18th Junior Balkan Mathematical Olympiad
June 21-26, Ohrid, R. Macedonia

20-та Македонска математичка олимпијада

17-та JMMO

Library Olympiads-ARMAGANKA

Mathematical Olympiads

Macedonian Mathematical Olympiads
Balkan Mathematical Olympiads
European Girl's Mathematical Olympiad
European Mathematical Cup

Aleksa Malcheski, Ph.D.
Risto Malcheski, Ph.D.
Slagjana Brsakoska, Ph.D.
Velinov Daniel, Ph.D.
Samoil Malcheski, ph.D.
Zoran Misajleski, Ph.D.
Martin Shoptrajanov, Ph.D.
Bojan Prangoski, Ph.D.
Pavel Dimovski, Ph.D.
Tomi Dimovski, M.Sc.
Vesna Andova, Ph.D.
Methodi Glavche, Ph.D

Union of mathematicians of Macedonia-Armaganka
Macedonia, Skopje 2021

President: Aleksa Malcheski
Publisher: Union of mathematicians of Macedonia-Armaganka
Street:. Ul.2. br. 107A
Nas. Vizbegovo, Butel, Skopje, Republic of Macedonia

CIP - Каталогизација во публикација
Национална и универзитетска библиотека "Св. Климент Охридски", Скопје

51(079.1)

MATHEMATICAL olympiads : Macedonian mathematical olympiads :
Balkan mathematical olympiads / [Aleksa Malčeski ... и др.]. -
Skopje : Macedonian mathematical society-Armaganka, 2021. - 24 стр. : ilustration.
; 24 см. - (Library Olimpiads)

Authors: Aleksa Malčeski, , Slagjana Brsakoska, Risto Malcheski, Daniel Velinov, Zoran Misajleski , Samoil Malcheski,Martin Shoptrajanov, Bojan Prangoski, Pavel Dimovski, Tomi Dimovski, Vesna Andova, Methodi Glavche,

ISBN 978-9989-646-48-5 (book)
ISBN 978-9989-646-18-8 (ed.)

1. Malčeski, Aleksa [author]
a) Mathematics – problems on competitions

COBISS.MK-ID 94225674

Foreword

This year in the Macedonia competitions of all levels in primary/secondary and high-school were held: school, regional, state and Olympiads.

After rigorous selection processes the BMO and JBMO teams were formed. The latter Olympiads took place in a pleasant atmosphere in the Republic of Bulgaria and the Macedonia, respectively.

After the IMO team selection test, the team to the IMO 2014 was formed. This year the IMO is taking place in Cape Town, the SAU.

The content of this book consists of the mathematical competitions that already took place in Macedonia and the Balkan region, as well as the solutions.

IMO

THAILAND 2015

3[rd] EUROPEAN MATHEMATICAL CUP,
December 6 - 14 2014
Junior Category

Problems and solutions

Problem 1. Which of the following claims are true, and which of them are false? *If a fact is true you should prove it, if it isn't, find a counterexample.*

a) Let a,b,c be real numbers such that $a^{2013}+b^{2013}+c^{2013}=0$. Then $a^{2014}+b^{2014}+c^{2014}=0$.

b) Let a,b,c be real numbers such that $a^{2014}+b^{2014}+c^{2014}=0$. Then $a^{2015}+b^{2015}+c^{2015}=0$.

c) Let a,b,c be real numbers such that $a^{2013}+b^{2013}+c^{2013}=0$ and $a^{2015}+b^{2015}+c^{2015}=0$. Then $a^{2014}+b^{2014}+c^{2014}=0$.

Solution: Fiorstly, we know that for every real number x, $x^2 \geq 0$ holds.

The key idea in this problem is to that the expression $a^{2014}+b^{2014}+c^{2014}$ is sum of squares (which are nonnegative numbers). Thus $a^{2014}+b^{2014}+c^{2014}=0 \Leftrightarrow a=b=c=0$.

a) No: It is sufficient to find three real numbers whose sum equals 0, and then take their 2013^{th} roots. For example $a=\sqrt[2013]{1}$, $b=\sqrt[2013]{2}$, $c=\sqrt[2013]{-3}$.

b)YES: From the key idea we conclude $a=b=c=0$ and then we conclude $a^{2015}+b^{2015}+c^{2015}=0+0+0=0$

c) NO: Again we have to find a counterexample, for instance $a=1, b=0, c=-1$.

Problem 2. In each vertex of a regular n-gon $A_1, A_2 \dots A_n$ there is a unique pawn. In each step it is allowed:

1. to move all pawns one step in the clockwise direction or

2. to swap the pawns at vertices A_1 and A_2.

Prove that by a finite series of such steps it is possible to swap the pawns at vertices:

a) A_i and A_{i+1} for any $1 \leq i < n$ while leaving all other pawns in their initial place

b) A_i and A_j for any $1 \leq i < j \leq n$ leaving all other pawns in their initial place.

Solution. We denote a pawn that was initially at point A_i as i. We will prove that a) and then use it to show part b).

a) We apply first operation $i-1$ times which will bring i and $i+1$ as they are at points A_1 and A_2 and move every other pawn $i-1$ steps in clokwise direction.

We can now apply second operation to swap i and $i+1$ as they are at points A_1 and A_2. This does not affect the position of any other pawn.

We now apply first operation $n-i+1$ times returning pawn $k \neq i, i+1$ to point A_k while moving pawn i to A_{i+1} and pawn $i+1$ to A_i which is exactly what we wanted.

b) We present 2 possible solutions, one using induction and one not using induction.

Solution not using induction

By using the previous problem we can swap pawns $(i,i+1)$ as they are at points (A_i,A_{i+1}) then $(i,i+2)$ as they are at points (A_{i+1},A_{i+2}) and carry on until we swap (i,j) as they were at points (A_{j-1},A_j). This brings us to the state where i is at A_j and each $i+1\le k\le j$ is at point A_{k-1}.

We can now apply part a to swap j with $j-1$ and siilarly carry on till we swap j with $i+1$. This will place j at A_i and move each $i+1\le k\le j-1$ to A_k.

This brings us to the state we swapped pawns i and j leaving others where they were just as was desired.◊

Solution using induction

We use induction on n for the following claim:

We can swap any two pawns $1\le i<j\le k$.

We note that the basis is exactly part a.

We assume we the calim holds for some k.

Hence we can swap any pawns $1\le i<j\le k$ and only need to show that we can swap i and $k+1$ for any $1\le i\le k$. This follows as we can swap i and k then k and $k+1$ by part a). Then again $k+1$ and i as they are now on points A_k and A_i.

Problem 3. Let $\triangle ABC$ be a triangle. The external and internal angle bisectors of $\angle CAB$ intersect side BC at D and E, respectively. Let F be a point on the segment BC. The circumcircle of triangle $\triangle ADF$ intersects AB and AC at I and J, respectively. Let N be the mid-point of IJ and H the foot of E on DN. Prove that E is the incenter of triangle $\triangle AHF$.

Solution . Denote by ω the circumcircle of $\triangle AHF$.

The key idea in the problem is to introduce a new point X which we define as the second intersection of DN and ω. We now note that the $\angle JAD=\angle CAD=90^\circ\pm\frac{\alpha}{2}$ where $\alpha=\angle CAB$. As AD is an external bisector of $\angle CAB$.

The $\pm$ signs depend on the picture and student shouldn't be deducated any points for not noticing this.

Hence we have either $\angle JAD=\angle BAD$ or $\angle JAD+\angle IAD=180^\circ$ so in both cases $DI=DJ$.

Now as N is midpoint of IJ this means that DN is bisector of IJ and hence passes through the centre of the. This shows that DX is a diameter of ω and $EH\parallel IJ$.

We also notice that $\angle EAD=90^\circ$ as angle between bisectors and $\angle XAD=90^\circ$ as DX is a diameter. Hence X,A,E are collinear.

Now this gives us $\angle DHE=\angle XHE=90^\circ$ and $\angle XFE=\angle DFE=90^\circ$ as DX is a diameter of ω and finally again $\angle EAD=90^\circ$. All this gives us that quadrilaterals $XFEH$ and $ADEH$ are cyclic.

Final step is to use some angle chasing to get $\angle AHE=\angle ADH=\angle AXF=\angle EXF=\angle EHF$ where first, second and fourth equalities are due to cyclicity of $ADEH$, $ADXF$ and $XFEH$ respectively. Also $\angle DFH=\angle EFH=\angle EXH=\angle AFD=\angle AFE$ where the second and fourth equalities are due to cyclicity of $XFEH$ and $ADXF$. This shows E is the incenter of $\triangle AFH$ as desired.

Problem 4. Find all infinite sequences $a_1,a_2,a_3,...$ of positive integers such that

a) $a_{nm}=a_na_m$, for all positive integers n,m, and

b) there are infinitely many positive integers n such that $\{1,2,...,n\}=\{a_1,a_2,...,a_n\}$.

Solution. Instead of sequence a_n, we'll use notation with the function $f(n)$ with same properties.

There exists only one such function: $f(n)=n$. We'll solve the problem with many separate facts.

Fact 1: $f(1)=1$

Proof. According to a) it holds $f(1)=f(1)f(1)=f(1)^2$. Since $f(1)$ is positive integer, it can't be $f(1)=0$, so it must be $f(1)=1$.

Fact 2: Function f is bijective.

Proof. Firstly, we'll show that f is injective. Let $a\neq b$ be two arbitrary positive integers and let's assume $f(a)=f(b)$. Since $\{1,2,...,n\}=\{f(1),f(2),...,f(n)\}$ holds for infinitely any positive integers n, it holds for some integer grether than a and b. Then, since $f(a)=f(b)$, set $\{f(1),f(2),...,f(n)\}$ contains $n-1$ or less (different) elements, but according to b), it contains n elements.

Secondly, we'll show that f is surjective. Let c be arbitrary integer and let's assume that $f(n)\neq c$ for all positive integers n. Similarly as in first part of proof, let's take positive integer n such that $\{1,2,...,n\}=\{f(1),f(2),...,f(n)\}$ holds. Since $c\in\{1,2,...,n\}$, c is also element of the set $\{f(1),f(2),...,f(n)\}$ holds. Since $c\in\{1,2,...,n\}$, c is also element of the set $\{f(1),f(2),...,f(n)\}$, so there exists positive integer $m\leq n$ such that $f(m)=c$.

Fact 3: Positive integer n is prime if and only if $f(n)$ is prime.

Proof. Let's assume that n is prime, but $f(n)$ isn't. Then it must be $f(n)=a'b'=f(a)f(b)=f(ab)$, where a',b' are positive integers greather 1, and a,b are unique positive integers such that $f(a)=a'$, $f(b)=b'$ (they exist since f is bijective). Since f is injective, $f(1)=1$ and a',b' are not equal to 1, integers a,b are also not equal to 1. Since f is injective and $f(n)=f(ab)$, we have $n=ab$, so n is complete.

Let's assume that $f(n)$ is prime, but n isn't. Then there exist positive integers a,b greather than one such that $n=ab$. From there we have $f(n)=f(ab)=f(a)f(b)$. Again from injective of f and $f(1)=1$, we see that $f(n)$ is product of two integers grether than 1.

Fact 4: If $n=p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}$ is unique factorization of positive integer n, then

$$f(n)=f(p_1)^{\alpha_1}f(p_2)^{\alpha_2}...f(p_k)^{\alpha_k}$$

is unique factorization of positive integer $f(n)$.

Proof. From multiple use of the condition a) we get identity $f(n)=f(p_1)^{\alpha_1}f(p_2)^{\alpha_2}...f(p_k)^{\alpha_k}$. From fact 3, numbers $f(p_i)$ are prime. Since f is injective, none of two numbers $f(p_i)$ and $f(p_j)$ are equal.

Fact 5: (Technical result) For all positive integers $y<x$ there exist positive integer $n_\circ$ such that for all positive integers $n>n_\circ$ holds inequality

$$y^{n+1}<x^n.$$

Proof. It is sufficient to prove the fact only for consecutive integers y and $y+1$ (because we'll have $y^{+1}<(y+1)^n\leq x^n$. By binomial theorem we have

$$(y+1)^n\geq y^n+ny^{n-1}=y^{n-1}(y+n).$$

Thus if we define $n_\circ=y^2-y+1$, then for all $n\geq n_\circ$ we have

$$(y+1)^n\geq y^{n-1}(y+n)\geq y^{n-1}(y+n_\circ)=y^{n-1}(y^2+1)>y^{n+1}.$$

Another proof. Inequality is equivalent to

$$\left(\frac{y}{x}\right)^n>y.$$

The fact follows from the fact that the expression on the left hand side is increasing and it is unbounded, while the right hand side is fixed.

Fact 6: For all prime numbers p we have $f(p) \le p$.

Proof. Let $p_1, p_2, ..., p_n, ...$ be the increasing sequence $2,3,5,7,...$ of all primes. Let's take arbitrary prime number p_n . From the Fact 3 we have that $f(p_n)$ is also prime. Let's take positive integer $n_\circ$ as the integer from the Fact 5, for positive integers $y = p_n < p_{n+1} = x$. Since b) holds for infinitely many positive integers, it holds for some positive integer N such that $\{1,2,...,N\} = \{f(1), f(2),..., f(N)\}$, and such that $N \ge p_\circ^{n_\circ}$. Let α be the greatest positive integer such that $p_n^\alpha \le N$. From defintions of N and α we have $\alpha \ge n_\circ$.

In set $\{1,2,...,N\}$ we'll observe all positive integers which are α^{th} power of a prime number. Since $N \ge p_n^\alpha$, we have that p_n^α is in that set. It is easy to see that all numbers $p_1^\alpha, p_2^\alpha,..., p_{n-1}^\alpha$ are also in that set. On the contrary, number p_{n+1}^α is not in that set, because from the definition of α and N respectively we have $N < p_n^{\alpha+1} \le p_{n+1}^\alpha$ (remember Fact 5 and $\alpha \ge n_\circ$). Similarly, neither of the numbers p_m^α (for $m > n$) is not in the set $\{1,2,...,N\}$.

Let us now observe all positive integers which are α^{th} power of a prime and they are in the set $\{f(1), f(2),..., f(N)\}$. According to Fact 4, we have that $f(n)$ is α^{th} power of a prime. From that and from previous paragraph we conclude that only such numbers are $f(p_1^\alpha), f(p_2^\alpha),..., f(p_n^\alpha)$.

Now we have $\{p_1^\alpha, p_2^\alpha,..., p_n^\alpha\} = \{f(p_1^\alpha), f(p_2^\alpha),..., f(p_n^\alpha)\}$. Thus $f(p_n^\alpha) \in \{p_1^\alpha, p_2^\alpha,..., p_n^\alpha\}$, so $f(p_n^\alpha) = p_i^\alpha$ for some $1 \le i \le n$, which implies $f(p_n^\alpha) = p_i^\alpha$ for some $1 \le i \le n \Rightarrow f(p_n) = p_i \le p_n$, which completes the proof.

Fact 7: For every positive integer we have $f(n) = n$.

Proof. From Fact 3 we have $f(p)$ if and only if p is prime. Let $p_1, p_2, ..., p_n, ...$ be the increasing sequence $2,3,5,7,...$ of all prime numbers From fact 6 we have $f(p_1) \le p_1 \Rightarrow f(2) = 2$. For $n \ge 2$, inductively and from injectivity of f we have $f(p_n) > p_{n-1}$ and from Fact 6 we have $f(p_n) \le p_n$, thus is must be $f(p_n) = p_n$, for all positive integer n .

Now for arbitrary positive integer n from Fact 4 we have

$$f(n) = f(p_1)^{\alpha_1} f(p_2)^{\alpha_2} ... f(p_k)^{\alpha_k} = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k} = n$$

which completes our proof.

Remark: We can prove Fact 6 differently(without using Fact 5). We observe numbers $1 \cdot 2 \cdot ... \cdot n$ and $f(1) \cdot f(2) \cdot ... \cdot f(n)$, and their unique factorization. They concide for infinitely many positive integers n . For fixed primes p, q , if we take sufficiently great n , we can use well-known formula for $\nu_p(n!)$ to prove that $\nu_p(n!) > \nu_q(n!)$ for all $q > p$ (here positive integer n depends on p, q).

3[rd] EUROPEAN MATHEMATICAL CUP,
December 6 - 14 2014

Senior Category

Problems and solutions

Problem 1. Prove that there are infinitely many positive integers which can't be expressed as $a^{d(a)}+b^{d(b)}$ where a and b are positive integers.
For positive integer a expression $d(a)$ denotes the number of positive divisors of a.

Solution. We will show that $a^{d(a)}$ is a square of an integers for every positive integer a.

If a is a square of en integer, any its power is also square of an integer.

If a is not a perfect square, number of it's positive divisors is even. We can prove this by pairing divisors of a as d and $\frac{a}{d}$. A divisor d won't be paired with itself because that would imply $a=d^2$. This proves that $d(a)$ is even and hence $a^{d(a)}$ is a perfect square for every positive integer a.

The extension in the problem is hence a sum of two squares. Every number of the form $4t+3$ can't be writen as a sum of two squares because 0 and 1 are the only quadratic residus modulo 4, so it impossible for a sum of two squares to give remainder 3 modulo 4.

Problem 2. Jeck and Lisa are playing a game on an $m\times n$ board, with $m,n>2$. Lisa starts by putting a knight onto the board. Then in turn Jeck and Lisa puta new piece onto the board according to the following rules:

1.Jeck puts a queen on an empty square that is two squares horizontally and one square vertically, or alternatively one square horizontally and two squares vertically, away from Lisa's last knight.

2. Lisa puts a knight on an empty square that is on the same, row, column or diagonal as Jeck's last queen.

The one who is unable to put a piece on the board loses the game. For which pairs (m,n) does Lisa have a winning strategy?

Solution . We will show that Lisa has a winning strategy if and only if m and n are both odd.

Lisa's winning strategy

Suppose the game is played on an $m\times n$ board with m and n both odd. Then Lisa puts her knight in a corner and partitions the remaining squares of the board into "dominoes". In each turn Jeck has to put a queen in one of these dominoes and Lisa puts a knight on the other square of the domino. As the board is finite, Jeck can't keep finding new dominoes and so Lisa will win.

Jeck's winning strategy

Suppoose the game is played on an $m\times n$ board with m or n even. We shall that Jek is able to partition the board into pairs of squares that are two squares horizontally and one square vertically, or alternatively one square horizontally and two squares vertically, away from each other. In each tyrn Lisa has to put a knight in one of these and Jeck puts a queen on the other square of the pair. As the board is finite, Lisa can't keep finding new pairs and so Jeck will win. Now we prove that Jeck can make the required partition.

Case 1. Suppose $4 \mid m$ or $4 \mid n$. We know that any $k \times 4l$ board ($k \geq 2$) can be divided into 2×4 and 3×4 boards(firstly divide $k \times 4l$ board in l boards of dimensions $k \times 4$; after that every $k \times 4$ board divide in $\frac{k}{2}$ boards of dimensions 2×4, or in $\frac{k-3}{2}$ boards of dimensions 2×4 and one 3×4 board, dependently on parity of k). The following diagrams show that every 2×4 and every 3×4 board alows a required pratition. .

Case 2. Suppose $m, n \equiv 1,2 \pmod 4$. Any $(5+4l) \times (6+4l)$ board can be divided into a 5×6 board, a $4k \times l$ board, a $5 \times 4l$ board and a $4k \times 4l$ board. The following diagram shows that a 5×6 board allows a required partition.

According to case 1 a $4k \times 6$ board, a $5 \times 4l$ board and a $4k \times 4l$ board also allow a partition.

Case 3. Suppose $m, n \equiv 2,3 \pmod 4$. Any $(3+4k) \times (6+4l)$ board can be divided into a 3×6 board, a $4k \times 6$ board, a $3 \times 4l$ board and a $4k \times 4l$ board. The following diagram shows that a 3×6 board alows a required partition.

According to case 1 a $4k \times 6$ board, a $3 \times 4l$ board and a $4k \times 4l$ board also allow a partition.

Case 4. Suppose $m, n \equiv 2 \pmod 4$. Any $(6+4k) \times (6+4l)$ board can be divided into a 6×6 board, a $4k \times 6$ board, a $6 \times 4l$ board and a $4k \times 4l$ board. The 6×6 board can be partitioned in two 3×6 boards, which were already solved. According to case 1 a $4k \times 6$ board, a $6 \times 4l$ board and a $4k \times 4l$ board also allow a partition.

Problem 3. Let $ABCD$ be a cyclic quadrilateral with the intersection of internal angle bisectors of $\angle ABC$ and $\angle ADC$ lying on the diagonal AC. Let M be the midpoint of AC. The line parallel to BC that passes through D intersects the line BM in E and the circumcircle of $ABCD$ at F where $F \neq D$. Prove that $BCEF$ is a parallelogram.

Solution. We prove the problem in reverse as this is much nore natural in this problem.

We note that if $BCEF$ is a parallelogram then the diagonales are bisecting each other so the point $G \equiv BE \cap CF$ should be the midpoint of CE.

If G is the midpoint of CE then ΔGBC and ΔGEF are congruent as $CG = GF$ and $FE \parallel BC$ gives $\angle GEF = \angle GBC$ and $\angle GFE = \angle GCB$. Hence theis implies $BG = GE$ and in particula $BCEF$ is a parallelogram as its diagonals bisect each other. Hence G being midpoint of CF is equivalent to our problem.

As M is the midpoint of AC by the midline theorem applied to triangle ACF we have G is the midpoint of CG if and only if $MG \parallel AF$. Hence we only need to prove $BM \parallel AF$.

Now we futher notice that, using $FD \parallel BC$, this is equivalent to $\angle AFD = \angle MBC$.

We further see that $\angle AFD = \angle ABD$ as they are angles over the same chord. So our claim is equivalent to $\angle ABD = \angle MBC$.

We add that here depending on the relative position of F on the circles we might have $\pi - \angle AFD = \angle MBC$ but then $\pi - \angle AFD = \angle ABD$ so the final conclusion still holds.

We know that $\angle BDA = \angle BCM$ as they are angles over the same chord. Now this us that our claim is equivalent to the claim $\Delta BCM \sim \Delta BDA$.

The same angle equality shows that this is equivalent to $\frac{BC}{CM}=\frac{AD}{BD}$. Using the fact M is the midpoint of AC we have $CM=\frac{AC}{2}$ so our claim is equivalent to $2AD\cdot BC=BD\cdot AC$.

We further have by the angle bisector theorem applied to ΔABC and ΔCDA:

$$\frac{AB}{BC}=\frac{AI}{CI}=\frac{AD}{CD}.$$

So using this our claim is equivalent to $AB\cdot CD+AD\cdot BC=BD\cdot AC$ which we can recognise to the Ptolomeys theorem for cyclic quadrilaterals.

Problem 4. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all $x,y\in\mathbb{R}$ the following holds:

$$f(x^2)+f(2y^2)=(f(x+y)+f(y))(f(x-y)+f(y)).$$

Solution. Let $P(x,y)$ be the assertion $f(x^2)+f(2y^2)=(f(x+y)+f(y))(f(x-y)+f(y))$. $P(0,x)$ gives us

$$f(0)+f(2x^2)=2f(x)(f(x)+f(-x)) \tag{1}$$

and $P(0,-x)$ gives us

$$f(0)+f(2x^2)=2f(-x)(f(x)+f(-x)). \tag{2}$$

By combining (1) and (2) we get

$$f(x)^2=f(-x)^2. \tag{3}$$

$P(0,0)$ gives us $2f(0)=4f(0)^2$, thus we have two cases:

Case 1. $f(0)=\frac{1}{2}$.

$P(x,0)$ gives us

$$f(x^2)=\left(f(x)+\frac{1}{2}\right)^2-\frac{1}{2}, \tag{4}$$

while $P(-x,0)$, gives us

$$f(x^2)=\left(f(-x)+\frac{1}{2}\right)^2-\frac{1}{2} \tag{5}$$

Combinig (4) and (5) and using (3) we get

$$f(x)=f(-x) \tag{6}$$

The assertion $P(x^2,x^2)$ can be written as

$$f(x^4)+f(2x^4)=f(2x^2)+f(x^2)\left(\frac{1}{2}+f(x^2)\right) \tag{7}$$

For an arbitrary $x\in\mathbb{R}$, let us denote $a=f(x)$. Using (4) we get:

$$f(x^2)=\left(a+\frac{1}{2}\right)^2-\frac{1}{2},$$

$$f(x^4)=\left(f(x^2)+\frac{1}{2}\right)^2-\frac{1}{2}=\left(a+\frac{1}{2}\right)^4-\frac{1}{2}.$$

Using (1) and (6) we get

$$f(2x^2)=4f(x^2)-\frac{1}{2}=4a^2-\frac{1}{2},$$

$$f(2x^4)=4f(x^2)^2-\frac{1}{2}=4\left(\left(a+\frac{1}{2}\right)^2-\frac{1}{2}\right)^2-\frac{1}{2}.$$

Plugging the last 4 equations in (7) we get:

$$\left(a+\frac{1}{2}\right)^2+4\left(\left(a+\frac{1}{2}\right)^2-\frac{1}{2}\right)^2-1=\left(4a^2-1+\left(a+\frac{1}{2}\right)^2\right)\left(a+\frac{1}{2}\right)^2$$

which is equivalent to

$$\left(a+\frac{1}{2}\right)^2(4a-2)=0\,.$$

Therefore $a=\pm\frac{1}{2}$ and $f(x)=\pm\frac{1}{2}$. Now if we use (6) in (1) we get

$$f(0)+f(2x^2)=4(f(x))^2=1$$

so $f(2x^2)=\frac{1}{2}$ for every x, now using (6) we conclude $f(x)=\frac{1}{2}$ for all x which is easily checked to be a solution.

Case 2. $f(0)=0$.

We immediately see using $P(x,0)$ that

$$f(x^2)=f(x)^2\,. \tag{8}$$

By comparing $P(x,y)$ and $P(x,-y)$ and using (3) we get:

$$(f(y)-f(-y))(f(x+y)+f(x-y))=0$$

If there exists $c\in\mathbb{R}$ such that $f(c)\neq f(-c)$ we have for all x

$$f(x+c)=-f(x-c)$$

Plugging in $x+c$ in x here gives us:

$$f(x+2c)=-f(x)\,. \tag{9}$$

Specially, $f(2c)=0$. Now, $P(2c-y,y)$:

$$f((2c-y)^2)+f(2y^2)=(f(2c)+f(y))(f(2c-y)+f(y))\,,$$

$$(-f(-y))^2+f(2y^2)=f(y)f(2c-2y)+f(y)^2$$

$$f(2y^2)=f(y)f(2c-2y)=-f(y)f(-2y) \tag{10}$$

Let $S(x)$ denote the statement $(x\neq 0)\wedge(f(x)=f(-x)\neq 0)$. If there is no $d\in\mathbb{R}$ such that $S(d)$ then $f(x)=-f(-x)$ for all $x\in\mathbb{R}$. $P(0,x)$ gives us

$$f(2x^2)2f(x)(f(x)+f(-x))=0\,,$$

which gives us another solution $f(x)=0$. Now, let us assume that there exists $d\in\mathbb{R}$ such that $S(d)$ holds. Obviously, $S(-d)$ holds, as well. $P(0,d)$ gives us

$$f(2d\ \ )=4f(d)^2$$

and (10) gives us

$$f(2d^2)=-f(d)f(-2d)$$

$$f(-2d)=-4f(d)$$

$$f(2d)=-4f(-d)=-4f(d)=f(-2d)\,.$$

Therefore, $S(2d)$ also holds. Inductively, we deduce that $S(2^n d)$ holds for every $n\in\mathbb{N}$. Also, $f(2^n d)=(-4)^n f(d)$, which means that f is unbounded.

$P(x,c)$, using the fact $f(x^2)=f(x)^2$:

$$f(x)^2+f(2c^2)=f(x+c)f(x-c)+f(c)(f(x+c)+f(x-c))+f(c)^2\,,$$

and since $f(x+c)=-f(x-c)$ and $f(2c^2)=0$ (this follows from $P(0,c)$) we have

$$f(x)^2+f(x+c)^2=f(c)^2$$

which implies that f is bounded and that is contradiction. Therefore, there is no $c\in\mathbb{R}$ such that $f(c)=-f(c)$ and therefore

$$f(x)=f(-x)\text{, for all } x\in\mathbb{R}. \qquad (11)$$

$P(0,x)$:

$$f(2x^2)=4f(x)^2=4f(x^2).$$

Therefore, using (11)

$$f(2x)=4f(x)\text{, for all } x\in\mathbb{R}. \qquad (12)$$

$P(x,y)$ can now be written as follows:

$$f(x)^2+3f(y)^2=f(y)(f(x+y)+f(x-y))+f(x+y)f(x-y)$$

and similarly, $P(y,x)$ can be written as

$$f(y)^2+3f(x)^2=f(x)(f(x+y)+f(x-y))+f(x+y)f(x-y).$$

Subtracting the previopus two equalities

$$(f(x)-f(y))(2f(x)+2f(y)-f(x+y)-f(x-y))=0. \qquad (13)$$

Assume that for some $x,y\in\mathbb{R}$, $f(x)=f(y)=a$. Let $f(x+y)=b$ and $f(x-y)=c$.

Now we have

$$4a^2=bc+ab+ac \qquad (14)$$

$P(x+y,x-y)$:

$$f(x+y)^2+4f(x-y)^2=(f(2x)+f(x-y))(f(2y)+f(x-y))$$

i.e.

$$b^2+4c^2=(4a+c)^2. \qquad (15)$$

If we plug in $x\to x+y$, $y\to x-y$ in (13) we get

$$(f(x+y)-f(x-y))(2f(x+y)+2f(x-y)-f(2x)-f(2y))=0$$

i.e.

$$(b-c)(2b+2c-8a)=0.$$

If $b=c$ (15) gives us

$$5b^2=(4a+b)^2$$

$$b^2=4a^2+2ab$$

while (14) gives us

$$4a^2=b^2+2ab.$$

Thus, $ab=0$ and $a=b=c=0$ which implies $2a+2a-b-c=0$. On the other hand, if $b\neq c$ we also have $2a+2a-b-c=0$.

4th Girls European Mathematical Olympiad
2015, Minsk, Belarus

Problems

Day 1

Problem 1. Let ΔABC be an acute-angled triangle, and let D be the foot of the altitude from C. The angle bisector of $\angle ABC$ intersects CD at E and meets the circumcircle ω of triangle ΔADE again at F. If $\angle ADF = 45^\circ$, show that CF is tangent to ω.

Problem 2.A domino is a 2×1 or 1×2 tile. Determine in how many exactly n^2 dominoes can be placed without overlaping on a $2n\times 2n$ chessboard so that every 2×2 sqare contains at least two uncovered unit sqaures which lie in the same row or column.

Problem 3. Let m,n be integers greather than 1, and let $a_1,a_2,...,a_m$ be positive integers not greather than n^m. Prove that there exist positive integers $b_1,b_2,...,b_m$ not greather than n such that

$$\gcd(a_1+b_1,a_2+b_2,a_3+b_3,...,a_m+b_m)<n,$$

where $\gcd(x_1,x_2,...,x_m)$ denotes the greatest common divisor of $x_1,x_2,...,x_m$.

Day 2

Problem 4. Determine whether there exists an infinite sequence $a_1,a_2,a_3,...$ of positive integers which satisfies the equality

$$a_{n+2}=a_{n+1}+\sqrt{a_{n+1}+a_n}$$

for every positive integer n.

Problem 5. Let m,n be positive integers with $m>1$. Anastasia partitions the integers $1,2,...,2m$ into m pairs. Boris then chooses one integer from each pair and finds the sum of these chosen integers. Prove that Anastasia can select the pairs so that Boris cannot make his sum equal to n.

Problem 6. Let H be the orthocentre and G be the centroid of acute-angled triangle ΔABC with $AB\neq AC$. The line AG intersects the circumcircle of ΔABC at A and P. Let P' be the reflection of P in the line BC. Prove that $\angle CAB=60^\circ$ if and only if $HG=GP'$.

Solutions

Problem 1. *Let $\triangle ABC$ be an acute-angled triangle, and let D be the foot of the altitude from C. The angle bisector of $\angle ABC$ intersects CD at E and meets the circumcircle ω of triangle $\triangle ADE$ again at F. If $\angle ADF = 45^\circ$, show that CF is tangent to ω.* (Luxemburg)

Solution 1: Since $\angle CDF = 90^\circ - 45^\circ = 45^\circ$, the line DF bisects $\angle CDA$, and so F lies on the perpendicular bisector of segment AE, which meets AB at G. Let $\angle ABC = 2\beta$. Since $ADEF$ is cyclic, $\angle AFE = 90^\circ$, and hence $\angle FAE = 45^\circ$. Further, as BF bisects $\angle ABC$, we have $\angle FAB = 90^\circ - \beta$, and thus

$$\angle EAB = \angle AEG = 45^\circ - \beta \text{ and } \angle AED = 45^\circ + \beta,$$

so $\angle GED = 2\beta$. This implies that right-angled triangles $\triangle EDG$ and $\triangle BDC$ are similar, and so we have $\frac{|GD|}{|CD|} = \frac{|DE|}{|DB|}$. Thus the right-angled triangle $\triangle DEB$ and $\triangle DGC$ are similar, whence $\angle GCD = \angle DBE = \beta$. But $\angle DFE = \angle DAE = 45^\circ - \beta$, then $\angle GFD = 45^\circ - \angle DFE = \beta$. Hence $GDCF$ is cyclic, so $\angle GFC = 90^\circ$, whence CF is perpendicular to the radius FG of ω. It follows that CF is a tangent to ω, as required.

Solution 2: As $\angle ADF = 45^\circ$ line DF is an exterior bisector of $\angle CDB$. Since BF bisects $\angle DBC$ line CF is an exterior bisector of $\angle BCD$. Let $\angle ABC = 2\beta$, so $\angle ECF = (\angle DBC + \angle CDB)/2 = 45^\circ + \beta$. Hence $\angle CFE = 180^\circ - \angle ECF - \angle BCE - \angle EBC = 180^\circ - (45^\circ + \beta + 90^\circ - 2\beta + \beta) = 45^\circ$. It follows that $\angle FDC = \angle CFE$, then CF is tangent to ω.

Solution 3: Note that AE is a diameter of circumcircle of $\triangle ABC$ since $\angle CDF = 90^\circ$. From $\angle AEF = \angle ADF = 45^\circ$ it follows that triangle $\triangle AFE$ is right-angled and isosceles. Without loss of generality, let points A, E and F have coordinates $(-1,0), (1,0)$ and $(0,1)$ respectively. Points F, E, B are collinear, hence B have coordinates $(b, 1-b)$ for some $b \neq -1$. Let point C' be intersection of line tangent to circumcircle of $\triangle AFE$ at F with line ED. Thus C' have coordinates $(c,1)$ and from $C'E \perp AB$ we get $c = \frac{2b}{b+1}$. Now vector $\overrightarrow{BC'} = \left(\frac{2b}{b+1} - b, b\right) = \frac{b}{b+1}(1-b, b+1)$, vector $\overrightarrow{BF} = (-b, b) = b(-1,1)$ and vector $\overrightarrow{BA} = (-(b+1), -(1-b))$. Its clear that $(1-b, b+1)$ and $(-(b+1), -(1-b))$ are symmetric with respect to $\overrightarrow{FE} = (-1,1)$, hence BF bisects $\angle C'BA$ and $C' = C$ which completes the proof.

Solution 4: Again F lies on the perpendicular bisector of segment AE, so $\triangle AFE$ is right-angled and isosceles. Let M be an intersection of BC and AF. Note that $\triangle AMB$ is isosceles since BF is a bisector and altitude in this triangle. Thus BF is a symmetry line of $\triangle AMB$. Then $\angle FDE = \angle FEA = \angle MEF = 45^\circ$, $AF = FE = FM$ and $\angle DAE = \angle EMC$. Let us show that $EC = CM$. Indeed,

$$\angle CEM = 180^\circ - (\angle AED + \angle FEA + \angle MEF) = 90^\circ - \angle AED = DAE = \angle EMC.$$

It follows that $FMCE$ is a kite, since $EF = FM$ and $MC = CE$. Hence $\angle EFC = \angle CFM = \angle EDF = 45^\circ$, so FC is tangent to ω.

Solution 5: Let the tangent to ω at F intersect CD at C'. Let $\angle ABF = \angle FBC = \beta$. It follows that $\angle C'FE = 45^\circ$ since $C'F$ is tangent. We have

$$\frac{\sin\angle BDC}{\sin\angle CDF} \cdot \frac{\sin\angle DFC'}{\sin\angle C'FB} \cdot \frac{\sin\angle FBC}{\sin\angle CBD} = \frac{\sin 90^\circ}{\sin 45^\circ} \cdot \frac{\sin(90^\circ - \beta)}{\sin 45^\circ} \cdot \frac{\sin\beta}{\sin 2\beta} = \frac{2\sin\beta\cos\beta}{\sin 2\beta} = 1.$$

So by trig Cheva on triangle $\triangle BDF$, lines FC', DC and BC are concurent (at C), so $C = C'$. Hence CF is tangent to ω.

Problem 2. *A domino is a 2×1 or 1×2 tile. Determine in how many exactly n^2 dominoes can be placed without overlaping on a $2n \times 2n$ chessboard so that every 2×2*

sqare contains at least two uncovered unit sqaures which lie in the same row or column. (Turkey)

Solution. The answer is $\binom{2n}{n}^2$.

Divide the chessboard into 2×2 squares. There are exactly n^2 such squares on the chessboard. Each of these squares can have at most two unit squares covered by the dominos. As the dominos cover exactly $2n^2$ squares, each of them must have exactly two unit squares which are covered, and these squares must lie in the same row or column.

We claim that these two unit squares are covered by the same domino tile. Suppose that this is not the case for some 2×2 square and one of the tiles covering one of its unit squares sticks out to the left. Then considering one of the leftmost 2×2 squares in this division with this property gives a contradiction.

Now consider this $n\times n$ chessboard consisting of 2×2 squares of the original board. Define A,B,C,D as the following configurations on the original chessboard, where the gray squares indicate the domino tile, and consider the coverting this $n\times n$ chessboard with the letters A,B,C,D in such a way that the resulting configuration on the original chessboard satisfies the condition of the question.

Note that then a square below or to the right of the containing an A or B must also contain an A or B. Therefore the(possibly empty) region consisting of all squares containing a A or B abuts the lower right corner of the chessboard and is separated from the (possibbly empty) region consisting of all squares containing a C or D by path which goes from the lower left corner to the upper right corner of this chessboard and which moves up or right at each step.

A similar reasoning shows that the (posibly empty)region consisting of all squares containing an A or D abouts the lower left corner of the chessboard and is separated from the (possibly empty)region consisting of all squares containing a B or C by a path which goes from the upper left corner to the lower right corner of this chessboard and which moves down or right at each step.

Therefore the $n\times n$ chessboard is divided by these two paths into four (possibly empty) regions that consist respectively of all squares containing A or B or C or D. Conversely, choosing two such paths and filling the four regions separated by them with A s, B s, C s and D s counterclokwise starting at the bottom results in a placement of the dominos on the original board satisfying the condition of the question.

As each of these can be chosen in $\binom{2n}{n}$ ways, there are $\binom{2n}{n}^2$ ways the dominos can be placed.

Problem 3. *Let* m,n *be integers greather than* 1, *and let* $a_1,a_2,...,a_m$ *be positive integers not greather than* n^m. *Prove that there exist positive integers* $b_1,b_2,...,b_m$ *not greather than* n *such that*

$$\gcd(a_1+b_1,a_2+b_2,a_3+b_3,...,a_m+b_m)<n,$$

where $\gcd(x_1,x_2,...,x_m)$ *denotes the greatest common divisor of* $x_1,x_2,...,x_m$.

Solution 1. Suppose without of generality that a_1 is the smallest of the a_i. If $a_1\geq n^m-1$, then the problem is simple: either all the a_i are equal, or $a_1=n^m-1$ and $a_j=n^m$ for some j. In the forst case we can take (say) $b_1=1$, $b_2=2$, and the rest of the b_i can be arbitrary, and we have

$$\gcd(a_1+b_1,a_2+b_2,a_3+b_3,...,a_m+b_m)\leq\gcd(a_1+b_1,a_2+b_2)=1.$$

In the second case, we can take $b_1=1$, $b_j=1$, and the rest of the b_i arbitrary, and again

$$\gcd(a_1+b_1,a_2+b_2,a_3+b_3,...,a_m+b_m)\leq\gcd(a_1+b_1,a_j+b_j)=1.$$

So from now on we can suppose that $a_1\leq n^m-2$.

Now, let us suppose the desired $b_1,...,b_m$ do not exist, and seek a contradiction. Then, for any choice of $b_1,b_2,...,b_m\in\{1,2,...,n\}$, we have

$$\gcd(a_1+b_1, a_2+b_2, a_3+b_3, ..., a_m+b_m) \ge n .$$

Also, we have

$$\gcd(a_1+b_1, a_2+b_2, a_3+b_3, ..., a_m+b_m) \le a_1+b_1 \le n^m + n - 2 .$$

Thus there are at most $n^m - 1$ possible values for the gratest common divisor. However, there are n^m choices for the m -tuple $(b_1, ..., b_m)$. Then, by the pigeonhole principle, there are two m -tuples that yield the same values for the greatest common divisor, say d. But since $d \ge n$, for each i there can be at most one choice of $b_i \in \{1, 2, ..., n\}$ such that $a_i + b_i$ is divisible by d and therefore there can be at most one m -tuple $(b_1, ..., b_m)$ yielding d as the greatest common divisor. This is the desired contradiction.

Solution 2. Similarly to Solution 1 suppose that $a_1 \le n^m - 2$. The gcd of $a_1+1, a_2+1, ..., a_m+1$ is coprime with the gcd of $a_1+1, a_2+1, ..., a_m+1$, thus $a_1+1 \ge n^2$. Now change another 1 into 2 and so on. After $m-1$ changes we get $a_1+1 \ge n^m$ which gives us a contradiction.

Solution 3. We will prove stronger version of this problem:

For $m, n > 1$, let $a_1, a_2, ..., a_m$ be positive integers with at least one $a_i \le n^{2^{m-1}}$. Then there are integers $b_1, b_2, ..., b_m$, each equal to 1 or 2, such that $\gcd(a_1+b_1, a_2+b_2, ..., a_m+b_m) < n$.

Proof: Suppose otherwise. Then the 2^{m-1} integers $\gcd(a_1+b_1, a_2+b_2, ..., a_m+b_m)$ with $b_1 = 1$ and $b_i = 1$ or 2 for $i > 1$ are all pairwise coprime, since for any two of them, there is some $i > 1$ with $a_i + 1$ appearing in one and $a_i + 2$ in the other. Since each of these 2^{m-1} integers divides $a_1 + 1$, and each is $\ge n$ with at most one equal to n, it follows that $a_1 + 1 \ge n(n+1)^{2^{m-1}-1}$ so $a_1 \ge n^{2^{m-1}}$. The same is true for each a_i, $i = 1, 2, .., n$, a contradiction.

Remark: Clearly the $n^{2^{m-1}}$ bound can be strengthened as well.

Problem 4. Determine whether there exists an infinite sequence $a_1, a_2, a_3, ...$ of positive integers which satisfies the equality

$$a_{n+2} = a_{n+1} + \sqrt{a_{n+1} + a_n}$$

for every positive integer n.

Solution 1. The answer is no.

Suppose that there exist a sequence (a_n) of positive integers satisfying the given condition. We will show that this will lead to a contradiction.

For each $n \ge 2$ define $b_n = a_{n+1} - a_n$. Then, by assumption, for $n \ge 2$ we get $b_n = \sqrt{a_n + a_{n-1}}$ so that we have

$$b_{n+1}^2 - b_n^2 = (a_{n+1} + a_n) - (a_n + a_{n-1}) = (a_{n+1} - a_n) + (a_n - a_{n-1}) = b_n + b_{n-1} .$$

Since each a_n is a positive integer we see that b_n is positive integer for $n \ge 2$ and the sequence (b_n) is strictly increasing for $n \ge 3$. Thus $b_n + b_{n-1} = (b_n - b_{n-1})(b_n + b_{n-1}) \ge (b_{n+1} + b_n)$, whence $b_{n-1} \ge b_{n+1}$ - a contradiction to increasing of the sequence (b_i).

Thus we conclude that there exists no sequence (a_n) of positive integers satisfying the given condition of the problem.

Solution 2. Suppose that such a sequence exists. We will calculate its members one by one and get a contradiction.

From the equality $a_3 = a_2 + \sqrt{a_2 + a_1}$ it follows that $a_3 > a_2$. Denote positive integers $\sqrt{a_3 + a_2}$ by b and a_3 by a, then we have $\sqrt{2a} > b$. Since $a_4 = a + b$ and $a_5 = a + b + \sqrt{2a + b}$ are positive integers, then $\sqrt{2a + b}$ is positive integer.

Consider $a_6 = a+b+\sqrt{2a+b}+\sqrt{2a+2b+\sqrt{2a+b}}$. Number $c=\sqrt{2a+2b+\sqrt{2a+b}}$ must be positive integer, obviously it is greater than $\sqrt{2a+b}$. But

$$\left(\sqrt{2a+b}+1\right)^2 = 2a+b+2\sqrt{2a+b}+1 = 2a+2b+\sqrt{2a+b}+(\sqrt{2a+b}-b)+1 > c^2 .$$

So $\sqrt{2a+b} < c < \sqrt{2a+b}+1$ which is impossible.

Solution 3. We will show that there is no sequence (a_n) of positive integers which consists of $N>5$ members and satisfies

$$a_{n+2} = a_{n+1} + \sqrt{a_{n+1}+a_n} \tag{1}$$

for all $n=1,2,...,N-2$. Moreover, we will describe all such sequences with five members.

Since every a_i is a positive integer it follows from (1) that there exists such positive integer k (obviously k depends on n) that

$$a_{n+1}+a_n = k^2 \tag{2}$$

From (1) we have $(a_{n+2}-a_{n+1})^2 = a_{n+1}+a_n$, consider this equality as a quadratic equation with respect to a_{n+1} ,

$$a_{n+1}^2 - (2a_{n+2}+1)a_{n+1} + a_{n+2}^2 - a_n = 0 .$$

Obviously its solutions are $(a_{n+1})_{1/2} = \dfrac{2a_{n+2}+1 \pm \sqrt{D}}{2}$, where

$$D = 4(a_n + a_{n+2}) + 1 . \tag{3}$$

Since $a_{n+2} > a_{n+1}$ we have

$$a_{n+1} = \frac{2a_{n+2}+1-\sqrt{D}}{2} .$$

From the last equality, using that a_{n+1} and a_{n+2} are positive integers, we conclude that D is a square of some odd number i.e. $D=(2m+1)^2$ for some positive integer $m\in\mathbb{N}$, substitute this into (3):

$$a_n + a_{n+2} = m(m+1) . \tag{4}$$

Now adding a_n to both sides of (1) and using (2) and (4) we get $m(m+1)=k^2+k$ whence $m=k$. So

$$\begin{cases} a_n + a_{n+1} = k^2 \\ a_n + a_{n+2} = k^2 + k \end{cases} \tag{5}$$

for some positive integer k (recall that k depends on n).

Write equations (5) for $n=2$ and $n=3$, then for some positive integers k and l we get

$$\begin{cases} a_2 + a_3 = k^2 \\ a_2 + a_4 = k^2 + k \\ a_3 + a_4 = l^2 \\ a_3 + a_5 = l^2 + l \end{cases} . \tag{6}$$

Solution of this linear system is

$$a_2 = \frac{2k^2 - l^2 + k}{2} , \qquad a_3 = \frac{l^2 - k}{2} , \qquad a_4 = \frac{l^2 + k}{2} , \qquad a_5 = \frac{l^2 + 2l + k}{2} . \tag{7}$$

From $a_2 < a_4$ we obtain $k^2 < l^2$ hence $k < l$.

Consider a_6 :

$$a_6 = a_5 + \sqrt{a_5 + a_4} = a_5 + \sqrt{l^2 + l + k} .$$

Since $0<k<l$ we have $l^2 < l^2 + l + k < (l+1)^2$. So a_6 cannot be integer i.e. there is no such sequence with six or more mebers.

To find all required sequences with five members we must find positive integers a_2, a_3, a_4 and a_5 which satisfy (7) for some positive integers $k<l$. Its clear that k and l must be of the same parity. Vise

versa, let positive integers k,l be of the same party and satisfy $k<l$ then from (7) we get integers a_2,a_3,a_4 and a_5 then $a_1=(a_3-a_2)^2-a_2$ and it remains to verfy that a_1 and a_2 are positive i.e. $2k^2+k>l^2$ and $2(l^2-k^2-k)^2>2k^2-l^2+k$.

Solution 4: It is easy to see that (a_n) is increasing for large enough n . Hence

$$a_{n+1}<a_n+\sqrt{2a_n} \tag{1}$$

and

$$a_n<a_{n-1}+\sqrt{2a_{n-1}} \ . \tag{2}$$

Lets define $b_n=a_n+a_{n-1}$. Using AM-QM inequality we have

$$\frac{\sqrt{2a_n}+\sqrt{2a_{n-1}}}{2}\le\sqrt{\frac{2a_n+2a_{n-1}}{2}} \ . \tag{3}$$

Adding (1), (2) and (3):

$$b_{n+1}<b_n+\sqrt{2a_n}+\sqrt{2a_{n-1}}\le b_n+2\sqrt{b_n} \ .$$

Let $b_n=m^2$. Since (b_n) is increasing for large enough n we have:

$$m^2<b_{n+1}<m^2+2m<(m+1)^2 \ .$$

So, b_{n+1} can't be a perfect square, so we get contradiction.

Problem 5. Let m,n be positive integers with $m>1$. Anastasia partitions the integers $1,2,...,2m$ into m pairs. Boris then chooses one integer from each pair and finds the sum of these chosen integers. Prove that Anastasia can select the pairs so that Boris cannot make his sum equal to n . (Netherlands)

Solution 1: Define the following ordered partitions:

$P_1=(\{1,2\},\{3,4\},...,\{2m-1,2m\})$

$P_2=(\{1,m+1\},\{2,m+2\},....,\{m,2m\})$

$P_3=(\{1,2m\},\{2,m+1\},\{3,m+2\},...,\{m,2m-1\})$.

For each P_j we will compute the possible values for the expression $s=a_1+...+a_m$, where $a_i\in P_{j,i}$, are the chosen integers. Here $P_{j,i}$ denotes the i -th coordinate of the ordered partition P_j . We will denote by σ the number $\sum_{i=1}^{m} i=\frac{m^2+m}{2}$.

- Consider the partition P_1 and a certain choice with corresponding s . We find that

$$m^2=\sum_{i=1}^{m}(2i-1)\le s\le\sum_{i=1}^{m}2i=m^2+m \ .$$

Hence, if $n<m^2$ or $n>m^2+m$, this partition gives a positive answer.

- Consider the partition P_2 and a certaion choice with corresponding s . We find that

$$s\equiv\sum_{i=1}^{m} i\equiv\sigma \pmod m \ .$$

Hence, if $m^2\le n^2\le m^2+m$ and $n\not\equiv\sigma \pmod m$, this partition solves the problem.

- Consider the partition P_3 and a certain choice with corresponding s . We set

$$d_i=\begin{cases}0 \ , \ if & a_i=i\\ 1 \ , \ if & a_i\ne i\end{cases} .$$

We also put $d=\sum_{i=1}^{m} d_i$, and note that $0\le d\le m$. Note also that if $a_i\ne i$, then $a_i\equiv i-1 \pmod m$. Hence, for all $a_i\in P_{3,i}$ it holds that

$$a_i\equiv i-d_i \pmod m \ .$$

Hence,

$$s \equiv \sum_{i=1}^{m} a_i \equiv \sum_{i=1}^{m} (i-d_i) \equiv \sigma - d \pmod m ,$$

which can only be congruent to σ modulo m if all d_i are equal, which forces $s=\frac{m^2+m}{2}$ or $s=\frac{3m^2+m}{2}$. Since $m>1$, it holds that

$$\frac{m^2+m}{2} < m^2 < m^2+m < \frac{3m^2+m}{2}.$$

Hence if $m^2 \le n \le m^2+m$ and $n \equiv \sigma \pmod m$, then s cannot be equal to n, so partition P_3 suffices for such n.

Note that all n are treated in one of the cases above, so we are done.

Common notes for solutions 1B and 1C: Given the analysis of P_1 and P_2 as in the section 1A we may conclude(noting that $\sigma \equiv \frac{m(m+1)}{2} \pmod m$) that if m is odd then m^2 and m^2+m are the only candidates for counterexamples n, while if m is even then $m^2+\frac{m}{2}$ is the only candidate.

There are now various ways to proceed as alternatives to the partition P_3.

Solution 1B: Consider the partition $(\{1,m+2\},\{2,m+3\},.....,\{m-1,2m\},\{m,m+1\})$. We consider possible sums mod $m+1$. For the first $m-1$ pairs, the elements of each pair are congruent mod $m+1$, so the sum of one element of each pair is $\pmod{m+1}$ congruent to $\frac{1}{2}m(m+1)-m$, which is congruent to 1 if $m+1$ is odd and $1+\frac{m+1}{2}$ if $m+1$ is even. Now the elements of the last pair are congruent to -1 and 0, so any achievable value of n is congruent to 0 or 1 if $m+1$ is odd, and to 0 or 1 plus $\frac{m+1}{2}$ if $m+1$ is even. If m is even then $m^2+\frac{m}{2} \equiv 1+\frac{m}{2}$, which is not congruent to 0 or 1. If m is odd then $m^2 \equiv 1$ and $m^2+m \equiv 0$, neither of which can equal 0 or 1 plus $\frac{m+1}{2}$.

Solution 1C: Similarly, consider the partition $(\{1,m\},\{2,m+1\},...,\{m-1,2m-2\},\{2m-1,2m\})$ this considering sums of elements of pairs mod $m-1$. If $m-1$ is odd, the sum is congruent to 1 or 2; if $m-1$ is even, to 1 or 2 plus $\frac{m-1}{2}$. If m is even then $m^2+\frac{m}{2} \equiv 1+\frac{m}{2}$, and this can only be congruent to 1 or 2 when $m=2$. If m is odd, m^2 and m^2+m are congruent to 1 and 2, and these can only be congruent to 1 or 2 plus $\frac{m-1}{2}$ when $m=3$. Now the cases of $m=2$ and $m=3$ need considering separately(by finding explici partitions excluding each n).

Solution 2: This solution does not use modulo arguments. Use only P_1 from the solution 1A to conclude that $m^2 \le n \le m^2+m$. Now consider the partition $(\{1,2m\},\{2,3\},...,\{2m-2,2m-1\})$. If 1 is chosen from the first pair, the sum is at most m^2; if $2m$ is chosen, the sum is at least m^2+m. So either $n=m^2$ or $n=m^2+m$. Now consider the partition $(\{1,2m-1\},\{2,2m\},\{3,4\},\{5,6\},...,\{2m-3,2m-2\})$. Sums of one element from each of the last $m-2$ pairs are in the range from each $(m-2)m=m^2-2m$ to $(m-2)(m+1)=m^2-m-2$ inclusive. Sums of one element from each of the first two pairs are $3, 2m+1$ and $4m-1$. In the first case we have $n \le m^2-m+1 < m^2$ in the second $m^2+1 \le n \le m^2+m-1$ and in the third $n \ge m^2+2m-1 > m^2+m$. So these three p[artitions together have eliminated all n.

Problem 6. Let H be the orthocentre and G be the centroid of acute-angled triangle ΔABC with $AB \ne AC$. The line AG intersects the circumcircle of ΔABC at A and P. Let P' be the reflection of P in the line BC. Prove that $\angle CAB = 60^\circ$ if and only if $HG = GP'$.

Solution 1: Let ω be the circumcircle of ΔABC. Reflecting ω in line BC, we obtain circle ω' which, obviously, contains points H and P'. Let M be the midpoint of BC. As triangle ΔABC is acute-angled, then H and O lie inside this triangle.

Let us assume that $\angle CAB = 60^\circ$. Since

$$\angle COB = 2\angle CAB = 120^\circ = 180^\circ - 60^\circ = 180^\circ - \angle CAB = \angle CHB ,$$

hence O lies on ω'. Reflecting O in line BC, we obtain point O' which lies on ω and this point is the center of ω'. Then $OO' = 2OM = 2R\cos\angle CAB = AH$, so $AH = OO' = HO' = AO = R$, where R is the radius of ω and , naturally, of ω'. Then quadrilateral $AHO'O$ is a rhombus, so A and O' are symmetric to each other with respect to HO. As H,G and O are collinear (Euler line), then $\angle GAH = \angle HO'G$. Diagonals of quadrilateral $GOPO'$ intersects at M. Since $\angle BOM = 60^\circ$, so

$$OM = MO' = \operatorname{ctg} 60^\circ \cdot MB = \frac{MB}{\sqrt{3}} .$$

As $3\cdot MO \cdot MO' = MB^2 = MB \cdot MC = MP \cdot MA = 3MG \cdot MP$, then $GOPO'$ is a cyclic. Since BC is a perpendicular bisector of OO', so the circumcircle of quadrilateral $GOPO'$ is symmetrical with respect to BC. Thus P' also belongs to the circumcircle of $GOPO'$, hence $\angle GO'P' = \angle GPP'$. Note that $\angle GPP' = \angle GAH$ since $AH \parallel PP'$. And as it was proved $\angle GAH = \angle HO'G$, then $\angle HO'G = \angle GO'P'$. Thus triangles $\Delta HO'G$ and $\Delta GO'P'$ are eqaul and hence $HG = GP'$.

Now we will prove that if $HG = GP'$ then $\angle CAB = 60^\circ$. Reflecting A with respect to M, we get A'. Then, as it was said in the first part of solution, points B, C, H and P' belong to ω'. Also it is clear that A' belongs to ω'. Note that $HC \perp CA'$ since $AB \parallel CA'$ and hence HA' is a diameter of ω'. Obviously, the center O' of circle ω' is midpoint of HA'. From $HG = GP'$ it follows that $\Delta HGO'$ is equal to $\Delta P'GO'$. Therefore H and P' are symmetric with respect to GO'. Hence $GO' \perp HP'$ and $GO' \parallel A'P'$. Let HG intersect $A'P'$ at K and $K \not\equiv O$ since $AB \neq AC$. We conclude that $HG = GK$, because line GO' is midline of the triangle $\Delta HKA'$. Note that $2GO = HG$, since HO is Euler line of triangle ABC. So O is midpoint of segment GK. Because of $\angle CMP = \angle CMP'$, then $\angle GMO = \angle OMP'$. Line OM, that passes through O', is an external angle bisector of $\angle P'MA'$. Also we know that , then O' is the midpoint of arc $P'MA'$. It follows that quadrilateral $P'MO'A'$ is cyclic, then $\angle O'MA' = \angle O'P'A' = \angle O'A'P'$. Let OM and $P'A'$ intersect at T. Triangles $\Delta TO'A'$ and $\Delta A'O'M$ are similar, hence $\frac{O'A'}{O'M} = \frac{O'T}{O'A'}$. In the other words, $O'M \cdot O'T = O'A'^2$. Using Menelaus' theorem for triangle $\Delta HKA'$ and line TO', we obtain that

$$\frac{A'O'}{O'H} \cdot \frac{HO}{OK} \cdot \frac{KT}{TA'} = 3 \cdot \frac{KT}{TA'} = 1 .$$

It follows that $\frac{KT}{TA'} = \frac{1}{3}$ and $KA' = 2KT$. Using Menelaus' theorem for triangle $TO'A'$ and line HK we get

$$1 = \frac{O'H}{HA'} \cdot \frac{A'K}{KT} \cdot \frac{TO}{OO'} = \frac{1}{2} \cdot 2 \cdot \frac{TO}{OO'} = \frac{TO}{OO'} .$$

It means that $TO = OO'$, so $O'A'^2 = O'M \cdot O'T = OO'^2$. Hence $O'A' = OO'$ and consequently, $O \in \omega'$. Finally we conclude that $2\angle CAB = \angle BOC = 180^\circ - \angle CAB$, so $\angle CAB = 60^\circ$.

Solution 2: Let O' and G' denote the reflection of O and G respectively, with respect to the line BC. We then need to show $\angle CAB = 60^\circ$ iff $G'H' = G'P$. Note that $\Delta H'OP$ is isosceles and hence $G'H' = G'P$ is equivalent to G' lying on the busector $\angle H'OP$. Let $\angle H'AP = \varepsilon$. By the assumption $AB \neq AC$, we have $\varepsilon \neq 0$. Then $\angle H'OP = 2\angle H'AP = 2\varepsilon$, hence $G'H' = G'P$ iff $\angle G'OH' = \varepsilon$. But $\angle GO'H = \angle G'OH'$. Let D be the midpoint of OO'. It is known that $\angle GDO = \angle GAH = \varepsilon$. Let F be the midpoint of HG. Then $HG = FO$ (Euler line). Let $\angle GO'H = \delta$. We then have to show $\delta = \varepsilon$ iff $\angle CAB = 60^\circ$. But by similarity ($\Delta GDO \sim \Delta FO'O$) we have $\angle FO'O = \varepsilon$. Consider the circumcircles of the triangles $FO'O$ and $GO'H$. By the sine law and since the segments HG and FO are of equallength we deduce that the circumcircles of the triangles $FO'O$ and $GO'H$ are symmetric with respect to the perpendicular bisector of the segment FG iff $\delta = \varepsilon$. Obviously, O' is the common point of these two circles. Hence O' must be fixed after the symmetry about the perpendicular bisector of the segment FG $\delta=\varepsilon$ we have $\varepsilon=\delta$ iff $\Delta HOO'$ is isosceles. But $HO'=H'O=R$, and so

$$\varepsilon=\delta \ \Leftrightarrow \ OO'=R \ \Leftrightarrow \ OD=\frac{R}{2} \ \Leftrightarrow \ \cos\angle CAB=\frac{1}{2} \ \Leftrightarrow \ \angle CAB=60^\circ .$$

Solution 3: Let H' and G' denote the reflection of points H and G with respect to the line BC. It is known that H' belongs to the circumcircle of ΔABC. The equality $HG=GP'$ is equivalent to $H'G'=G'P$. As in the solution 2, it is equivalent to the statement that point G' belongs to the perpendicular bisector of $H'P$, which is equivalent to $OG'\perp H'P$, where O is the circumcenter of ΔABC.

Let points $A(a)$, $B(b)$, and $C(c=-\overline{b})$ belong to the unit circle in the complex plane. Point G have coordinate $g=(a+b-\overline{b})/3$. Since BC is parallel to the real axis point H' have coordinate $h'=\overline{a}=\frac{1}{a}$.

Point $P(p)$ belongs to the unit circle, so $\overline{p}=\frac{1}{p}$. Since a,p,g are collinear we have $\frac{p-a}{g-a}=\overline{\left(\frac{p-a}{g-a}\right)}$. After computation we get $p=\frac{g-a}{1-\overline{g}a}$. Since $G'(g)$ is the reflection of G with respect to the chord BC, we have $g'=b+(-\overline{b})-b(-\overline{b})\overline{g}=b-\overline{b}+\overline{g}$. Let $b-\overline{b}=d$. We have $\overline{d}=-d$. So

$$g=\frac{a+d}{3},\ \overline{g}=\frac{\overline{a}-d}{3},\ g'=d+\overline{g}=\frac{\overline{a}+2d}{3},\ \overline{g'}=\frac{a-2d}{3},\ \text{and}\ p=\frac{g-a}{1-\overline{g}a}=\frac{d-2a}{2+ad}. \qquad (1)$$

It is easy to see that $OG'\perp H'P'$ is equivalent to

$$\frac{g'}{h'-p}=-\overline{\left(\frac{g'}{h'-p}\right)}=-\frac{\overline{g'}}{\frac{1}{h'}-\frac{1}{p}}=\frac{\overline{g'}h'p}{h'-p}$$

since h' and p belong to the unit circle(note that $H'\neq P$ because $AB\neq AC$). This is equivalent to $g'=\overline{g'}h'p$ and from (1), after easy computations, this is equivalent to

$$a^2g^2+a^2+d^2+1=(a^2+1)(d^2+1)=0 .$$

We cannot have $a^2+1=0$, because then $a=\pm i$, but $AB\neq AC$. Hence $d=b-\overline{b}=\pm i$, and the pair $\{b,c=-\overline{b}\}$ is either $\left\{-\frac{\sqrt{3}}{2}+\frac{i}{2},\frac{\sqrt{3}}{2}+\frac{i}{2}\right\}$ or $\left\{-\frac{\sqrt{3}}{2}-\frac{i}{2},\frac{\sqrt{3}}{2}-\frac{i}{2}\right\}$. Both cases are eqivalent to $\angle BAC=60^\circ$ which completes the proof.

22-nd Macedonian Mathematical Olympiad
Fon University - Skopje
04.04.2015

1. Let AH_A, BH_B, CH_C be the heights in ΔABC. We draw perpendiculars p_A, p_B, p_C through the vertices A, B, C to H_BH_C, H_CH_A, H_AH_B, respectively. Prove that p_A, p_B, p_C pass through the same point.

2. Let a, b, c be positive real numbers for which $abc = 1$. Prove that

$$a^2b + b^2c + c^2a \geq \sqrt{(a+b+c)(ab+bc+ca)}.$$

3. The contestants of this year's MMO are "well" distributed in n columns (a distribution in columns is "well" if no two contestants in the same column are acquaintances), but the same cannot be obtained in less than n columns. Show that there exist contestants $M_1, M_2, \ldots, M_n$ for which the following hold:

(1) M_i is in the i-th column, for each $i = 1, 2, \ldots, n$;

(2) M_i and M_{i+1} are acquaintances, for each $i = 1, 2, \ldots, n-1$.

4. The circles k_1, k_2 intersect at points A and B. A line thorugh B intersects the circles k_1 and k_2 for the second time at points C and D, respectively, in such a way that C lies outside of k_2, and D lies outside of k_1. Let M be the point of intersection of the tangents to k_1 and k_2 drawn through C and D, respectively, and $AM \cap CD = \{P\}$. The tangent drawn through B to k_1 intersects AD in point L, and the tangent drawn through B to k_2 intersects AC in point K. Let $KP \cap MD = \{N\}$ and $LP \cap MC = \{Q\}$. Show that the quadrilateral $MNPQ$ is a parallelogram.

5. Determine all natural numbers m which have exactly three different prime divisors p, q and r, such that:

a) $p - 1 \mid m$, $qr - 1 \mid m$,

b) $q - 1 \nmid m$, $r - 1 \nmid m$, $3 \nmid q + r$.

Time: 4 hours 30 minutes
Each correctly solved problem is worth 8 points

SOLUTIONS

1. Let AH_A, BH_B, CH_C be the heights in ΔABC. We draw perpendiculars p_A, p_B, p_C through the vertices A, B, C to H_BH_C, H_CH_A, H_AH_B, respectively. Prove that p_A, p_B, p_C pass through the same point.

First solution: Let O be the center of the circumscribed circle around ΔABC. We will show that each of the lines p_A, p_B, p_C passes through O. (**2 points**) Because of symmetry, it is enough to show that $OC \perp H_AH_B$. (**1 point**) Let D be the point of intersection of these two lines. We restrict ourselves to the case where ΔABC is acute (since in the case of ΔABC being obtuse the argument is analogous (**1 point**)). It is enough to use the fact that $\measuredangle H_ACD = \measuredangle BCO = 90^\circ - \alpha$ (**2 points**) and $\measuredangle DH_AC = \measuredangle H_BH_AC = \alpha$ (the last equality follows from the fact that ABH_AH_B is inscribed). (**2 points**)

Second solution: According to Carnot's theorem, it is sufficient (and necessary) to show that

$$|AH_B|^2 - |AH_C|^2 + |BH_C|^2 - |BH_A|^2 + |CH_A|^2 - |CH_B|^2 = 0.$$

holds. (**4 points**).

For that purpose, it is sufficient to sum the obvious equalities: $|BH_C|^2 - |AH_C|^2 = |BC|^2 - |AC|^2$, $|CH_A|^2 - |BH_A|^2 = |AC|^2 - |AB|^2$, and $|AH_B|^2 - |CH_B|^2 = |AB|^2 - |BC|^2$. **(4 points)**

2. Let a,b,c be positive real numbers for which $abc = 1$. Prove that

$$a^2b + b^2c + c^2a \geq \sqrt{(a+b+c)(ab+bc+ca)}.$$

Solution. From the inequality between the arithmetical and geometrical mean we have

$$\left(a^2b+b^2c+c^2a\right)^2 \overset{(1)}{\geq} 3\left(a^2b\cdot b^2c + b^2c\cdot c^2a + c^2a\cdot a^2b\right) = 3abc\left(b^2a+c^2b+a^2c\right) = 3\left(b^2a+c^2b+a^2c\right). \textbf{ (2 points)}$$

$$\left(a^2b+b^2c+c^2a\right)^2 = \left(a^2b+b^2c+c^2a\right)\cdot\left(a^2b+b^2c+c^2a\right) \overset{(2)}{\geq} 3\sqrt[3]{a^3b^3c^3}\left(a^2b+b^2c+c^2a\right) = 3\left(a^2b+b^2c+c^2a\right). \textbf{ (2 points)}$$

$$\left(a^2b+b^2c+c^2a\right)^2 = \left(a^2b+b^2c+c^2a\right)\cdot\left(a^2b+b^2c+c^2a\right) \overset{(3)}{\geq} 3\sqrt[3]{a^3b^3c^3}\,3\sqrt[3]{a^3b^3c^3} = 9abc. \textbf{ (2 points)}$$

$$3\left(a^2b+b^2c+c^2a\right)^2 \geq 3\left(b^2a+c^2b+a^2c\right) + 3\left(a^2b+b^2c+c^2a\right) + 9abc \Leftrightarrow$$

$$\left(a^2b+b^2c+c^2a\right)^2 \geq a^2b+b^2c+c^2a+b^2a+c^2b+a^2c+3abc \Leftrightarrow \textbf{ (1 point)}$$

$$\left(a^2b+b^2c+c^2a\right)^2 \geq (a+b+c)(ab+bc+ca) \Leftrightarrow$$

$$a^2b+b^2c+c^2a \geq \sqrt{(a+b+c)(ab+bc+ca)}. \textbf{ (1 point)}$$

(1), (2) and (3) The inequality between arithmetical and geometrical mean.

3. The contestants of this year's MMO are "well" distributed in n columns (a distribution in columns is "well" if no two contestants in the same column are acquaintances), but the same cannot be obtained in less than n columns. Show that there exist contestants $M_1, M_2, \ldots, M_n$ for which the following hold:

(1) M_i is in the i-th column, for each $i = 1, 2, \ldots, n$;

(2) M_i and M_{i+1} are acquaintances, for each $i=1,2,\ldots,n-1$.

Solution. We will perform a rearrangement with respect to columns. First we move to the first column each contestant from the second column who doesn't have an acquaintance in the first column. (**1 point**) The new arrangement is "well", and therefore at least one contestant remains in the second column. Now we move to the second column each contestant from the third column who doesn't have an acquaintance among the remaining contestants in the second column. The new arrangement is "well", and therefore there is at least one contestant remaining in the third column. We continue this procedure. (**3 points**) In the end we move to the $(n-1)$-th column each contestant from the n-th column who doesn't have an acquaintance among the remaining ones in the $(n-1)$-th column. The new arrangement is again "well" and therefore at least one contestant remains in the n-th column. We denote such a contestant by M_n. (**1 point**). He must have an acquaintance M_{n-1} in the $(n-1)$-th column. Let us notice that M_{n-1} has not been moved (otherwise the initial arrangement is not "well"). Therefore M_{n-1} has an acquaintance M_{n-2} in the $(n-2)$-th column. We conclude analogously that M_{n-2} has not been moved. We proceed in this way and therefore we find contestants $M_1, M_2, \ldots, M_n$ for which (1) and (2) hold.(**3 points**)

4. The circles k_1, k_2 intersect at points A and B. A line thorugh B intersects the circles k_1 and k_2 for the second time at points C and D, respectively, in such a way that C lies outside of k_2, and D lies outside of k_1. Let M be the point of intersection of the tangents to k_1 and k_2 drawn through C and D, respectively, and $AM \cap CD = \{P\}$. The tangent drawn through B to k_1 intersects AD in point L, and the tangent drawn through B to k_2 intersects AC in point K. Let $KP \cap MD = \{N\}$ and $LP \cap MC = \{Q\}$. Show that the quadrilateral $MNPQ$ is a parallelogram.

Solution: Due to symmetry reasons, it is enough to show that $KP \| MC$ holds. First we show that the quadrilateral $ACMD$ is inscribed. (**1 point**) Namely, let us notice that B lies on the line segment $\overline{CD}$, and A and M are on different sides of the line CD. From $\measuredangle BDM = \measuredangle DAB$ and $\measuredangle BCM = \measuredangle BAC$, it follows that $\measuredangle DAC = \measuredangle DAB + \measuredangle BAC = \measuredangle BDM + \measuredangle BCM = 180^\circ - \measuredangle DMC$. (**2 points**)
Second, we will show that B and P lie on the same arc which passes through points A and K. (**1 point**)
For that purpose, we consider two cases:

first case: the point P lies on the segment BC; let us notice that points A and B are on the same side of the line KP. Let E denote the intersection of the lines KB and DM. We have the sequence of equalities $\measuredangle KBP = \measuredangle DBE = \measuredangle BDE = \measuredangle CDM = \measuredangle CAM = \measuredangle KAP$. Then, from $\measuredangle KBP = \measuredangle KAP$ it follows that the quadrilateral $AKPB$ is inscribed. (**1 point**)

second case: the point P lies on the segment AC; this time the points A and B are on different sides of the line KP. Again, let E be the intersection of KB and DM. We have the following sequence of equalities $180^\circ - \measuredangle KBP = \measuredangle DBE = \measuredangle BDE = \measuredangle CDM = \measuredangle CAM = \measuredangle KAP$, from where it follows that the quadrilateral $AKBP$ is inscribed. (**1 point**)

Therefore we get $\measuredangle APK = \measuredangle ABK = \measuredangle ADB = \measuredangle ADC = \measuredangle AMC$, with which we confirm that $KP \| MC$. (**2 points**)

Remark: The assertion also remains true without the restriction "...in such a way that C (resp. D) lies outside of k_2 (resp. k_1)**...**". In the remaining two cases the argument is analogous.

5. Determine all natural numbers m which have exactly three different prime divisors p, q and r, such that:

a) $p-1 \mid m$, $qr-1 \mid m$,

b) $q-1 \nmid m, r-1 \nmid m$, $3 \nmid q+r$.

Solution. The number m can be represented in the form $m=p^a q^b r^c$. The numbers q and r cannot be 2, because $2-1=1$ is a divisor of each natural number, from where it follows that they are odd, i.e. $qr-1$ is even, so therefore m must be even, i.e. $p=2$.(**1 point**)

From $\text{НЗД}(q,qr-1)=1=\text{НЗД}(r,qr-1)$ and $qr-1 \mid m$, it follows that $qr-1 \mid 2^a$, from where $qr-1=2^k$ for some natural number $k \le a$. This means that the number 2^k+1 has exactly two divisors. (**1 point**) The number k can be represented in the form $k=2^t s$, where t is the greatest power of 2 in k, and s is the greatest odd divisor of k. If $s>1$, then:

$$2^k+1=\left(2^{2^t}\right)^s+1=\left(2^{2^t}+1\right)\left(\left(2^{2^t}\right)^{s-1}-\left(2^{2^t}\right)^{s-2}+...-\left(2^{2^t}\right)+1\right).$$ (**2 points**)

This is possible only when $q=2^{2^t}+1$, $r=\left(2^{2^t}\right)^{s-1}-\left(2^{2^t}\right)^{s-2}+...-\left(2^{2^t}\right)+1$ or vice versa, but then $q-1=2^{2^t} \mid m$ or $p=2^{2^t} \mid m$, which is contradictory to the conditions of the exercise. (**1 point**)

It follows that $s=1$, i.e. $k=2^t$ is a power of 2 and $qr=2^{2^t}+1$, i.e. $2^{2^t}+1$ is the product of two prime numbers. For $t=0$ ($2^{2^t}+1=3$) this obviously is not the case, therefore $t>1$.(**1 point**)

From $2^{2^t}+1 \equiv 2^{2 \cdot 2^{t-1}}+1 \equiv \left(2^2\right)^{2^{t-1}}+1 \equiv 4^{2^{t-1}}+1 \equiv 1^{2^{t-1}}+1 \equiv 1+1 \equiv 2 \pmod 3$, it follows that $3 \nmid q$ and $3 \nmid r$. If $q \equiv r \pmod 3$, then $qr \equiv 1 \pmod 3$, therefore one must be congruent to one, and the other with two, but then their sum is divisible by three, which is contradictory to the last condition of the exercise, i.e. a number that satisfies the given conditions does not exist. (**2 points**)

19-th Macedonian Junior Mathematical Olympiad
Faculty of Mechanical Engineering-Skopje
06.06.2015

19-t a JMMO

1. Solve the equation $x^2+y^4+1=6^z$ in the set of integers.

2. A circle k with center at O and radius r and a line p which doesn't have a common point with k are given. Let E be the foot of the perpendicular from O to p. An arbitrary point M different from E is chosen on p and the two tangents are drawn from M to k which touch the circle k at points A and B. If H is the intersection of AB and OE, prove that $\overline{OH}=\frac{r^2}{\overline{OE}}$.

3. Prove that for positive real numbers a,b,c the following inequality holds:

$$(16a^2+8b+17)(16b^2+8c+17)(16c^2+8a+17)\geq 2^{12}(a+1)(b+1)(c+1).$$

When does equality hold?

4. Let ABC be an acute triangle and let k be the circle circumscribed around it. The point O in the interior of the triangle is such that $\overline{CE}=\overline{CF}$, where E and F are points on k and E lies on AO, and F lies on BO. Prove that O lies on the bisector of the angle at the vertex C if and only if the triangle is isosceles with base $\overline{AB}$.

5. Let A and B be two identical convex polygons, each having area 2015. The polygon A is divided into polygons $A_1,A_2,...,A_{2015}$ with positive area, and the polygon B into polygons $B_1,B_2,...,B_{2015}$ with positive area. The polygons $A_1,A_2,...,A_{2015},B_1,B_2,...,B_{2015}$ are colored with 2015 colors, in such a way that A_i is colored differently from A_j and B_i is colored differently from B_j, for $i\neq j$. After overlapping the polygons A and B, we calculate the sum of the areas of the parts that have the same color.

Prove that there exists a coloring of the polygons for which this sum is at least 1.

Solutions

1. Solve the equation $x^2+y^4+1=6^z$ in the set of integers.

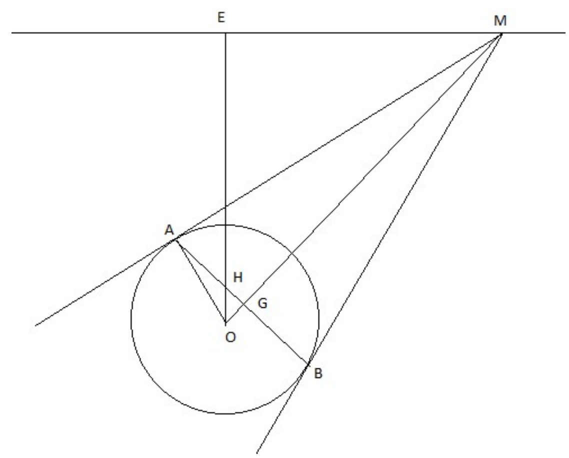
Solution. It is obvious that $z\geq 0$. If $z\geq 2$, then $x^2+y^4+1\equiv 0(\text{mod}\,4)$, i.e. $x^2+y^4\equiv 3(\text{mod}\,4)$. This is not possible because the remainders of squares of integers after division by 4 are 0 or 1. According to that, $0\leq z<2$.

If $z=0$, then $x=y=0$.

If $z=1$, then $x^2+y^4=5$, i.e. $(x,y)=\{(2,1),(-2,1),(2,-1),(-2,-1)\}$.

Therefore $(x,y,z)=\{(0,0,0),(2,1,1),(-2,1,1),(2,-1,1),(-2,-1,1)\}$ are the solutions of the given equation.

2. A circle k with center at O and radius r and a line p which doesn't have a common point with k are given. Let E be the foot of the perpendicular from O to p. An arbitrary point M different from E is chosen on p and the two tangents are drawn from M to k which touch the circle k at points A and B. If H is the intersection of AB and OE, prove that $\overline{OH}=\frac{r^2}{\overline{OE}}$.



Solution. Let G be the intersection of OM and AB. Since $\Delta OGH \sim \Delta OEM$ we get $\frac{\overline{OG}}{\overline{OH}}=\frac{\overline{OE}}{\overline{OM}}$ hence $\overline{OE}\cdot\overline{OH}=\overline{OM}\cdot\overline{OG}$. On the other hand, since $\Delta AOG \sim \Delta MOA$, we have $\frac{\overline{OA}}{\overline{OG}}=\frac{\overline{OM}}{\overline{OA}}$. Therefore $\overline{OM}\cdot\overline{OG}=\overline{OA}^2$.

We get $\overline{OH}=\overline{OA}^2/\overline{OE}$.

Remark. The equality $\overline{OE}\cdot\overline{OH}=\overline{OM}\cdot\overline{OG}$ can also be obtained as a power of a point since $GMEH$ is inscribed.

3. Prove that for positive real numbers a,b,c the following inequality holds:

$$(16a^2+8b+17)(16b^2+8c+17)(16c^2+8a+17)\geq 2^{12}(a+1)(b+1)(c+1).$$

When does equality hold?

Solution. By twice using the inequality between the arithmetical mean and geometrical mean we get

$$(16a^2+8b+17)=(16a^2+1+8b+16)\geq 8a+8b+16=8(a+b+2)=8(a+1+b+1)\geq$$
$$\geq 8\cdot 2\sqrt{(a+1)(b+1)}=2^4\sqrt{(a+1)(b+1)}\text{ . (1)}$$

Analogously we have

$$(16b^2+8c+17)\geq 2^4\sqrt{(b+1)(c+1)}\text{ (2)}$$
$$(16c^2+8a+17)\geq 2^4\sqrt{(c+1)(a+1)}\text{ (3).}$$

If we multiply the three inequalities we get

$$(16a^2+8b+17)(16b^2+8c+17)(16c^2+8a+17)\geq 2^{12}(a+1)(b+1)(c+1).$$

In (1) equality is obtained when $16a^2=1$ and $a=b$, i.e. $a=b=\frac{1}{4}$. By an analogous argument for (2) and (3) we get $a=b=c=\frac{1}{4}$.

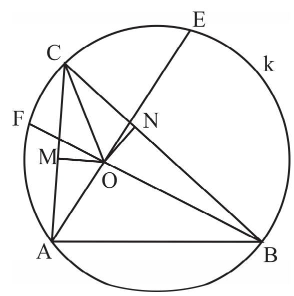
4. Let ABC be an acute triangle and let k be the circle circumscribed around it. The point O in the interior of the triangle is such that $\overline{CE}=\overline{CF}$, where E

and F are points on k and E lies on AO, and F lies on BO. Prove that O lies on the bisector of the angle at the vertex C if and only if the triangle is isosceles with base $\overline{AB}$.

Solution. From $\overline{CE}=\overline{CF}$ it follows that $\sphericalangle CAE=\sphericalangle CBF$ (1), as inscribed angles subtending equal chords.

Let us assume first that the triangle is isosceles. Then, from the fact that O lies in the interior of ABC and (1), it follows that $\sphericalangle BAO=\sphericalangle BAC-\sphericalangle CAO=\sphericalangle ABC-\sphericalangle CBO=\sphericalangle ABO$, from where it follows that the triangle ABO is isosceles with base $\overline{AB}$, i.e. $\overline{AO}=\overline{BO}$ (2). From the fact that the triangle ABC is isosceles, it follows that $\overline{AC}=\overline{BC}$ (3). From (1), (2) and (3) it follows that $\Delta AOC\cong\Delta BOC$, from where we have $\sphericalangle ACO=\sphericalangle BCO$, i.e. O lies on the bisector of the angle at the vertex C.

Remark: $\Delta AOC\cong\Delta BOC$ does not follow directly from $\sphericalangle CAE=\sphericalangle CBF$, $\overline{AC}=\overline{BC}$ and $\overline{CO}$ is a common side.



Let's assume now that point O lies on the bisector of the angle at the vertex C and let M and N be the feet of the perpendiculars from O to the sides AC and BC respectively. The right-angled triangles CON and COM are congruent because $\sphericalangle ACO=\sphericalangle BCO$ and $\overline{CO}$ is a common side, so $\overline{CN}=\overline{CM}$ (4) and $\overline{ON}=\overline{OM}$ (5). The right-angled triangles BON and AOM are congruent from (1) and (5), so $\overline{BN}=\overline{AM}$ (6). By adding (4) and (6) we get that $\overline{AC}=\overline{BC}$ (the points M and N lie in the interior of the sides, since the triangle is acute).

5. Let A and B be two identical convex polygons, each having area 2015. The polygon A is divided into polygons $A_1,A_2,...,A_{2015}$ with positive area, and the polygon B into polygons $B_1,B_2,...,B_{2015}$ with positive area. The polygons $A_1,A_2,...,A_{2015}$, $B_1,B_2,...,B_{2015}$ are colored with 2015 colors, in such a way that A_i is colored differently from A_j and B_i is colored differently from B_j, for $i\neq j$. After overlapping the polygons A and B, we calculate the sum of the areas of the parts that have the same color.

Prove that there exists a coloring of the polygons for which this sum is at least 1.

Solution. After the overlapping of the polygons we get $C_{ij}=A_i\cap B_j$, $i,j\in\{1,2,...,2015\}$. The polygons $B_1,B_2,...,B_{2015}$ can be colored in $2015!$ ways.

Let a coloring of $A_1,A_2,...,A_{2015}$ be given. For an arbitrary coloring of $B_1,B_2,...,B_{2015}$ which we denote by n, let O_n be the sum of the areas of the parts from the two polygons colored with the same color. Then $O_n=\sum_{i,j=1}^{2015}c_{ij}P(C_{ij})$, where $c_{ij}=1$ if A_i and B_j are colored with the same color and $c_{ij}=0$ in the other cases. We get that $\sum_{n=1}^{2015!}O_n=\sum_{i,j=1}^{2015}d_{ij}P(C_{ij})$, where d_{ij} is the number of colorings of

$B_1, B_2, ..., B_{2015}$ in which A_i and B_j have the same color. It is obvious that $d_{ij} = 2014!$. According to that, $\sum_{n=1}^{2015!} O_n = \sum_{i,j=1}^{2015} d_{ij} P(C_{ij}) = 2014! \cdot 2015 = 2015!$.

Finally, from $O_1 + O_2 + ... + O_{2015!} = 2015!$ it follows that there exists $k \in \{1, 2, ..., 2015!\}$ for which $O_k \geq 1$, which was to be proven.

32 th Balkan Mathematical Olympiads 2015
03.05-08.05.2015, Athens, Greece

Problem 1. Let a, b and c be positive real numbers. Prove that

$$a^3b^6 + b^3c^6 + c^3a^6 + 3a^3b^3c^3 \geq abc(a^3b^3 + b^3c^3 + c^3a^3) + a^2b^2c^2(a^3 + b^3 + c^3)$$

Problem 2. Let ABC be a scalene triangle with incentre I and circumcircle ω. The lines AI, BI, CI intersect ω for the second time at the points D, E, F, respectively. The lines through I parallel to the sides BC, AC, AB intersect the lines EF, DF, DE at the points K, L, M, respectively. Prove that the points K, L, M are collinear.

Problem 3. A juri of 3366 film critics are judging the Oscars. Each critic makes a single vote for the favorite actor, and a single vote for his favourite actress. In turns out that for every integer $n \in \{1, 2, 3, ..., 100\}$ there is an actor or actress who has been voted for exactly n times. Show that there are two critics who voted for the same actor for the same actress.

Problem 4. Prove that among any 20 consecutive positive integers there exists an integer d such that for each positive integer n we have the inequality

$$n\sqrt{d}\{n\sqrt{d}\} > \frac{5}{2}$$

where $\{x\}$ denotes the fractinal part of the real number x. The fractional part of a real number x is x minus the greather less than or equal to x.

Solutions

Problem 1. Let a,b and c be positive real numbers. Prove that

$$a^3b^6+b^3c^6+c^3a^6+3a^3b^3c^3 \ge abc(a^3b^3+b^3c^3+c^3a^3)+a^2b^2c^2(a^3+b^3+c^3)$$

Solution. After dividing both sides of the given inequality by $a^3b^3c^3$ it becomes

$$\left(\frac{b}{c}\right)^3+\left(\frac{c}{a}\right)^3+\left(\frac{a}{b}\right)^3+3 \ge \left(\frac{a}{c}\cdot\frac{b}{c}+\frac{b}{a}\cdot\frac{c}{a}+\frac{c}{b}\cdot\frac{a}{b}\right)+\left(\frac{a}{b}\cdot\frac{a}{c}+\frac{b}{a}\cdot\frac{c}{a}+\frac{c}{a}\cdot\frac{c}{b}\right) \quad (1)$$

Set

$$\frac{b}{a}=\frac{1}{x},\ \frac{c}{b}=\frac{1}{y},\ \frac{a}{c}=\frac{1}{z}. \quad (2)$$

Then we have that $xyz=1$, and by substituting (2) into (1), we find that

$$x^3+y^3+z^3+3 \ge \left(\frac{y}{z}+\frac{z}{x}+\frac{x}{y}\right)+\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right). \quad (3)$$

Multiplying the inequality (3) by xyz, and using the fact that $xyz=1$, the inequality is equivalent to

$$x^3+y^3+z^3+3xyz-xy^2-yz^2-zx^2-yx^2-zy^2-xz^2 \ge 0. \quad (4)$$

Finally, notice that by the special case of Shur's inequality

$$x^r(x-y)(x-z)+y^r(y-x)(y-z)+z^r(z-y)(z-x)\ge 0,\ x,y,z\ge 0,\ r>0,$$

with $r=1$ there holds

$$x(x-y)(x-z)+y(y-x)(y-z)+z(z-y)(z-x)\ge 0 \quad (5)$$

which after expansion actually coincides with the congruence (4).

Remark 1. The inequality (5) immediately follows by supposing (without loss of generality) that $x\ge y\ge z$, and then writing the left side of the inequality (5) in the form

$$(x-y)(x(x-z)-y(y-z))+z(y-z)(z-x),$$

which is obviously ≥ 0.

Remark 1. One can obtain the relation (4) using also the substitution $x=ab^2, y=bc^2$ and $z=ca^2$.

Problem 2. Let ABC be a scalene triangle with incentre I and circumcircle ω. The lines AI,BI,CI intersect ω for the second time at the points D,E,F, respectively. The lines through I parallel to the sides BC,AC,AB intersect the lines EF,DF,DE at the points K,L,M, respectively. Prove that the points K,L,M are collinear.

Solution. First we will prove that KA is tangent to ω.

Indeed, it is a wll-known fact that $FA=FB=FI$ and $EA=EC=EI$, so FE is the perpendicular bisector of AI. It follows that $KA=KI$ and

$$\angle KAF=\angle KIF=\angle FCB=\angle FEB=\angle FEA,$$

so KA is tangent to ω. Similarly we can prove that LB,MC are tangent to ω as well.

Let A',B',C' the intersections of AI,BI,CI with BC,CA,AB resepectively. From Pascal's Theorem on the cyclic hexagon $AACDEB$ we get K,C',B' colinear. Similarly L,C',A' collinear and M,B',A' collinear.

Then from Desargues' Theorem for ΔDEF, $\Delta A'B'C'$ which are perspective from the point I we get K,L,M of the intersection of their corresponding sides are collinear as wanted.

Remark(P.S.C.). After proving that KA,LB,MC are tangent to ω, we can argue as follows:

It readily follows that $\Delta KAF \sim \Delta KAE$ and so $\frac{KA}{KE}=\frac{KF}{KA}=\frac{AF}{AE}$, thus $\frac{KF}{KE}=\left(\frac{AF}{AE}\right)^2$. In a similar way we can find that $\frac{ME}{MD}=\left(\frac{CE}{CD}\right)^2$ and $\frac{LD}{LF}=\left(\frac{BD}{BF}\right)^2$. Multiplying we obtain $\frac{KF}{KE}\cdot\frac{ME}{MD}\cdot\frac{LD}{LF}=1$, so by the converse of Menelaus theorem applied in the triangle DEF we get that the points K,L,M are collinear.

Problem 3. A juri of 3366 film critics are judging the Oscars. Each critic makes a single vote for the favorite actor, and a single vote for his favourite actress. In turns out that for every integer $n\in\{1,2,3,...,100\}$ there is an actor or actress who has been voted for exactly n times. Show that there are two critics who voted for the same actor for the same actress.

Solution. Let us assume that every votes for a different pair of actor and actress. We'll arrive at a contradiction proving the required result. Indeed:

Call the vote of each critic, i.e. his choice for the pair of an actor and and an actress, as a double-vote, and call as a single-vote each one of the two choices he makes, i.e. the one for an actor and the other one for an actress. In this terminology, a double-vote corresponds to two single-votes.

For each $n=34,35,...,100$ let us pick out one actor or one actress who has been voted by exactly n critics (i.e. appears in exactly n -single votes) and call S the set of these movie stars. Calling a,b the number of men and women in S, we have $a+b=67$.

Now let S_1 be the set of double-votes, each having exactly one of its two corresponding single-votes in S, and let S_2 be the set of double-votes with both its single-votes in S. If s_1,s_2 are the number of elements in S_1,S_2 respectively, we have that the number of all double-votes with at least one single-vote in S is s_1+s_2, whereas the number of all double-votes with both single votes in S is $s_2\le ab$.

Since all double-votes are distinct, there must exist at least s_1+s_2 critics. But the number of all single-votes in S is $s_1+2s_2=34+35+...+100=4489$, and moreover $s\le ab$. So there exist at least $s_1+s_2=s_1+2s_2-s_2\ge 4489-ab$ critics.

Now notice that as $a+b=67$, the maximum value of ab with a,b integers is obtained for $\{a,b\}=\{33,34\}$, so $ab\le 33\cdot 34=1122$. A quick proof of this is the following:

$$ab=\frac{(a+b)^2-(a-b)^2}{4}=\frac{67^2-(a-b)^2}{4},$$

which is maximed(for not equal integers a,b as $a+b=67$) whenever $|a-b|=1$, thus for $\{a,b\}=\{33,34\}$.

Thus there exist at least $4489-1122=3367$ critics which is a contradiction and we are done.

Remark. We are going here to give some motivation about the choice of number 34, used in the above solution.

Let us assume that every critic votes for a different pair of actor and actress. One can again start by picking out one actor or one actress who has been voted by exactly n critics for $n=k,k+1,...,100$. Then $a+b=100-k+1=101-k$ and the number of all single-votes is $s_1+2s_2=k+k+1+...+100=5050-\frac{k(k-1)}{2}$, so there exist at least $s_1+s_2=s_1+2s_2-s_2\ge 5050-\frac{k(k-1)}{2}-ab$ and

$$ab=\frac{(a+b)^2-(a-b)^2}{4}=\frac{(101-k)^2-(a-b)^2}{4}\le\frac{(101-k)^2-1}{4}.$$

After all, the number of critics is at least

$$5050-\frac{k(k-1)}{2}-\frac{(101-k)^2-1}{4}.$$

In order to arrive at a contradiction we have to choose k such that

$$5050-\frac{k(k-1)}{2}-\frac{(101-k)^2-1}{4}\geq 3367,$$

and solving the inequality with respect to k, the only value that makes the last one true is $k=34$.

Problem 4. Prove that among any 20 consecutive positive integers there exists an integer d such that for each positive integer n we have the inequality

$$n\sqrt{d}\{n\sqrt{d}\}>\frac{5}{2}$$

where $\{x\}$ denotes the fractinal part of the real number x. The fractional part of a real number x is x minus the greather less than or equal to x.

Solution. Among the given numbers there is a number of the form $20k+15=5(4k+3)$. We shall prove that $d=5(4k+3)$ satisfies the statement's condition. Since $d\equiv -1(\text{mod } 4)$, it follows that d is not a perfect square, and thus for any $n\in\mathbb{N}$ such that $a+1>n\sqrt{d}>a$, that is, $(a+1)^2>n^2d>a^2$. Actually, we are going to prove that $n^2d\geq a^2+5$. Indeed:

It is known that each positive integer of the form $4s+3$ has a prime divisor of the same form. Let $p\,|\,4k+3$ and $p\equiv -1\,(\text{mod } 4)$. Because of the form of p, the numbers a^2+1^2 and a^2+2^2 are not divisible by p, and since $p\,|\,n^2d$, it follows that $n^2d\neq a^2+1, a^2+4$. On the other hand, $5\,|\,n^2d$, and since $5\nmid a^2+2, a^2+3$, we conclude $n^2d\neq a^2+2, a^2+3$. Since $n^2d>a^2$ we must have $n^2d\geq a^2+5$ as claimed. Therefore

$$n\sqrt{d}\{n\sqrt{d}\}=n\sqrt{d}(n\sqrt{d}-a)\geq a^2+5-a\sqrt{a\ \ +5}>a^2+5-\frac{a^2+a^2+5}{2}=\frac{5}{2},$$

which was to be proved.

Junior Balkan Mathematical Olympiad 2015

24.06-29.06.2014, Belgrade, Serbia

Problems

Problem 1. Find all prime numbers a,b,c and positive integers k satisfying the equation

$$a^2+b^2+16c^2=9k^2+1$$

Problem 2. Let a,b,c be positive real numbers such that $a+b+c=3$. Find the minimum value of the expression

$$A=\frac{2-a^3}{a}+\frac{2-b^3}{b}+\frac{2-c^3}{c}$$

Problem 3. Let ABC be an acute triangle. The lines l_1 and l_2 are perpendicular to AB at the points A and B respectively. The perpendicular lines from the midpoint M of AB to the lines AC and BC intersect l_1 and l_2 at the points E and F, respectively. If D is the intersection point of the lines EF and MC, prove that

$$\angle ADB=\angle EMF.$$

Problem 4. An L-shape is one of the following four pieces, each consisting of three unit squares:

A 5×5 board, consisting of 25 unit squares, a positive integer $k\leq 25$ and an unlimited supply of L-shapes are given. Two players, A and B, play the following game:starting with A they alternatively mark a previously unmarked unit square until they mark a total of k unit squares.

We say that a placement of L-shapes on unmarked unit squares is called good if the L-shapes do not overlap and each of them covers exactly three unmarked unit squares of the board. B wins if every good placement of L-shapes leaves uncovered at least three unmarked unit squares. Determine the minimum value of k for which B has a winning strategy.

Solutions

Problem 1. Find all prime numbers a,b,c and positive integers k satisfying the equation

$$a^2+b^2+16c^2=9k^2+1$$

Solution. The relation $9k^2+1\equiv 1 \pmod 3$ implies

$$a^2+b^2+16c^2\equiv 1 \pmod 3 \quad \Leftrightarrow \quad a^2+b^2+c^2\equiv 1 \pmod 3.$$

Since $a^2\equiv 0,1 \pmod 3$, $b^2\equiv 0,1 \pmod 3$, $c^2\equiv 0,1 \pmod 3$, we have:

a^2	0	0	0	0	1	1	1	1
b^2	0	0	1	1	0	0	1	1
c^2	0	1	0	1	0	1	0	1
$a^2+b^2+c^2$	0	1	1	2	1	2	2	0

From the previous table it follows that two of three prime numbers a,b,c are equal to 3.

Case 1. $a=b=3$. We have

$$a^2+b^2+16c^2=9k^2+1 \quad \Leftrightarrow \quad 9k^2-16c^2=17 \quad \Leftrightarrow \quad (3k-4c)(3k+4c)=17$$

If $\begin{cases} 3k-4c=1 \\ 3k+4c=17 \end{cases}$, then $\begin{cases} c=2 \\ k=3 \end{cases}$ and $(a,b,c,k)=(3,3,2,3)$.

If $\begin{cases} 3k-4c=-1 \\ 3k+4c=-17 \end{cases}$, then $\begin{cases} c=2 \\ k=-3 \end{cases}$ and $(a,b,c,k)=(3,3,2,-3)$.

Case 2. $c=3$. If $(3,b_\circ,c,k)$ is a solution of the given equation, then $(b_\circ,3,c,k)$ is a solution, too.

Let $a=3$. We have

$$a^2+b^2+16c^2=9k^2+1 \quad \Leftrightarrow \quad 9k^2-b^2=152 \quad \Leftrightarrow \quad (3k-b)(3k+b)=152.$$

Both factors shall have the same parity and we obtain only 4 cases:

If $\begin{cases} 3k-b=2 \\ 3k+b=76 \end{cases}$, then $\begin{cases} b=37 \\ k=13 \end{cases}$ and $(a,b,c,k)=(3,37,3,13)$.

If $\begin{cases} 3k-b=4 \\ 3k+b=38 \end{cases}$, then $\begin{cases} b=17 \\ k=7 \end{cases}$ and $(a,b,c,k)=(3,17,3,7)$.

If $\begin{cases} 3k-b=-76 \\ 3k+b=-2 \end{cases}$, then $\begin{cases} b=37 \\ k=-13 \end{cases}$ and $(a,b,c,k)=(3,37,3,-13)$.

If $\begin{cases} 3k-b=-38 \\ 3k+b=-4 \end{cases}$, then $\begin{cases} b=17 \\ k=-7 \end{cases}$ and $(a,b,c,k)=(3,17,3,-7)$.

In addition, $(a,b,c,k)\in\{(37,3,3,13),(17,3,3,7),(37,3,3,-13),(17,3,3,-7)\}$.

So, the given equation has 10 solutions:

$S=\{(37,3,3,13),(17,3,3,7),(37,3,3,-13),(17,3,3,-7),(3,37,3,13),(3,17,3,7),(3,37,3,-13),(3,17,3,-7),(3,3,2,3),(3,3,2,-3)\}$.

Problem 2. Let a,b,c be positive real numbers such that $a+b+c=3$. Find the minimum value of the expression

$$A=\frac{2-a^3}{a}+\frac{2-b^3}{b}+\frac{2-c^3}{c}$$

Solution. We rewrite A as follows:

$$A=\frac{2-a^3}{a}+\frac{2-b^3}{b}+\frac{2-c^3}{c}=2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-a^2-b^2-c^2=2\frac{ab+bc+ca}{abc}-(a^2+b^2+c^2)=$$
$$=2\frac{ab+bc+ca}{abc}-((a+b+c)^2-2(ab+bc+ca))=2\frac{ab+bc+ca}{abc}-(9-2(ab+bc+ca))=$$
$$=2\frac{ab+bc+ca}{abc}+2(ab+bc+ca)-9=2(ab+bc+ca)\left(\frac{1}{abc}+1\right)-9$$

Recall now the well-known inequality $(x+y+z)^2\geq 3(xy+yz+zx)$ and set $x=ab$, $y=bc$, $z=ca$, to obtain $(ab+bc+ca)^2\geq 3abc(a+b+c)=9abc$ where we have used $a+b+c=3$. By taking the square roots on both sides of the last one we obtain:

$$ab+bc+ca\geq 3\sqrt{abc}\ . \qquad (1)$$

Also by using AM-GM inequality we get that

$$\frac{1}{abc}+1\geq 2\sqrt{\frac{1}{abc}}\ . \qquad (2)$$

Multiplication of (1) and (2) gives

$$(ab+bc+ca)\left(\frac{1}{abc}+1\right)\geq 3\sqrt{abc}\,2\sqrt{\frac{1}{abc}}=6\ .$$

So $A\geq 2\cdot 6-9=3$ aand the equality holds if and only if $a=b=c=1$, so the minimum value is 3.

Remark. Note that if $f(x)=\frac{2-x^3}{x}$, $x\in(0,3)$ then $f''(x)=\frac{4}{x^3}-2$, so the function is convex on $x\in(0,\sqrt[3]{2})$ and concave on $x\in(\sqrt[3]{2},3)$. This means that we cannot apply Jensen's inequality.

Problem 3. Let ABC be an acute triangle. The lines l_1 and l_2 are perpendicular to AB at the points A and B respectively. The perpendicular lines from the midpoint M of AB to the lines AC and BC intersect l_1 and l_2 at the points E and F, respectively. If D is the intersection point of the lines EF and MC, prove that

$$\angle ADB=\angle EMF\ .$$

Solution1. Let the circles with diameter EM and FM intersect for second time at D' and let them intersect the sides CA, CB at points G,K respectively. Since

$$\angle ED'M=\angle FD'M=90^\circ\ ,$$

we have that E,D',F are collinear.

Since EM is a diameter and AG is a chord perpendicular to it, we have that $MG=MA$ and similarly $MK=MB$. Since $MA=MB$, it follows that $AGKB$ is cyclic.

From the above we have that $CG\cdot CA=CK\cdot CB$ and this means that C has equal power to the two circles, so it is on the radical axis of them, so C,D',M are collinear. From the above it follows that $D'\equiv D$.

Finally, from the cyclic quadrilaterals $EAMD$ and $DMBF$ we have that

$$\angle ADB=180^\circ-\angle EDA-\angle BDF=180^\circ-\angle AME-\angle BMF=\angle EMF\ .$$

Solution 2. Let H,G be the points of intersection of ME,MF with AC,BC respectively. From the similarity of triangles ΔMHA and ΔMAE we get

$$\frac{MH}{MA}=\frac{MA}{ME}\ ,$$

thus,

$$MA^2=MH\cdot ME\ . \qquad (1)$$

Similarly, from the similarity of triangles ΔMBG and ΔMFB we get

$$\frac{MB}{MF}=\frac{MG}{MB}$$

thus,

$$MB^2=MF\cdot MG\ . \qquad (2)$$

Since $MA=MB$, from (1) and (2) we have that the points E,H,G,F are concyclic.

Therefore, we get that $\angle FEH=\angle FEM=\angle HGM$. Also, the quadrilateral $CHMG$ is cyclic, so $\angle CMH=\angle HGC$. We have

$\angle FEH + \angle CMH = \angle HGM + \angle HGC = 90^\circ$.

Thus $CM \perp EF$. Now, from the cyclic quadrilaterals $FDMB$ and $EDMA$, we get that $\angle DFM = \angle DBM$ and $\angle DEM = \angle DAM$. Therefore, the riangles ΔEMF and ΔADB are similar, so $\Delta ADB = \angle EMF$. Even more

$\angle ADB = \angle ADM + \angle MDB = \angle AEM + \angle MFB = \angle CAB + \angle CBA$.

Problem 4. An L-shape is one of the following four pieces, each consisting of three unit squares:

A 5×5 board, consisting of 25 unit squares, a positive integer $k \leq 25$ and an unlimited supply of L-shapes are given. Two players, A and B, play the following game:starting with A they alternatively mark a previously unmarked unit square until they mark a total of k unit squares.

We say that a placement of L-shapes on unmarked unit squares is called good if the L-shapes do not overlap and each of them covers exactly three unmarked unit squares of the board. B wins if every good placement of L-shapes leaves uncovered at least three unmarked unit squares. Determine the minimum value of k for which B has a winning strategy.

Solution. We will show that player A wins if $k = 1, 2$ or 3 , but player B wins if $k = 4$. Thus the smallest k for which B has a winning strategy exists and is equal to 4 .

If $k = 1$, player A marks the upper left corner of the square and then fills it as follows.

If $k = 2$, player A marks the upper left corner of the square. Whatever square player B marks, then player A can fill in the square in exactly the same patern as above except that he doesn't put the trimino which covers the marked square of B . Player A wins because he has left only two unmarked squares uncovered.

For $k = 3$, player A wins by following the same strategy. When he has to mark a square for the second time, he marks any yet unmarked square of the triomino that covers the marked square of B .

Let us now show that for $k = 4$ player B winning strategy. Since there will be 21 unmarked squares, player A will need to cover all of them with seven L-shaped triominoes. We can assume that in his first move, player A does not mark any square in the bottom two rows of the chesboard(otherwise just rotate the chessboard). In his first move player B marks the square labeled 1 in the following figure.

If player A in his next move marks the squares 2 then player B marks the square labeled 5. Player B wins as the squre labeled 3 is left unmarked but cannot be covered with an L-shaped triomino.

Finally, if player A in his next move marks one of the squares labeled 3 or 4, player B marks the other of these two squares. Player B wins as the square labeled 2 is left unmarked but cannot be covered with an L-shaped triomino.

Since we have covered all possible cases, player B wins when $k = 4$.


THAILAND 2015
IMO
4-16 July 201
In Chiang Mai, Th
Hosted by The Institute for the Promotion of Teaching Science and Technology
University • The Mathematical Association of Thailand under the Patronage of H
• The Promotion of Academic Olympiad and Development of Science Education F
Patronage of Her Royal Highness Princess Galyani Vadhana Krom Luang Naradhiwas R
IPST

18th Junior Balkan
Mathematical Olympiad
June 21-26, Ohrid, R. Macedonia

20-та Македонска
математичка олимпијада

17-та JMMO