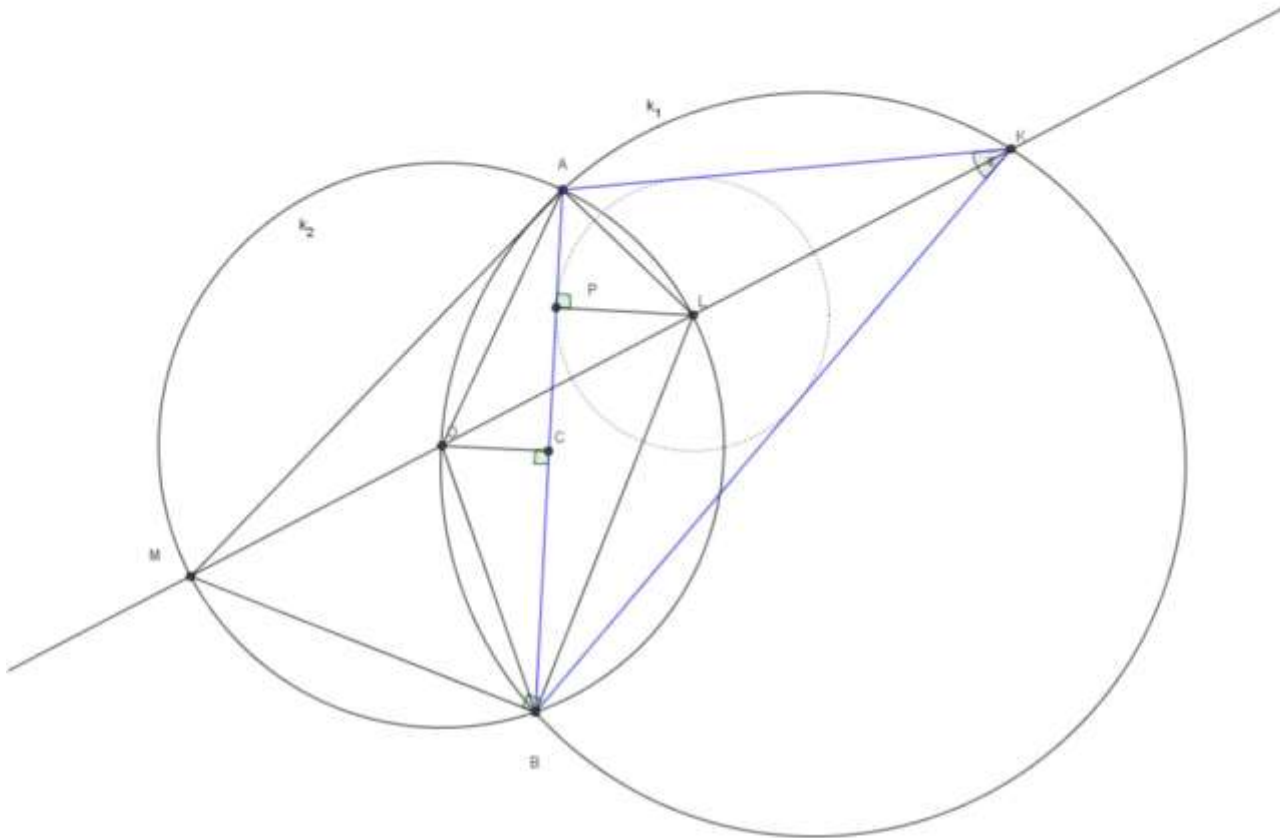
$$\frac{z^2x + z^2y}{2} \geq z^2\sqrt{xy}.$$

Summing up inequalities from above finishes the proof.

**Problem 3.** Circles  $k_1$  and  $k_2$  intersect in points  $A$  and  $B$ , such that  $k_1$  passes through the center  $O$  of the circle  $k_2$ . The line  $p$  intersects  $k_1$  in points  $K$  and  $O$  and  $k_2$  in points  $L$  and  $M$ , such that the point  $L$  is between  $K$  and  $O$ . The point  $P$  is orthogonal projection of the point  $L$  to the line  $AB$ . Prove that the line  $KP$  is parallel to the  $M$ -median of the triangle  $ABM$ .

(Matko Ljulj)

**Solution.** Let the point  $C$  be the midpoint of the line segment  $\overline{AB}$ . We have to prove  $MC \parallel KP$ .



Let us introduce angle  $\alpha := \angle BKA$ . Notice that

$$\angle BLA = 180 - \angle BMA = 180 - \frac{1}{2}\angle BOA = 180 - \frac{1}{2}(180 - \angle BKA) = 90 + \frac{1}{2}\alpha.$$

Also, notice that the point  $O$  is midpoint of the arc  $\widehat{AB}$ . Thus the line  $KO$  is bisector of the angle  $\angle BKA$ . From the two claims above, we deduce that  $L$  is incenter of the triangle  $ABK$ . Moreover, notice that  $\overline{ML}$  is diameter of the circle  $k_2$ , thus  $\angle ABM = 90$ . Since  $BL$  is angle bisector of the angle  $\angle ABK$ , we deduce that  $BM$  is exterior angle bisector of the same angle. Thus, since  $M$  lies on angle bisector  $KM$  and exterior angle bisector  $BM$ ,  $M$  is the center of the excircle for the triangle  $ABK$ . Thus, we have to prove that the line passing through the incenter  $L$  of the triangle  $ABK$  and point of the tangency of incircle of the same triangle is parallel to the line passing through the center of the excircle  $M$  and the midpoint  $C$  of the line segment  $\overline{AB}$ . This is a well known lemma, which completes the proof.

**Problem 4.** A group of mathematicians is attending a conference. We say that a mathematician is  $k$ -content if he is in the room with at least  $k$  people he admires or if he is admired by at least  $k$  other people in the room. It is known that when all participants are in the same room then they are all at least  $3k + 1$ -content. Prove that you can assign everyone into one of the 2 rooms in a way that everyone is at least  $k$ -content in his room and neither room is empty. *Admiration is not necessarily mutual and no one admires himself.*

(Matija Bucić)

**Solution.** We will for simplicity and clarity of presentation use some basic graph theoretic terms, this is in no way essential.

We represent the situation by a directed graph (abbr. digraph)  $G(V, E)$  where each vertex  $v \in V(G)$  represents a mathematician and each edge  $e \in E(G)$  represents an admiration relation. Given  $v \in V(G)$  we define out-degree of  $v$  denoted  $o(v)$  as the number of edges starting in  $v$  (so the number of mathematicians  $v$  admires) and in-degree  $i(v)$  as the number of edges ending in  $v$  (so the number of mathematicians who admire  $v$ ). Given  $X \subseteq V$  by  $G(X)$  we denote the induced subgraph (a graph with vertex set  $X$  and edges inherited from  $G$ ). We say that a digraph is a  $k$ -digraph if for every  $v \in V(G)$  we have  $i(v) \geq k$  or  $o(v) \geq k$ .

So the question can be reformulated as: Given  $G$  is a  $3k + 1$ -digraph we can split its vertices into 2 vertex disjoint classes such that each induced subgraph on class is a  $k$  digraph.

We call a subset  $X$  of vertices of  $G$   $k$ -tight if for any  $Y \subseteq X$  we have a vertex  $v \in Y$  such that  $i_{G(Y)}(v) \leq k$  and  $o_{G(Y)}(v) \leq k$ . A partition of  $V$ ,  $(A_1, A_2)$  is feasible if  $A_1$  is  $k$ -tight and  $A_2$  is  $k$ -tight.

We first assume there are no feasible partitions.

In this case consider a minimal size subset  $A_1 \subseteq V(G)$  subject to  $G(A_1)$  being a  $k$ -digraph, we define  $A_2 \equiv V(G) - A_1$ . Given a subset  $X \subset A_1$ ,  $G(X)$  is not a  $k$ -digraph so there is a vertex  $v \in X$  such that  $o_{G(X)}(v) < k$  and  $i_{G(X)}(v) < k$  which shows that any proper subset of  $A_1$  satisfies the condition of  $k$ -tightness. For the case of  $X \equiv A_1$  by removing any vertex  $v \in A_1$  the graph  $G' \equiv G(A_1 - \{v\})$ , by minimality assumption on  $A_1$ , must contain a vertex  $w$  such that  $o_{G'}(w) < k$  and  $i_{G'}(w) < k$  so as there is only one extra vertex in  $G(A_1)$ , namely  $v$   $o_{G(A_1)}(w) \leq k$ ,  $i_{G(A_1)}(w) \leq k$ . In particular this shows  $A_1$  is  $k$ -tight.

This implies  $A_2$  is not  $k$ -tight by our assumption so there exists an  $A'_2 \subseteq A_2$  such that  $A'_2$  is a  $k + 1$  digraph. Now applying the following proposition to extend the pair  $(A_1, A'_2)$  to a full partition which satisfies the conditions of the problem.

Given disjoint subsets  $A, B \subseteq V(G)$  we say  $(A, B)$  is a solution pair if both  $G(A)$  and  $G(B)$  are  $k$ -digraphs.

**Proposition:** If a  $2k + 1$  digraph  $G$  admits a solution pair it admits a partition with both induced graphs of both classes being  $k$ -digraphs.

**Proof.** Take a maximal solution pair  $(A, B)$ , the condition in the lemma guaranteeing it exists. Let  $C = V(G) - (A \cup B)$ , if  $C$  is empty we are done so assume  $|C| > 0$ . By our assumption  $(A, B \cup C)$  is not a solution pair so there is some  $x \in C$  such that  $o_{G(B \cup C)}(x), i_{G(B \cup C)}(x) < k$  so as  $G$  is  $2k + 1$  digraph  $i_G(x) \geq 2k + 1$  or  $o_G(x) \geq 2k + 1$  so either  $o_{G(A \cup \{x\})}(x) > k + 1$  or  $i_{G(A \cup \{x\})}(x) > k + 1$  so in particular  $(A \cup \{x\}, B)$  is a solution pair contradicting maximality and completing our argument. ■

Hence we are left with the case in which we have at least one feasible partition. We pick the feasible partition  $(A, B)$  maximizing  $w(A, B) = |E(G(A))| + |E(G(B))|$ . The fact that  $A$  is  $k$ -tight implies there is an  $x$  with  $o_{G(A)}(x) \leq k$ ,  $i_{G(A)}(x) \leq k$  so  $x$  needs to have at least  $k + 1$  edges in or out of  $B$  so  $|B| \geq k + 1$  and by symmetry  $|A| \geq k + 1$ .

We now prove that there exist an  $X \subseteq A$  such that  $G(X)$  is a  $k$ -digraph, by contradiction. Assuming the opposite we notice that for any  $x \in B$ ,  $B - \{x\}$  is still  $k$ -tight while  $B$  being  $k$ -tight implies there is an  $x \in B$  such that  $o_{G(B)}(x) \leq k$ ,  $i_{G(B)}(x) \leq k$  so for this  $x$  we have  $A \cup \{x\}$  is also  $k$ -tight. Hence, for  $A' = A \cup \{x\}$  and  $B' = B - \{x\}$ ,  $(A', B')$  is a feasible partition. We considering the change in edges which moving  $x$  causes we have  $w(A', B') - w(A, B) \geq 3k + 1 - k - k - k = 1$  as we know  $i_G(x) \geq 3k + 1$  or  $o_G(x) \geq 3k + 1$  so moving  $x$  from  $B$  to  $A$  increases number of edges in  $A$  by at least  $3k + 1 - k$  while the choice of  $x$  in  $B$  means we lose at most  $k + k$  edges in  $B$ . This is a contradiction to maximality of  $(A, B)$ .

Analogously we can find  $Y \subseteq B$  with  $G(Y)$  a  $k$ -digraph. Now applying the above proposition yet again we are done. ■

*Remark:* The same argument with slightly modified weight function can be used to show the result for non symmetric rooms, in particular if the graph is a  $k + l + \max(k, l) + 1$  digraph it can be partitioned into  $k$ - digraph and  $l$  digraph parts.